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# Semisimple Corings

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**Corings.**(Following *Sweedler*, 1975) Let $A$ be a ring. An $A$–*coring* is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \epsilon_{\mathfrak{C}})$ which consists of uan $A$–bimodule $\mathfrak{C}$ and two homomorphism of $A$–bimodules

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C} \qquad \epsilon_{\mathfrak{C}} : \mathfrak{C} \longrightarrow A \tag{1}$$

such that the diagrams

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ {\scriptstyle \Delta_{\mathfrak{C}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{C} \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ \mathfrak{C} \otimes_A \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mathfrak{C} \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & \mathfrak{C} \otimes_A A \end{array} \qquad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \epsilon_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} \\ & & A \otimes_A \mathfrak{C} \end{array}$$

commute.

**Example.** *Sweedler's canonical coring*. Consider $B \leq A$ a subring.

Bimodule:

$$A \otimes_B A, \quad a(a' \otimes a'')a''' = aa' \otimes a''a'''$$

Comultiplication:

$$\Delta : A \otimes_B A \longrightarrow A \otimes_B A \otimes_A A \otimes_B A$$

$$a \otimes a' \longmapsto a \otimes 1 \otimes 1 \otimes a'$$

Counity:

$$\epsilon : A \otimes_B A \longrightarrow A, \quad a \otimes a' \longmapsto aa'$$

**Example.** *Idempotent coring.*

Bimodule: A twosided ideal $I$ such that $I^2 = I$ and ${}_A A/I$ or $A/I_A$ is flat.

Comultiplication: The canonical isomorphism $I \cong I \otimes_A I$.

Counity: The inclusion $I \subseteq A$.

**Example.** *Coring stemming from an entwining structure (Brzeziński-Takeuchi)*

$(A, C)_\varphi$ an entwining structure over a commutative ring $K$, with $A$ a $K-$algebra, $C$ a $K-$coalgebra and $\varphi : C \otimes_K A \to A \otimes_K C$ the entwining morphism.

Bimodule: $A \otimes_K C$, $a(a' \otimes_K c)a'' = aa'\varphi(c \otimes a'')$.

Comultiplication: the composite

$$A \underset{K}{\otimes} C \xrightarrow{A \underset{K}{\otimes} \Delta_C} A \underset{K}{\otimes} C \underset{K}{\otimes} C \cong A \underset{K}{\otimes} C \underset{A}{\otimes} A \underset{K}{\otimes} C$$

Counity: $A \otimes_K \epsilon_C : A \otimes_K C \to A \otimes_K K \cong A.$

**Comodule categories.** Given an $A$–coring $\mathfrak{C}$, the category $\mathcal{M}^{\mathfrak{C}}$ of all right $\mathfrak{C}$–comodules is defined as follows.

Objects: pairs $(M, \rho_M)$, with $M_A$ a module, and $\rho_M : M \to M \otimes_A \mathfrak{C}$ a morphism of $A$–modules such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle M \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{\rho_M \underset{A}{\otimes} \mathfrak{C}} & M \underset{A}{\otimes} \mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ & \searrow^{\cong} & \downarrow{\scriptstyle M \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & M \underset{A}{\otimes} A \end{array}$$

commute.

Morphisms: a morphism $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism of $A$–modules $f : M \to N$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{f \underset{A}{\otimes} \mathfrak{C}} & N \underset{A}{\otimes} \mathfrak{C} \end{array}$$

$\mathcal{M}^{\mathfrak{C}}$ is an additive category with inductive limits, but it is not abelian in general (kernels can fail).

$$\begin{array}{c} \mathcal{M}^{\mathfrak{C}} \\ U \Big\downarrow \Big\uparrow -\underset{A}{\otimes}\mathfrak{C} \\ \mathcal{M}_A \end{array}$$

the forgetful functor $U$ has a right adjoint $-\otimes_A \mathfrak{C}$

**Theorem.** *The following are equivalent.*
*(i) $\mathcal{M}^{\mathfrak{C}}$ is abelian and $U$ is left exact;*
*(ii) $\mathcal{M}^{\mathfrak{C}}$ is a Grothendieck category and $U$ is left exact;*
*(iii) ${}_A\mathfrak{C}$ is flat.*

**Remark.** *$\mathcal{M}^{\mathfrak{C}}$ can be abelian without ${}_A\mathfrak{C}$ flat.*

**Example.** Let ${}_RB_S$ a bimodule, $A = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}$, and $I = I^2 = \begin{pmatrix} R & B \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{M}^I \sim \mathcal{M}_R$ but ${}_AI$ is no flat unless ${}_RB$ is.

## Example worked out.

$$A = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}, I = \begin{pmatrix} R & B \\ 0 & 0 \end{pmatrix}$$

Objects of $\mathcal{M}_A$: $M = (M', M'', \mu)$, $M' \in \mathcal{M}_R, M'' \in \mathcal{M}_S$ and $\mu : M' \otimes_R B \to M''$ is $S$–linear.
Morphisms of $\mathcal{M}_A$:

$$(f', f'') : (M'_R, M''_S, \mu) \to (N'_R, N''_S, \nu)$$

making commute
$$\begin{array}{ccc} M' \otimes_R B & \xrightarrow{\mu} & M'' \\ \downarrow{\scriptstyle f\otimes B} & & \downarrow{\scriptstyle f''} \\ N' \otimes_R N & \xrightarrow{\nu} & N'' \end{array}$$

$- \otimes_A I \cong F$, where $F : \mathcal{M}_A \to \mathcal{M}_A$ is given by

$$F(M', M'', \mu) = (M', M' \otimes_R B, 1)$$

and

$$F(M) = (M', M' \otimes_R B, 1) \xrightarrow{(1,\mu)} (M', M'', \mu) = M$$

is natural. Using this, we have $\mathcal{M}^I$ consists of the modules $(M', M'', \mu)$ such that $\mu$ is isomorphism, and the functor $\mathcal{M}^I \to \mathcal{M}_A$ which sends $(M', M'', \mu)$ onto $M'$ is an equivalence.

## Two convolution rings

$$^*\mathfrak{C} = \mathrm{Hom}({}_A\mathfrak{C}, A) \qquad (f *_l g = f \circ (\mathfrak{C} \underset{A}{\otimes} g) \circ \Delta)$$

$$\mathfrak{C}^* = \mathrm{Hom}(\mathfrak{C}_A, A) \qquad (f *_r g = g \circ (f \underset{A}{\otimes} \mathfrak{C}) \circ \Delta)$$

## Two pairs of rings

$$\begin{array}{ccccc} \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) & & f & & \mathrm{End}(\mathfrak{C}_{\mathfrak{C}})^{op} \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ {}^*\mathfrak{C} & & \epsilon \circ f & & \mathfrak{C}^* \end{array}$$

Thus, we have a bimodule structure ${}_{^*\mathfrak{C}}\mathfrak{C}_{\mathfrak{C}^*}$.

## Rational modules.

We have a functor $\mathcal{M}^{\mathfrak{C}} \to {}_{^*\mathfrak{C}}\mathcal{M}$, which makes $M \in \mathcal{M}^{\mathfrak{C}}$ a module ${}_{^*\mathfrak{C}}M$ with the action $\varphi m = \sum m_0 \varphi(m_1)$.

Try to reverse the process: Let $M \in {}_{^*\mathfrak{C}}\mathcal{M}$, an element $m \in M$ is said to be *rational* if $\varphi m = \sum m_i \varphi(c_i)$ for every $\varphi \in {}^*\mathfrak{C}$ and some $(m_i, c_i) \in M \times \mathfrak{C}$.

Define the coaction $M \to M \otimes_A \mathfrak{C}$ which sends $m$ onto $\sum_i m_i \otimes_A c_i$. This is mathematically sound whenever ${}_A\mathfrak{C}$ is required to be projective.

This defines a functor $Rat^l : {}_{^*\mathfrak{C}}\mathcal{M} \to \mathcal{M}^{\mathfrak{C}}$ defined as

$$Rat^l(M) = \{m \in M \mid m \text{ is rational}\}$$

which allows to recognize $\mathcal{M}^{\mathfrak{C}}$ as isomorphic to a full subcategory of ${}_{^*\mathfrak{C}}\mathcal{M}$.

**Theorem.** *Let $\mathfrak{C}$ be an $A$–coring. The following statements are equivalent:*

*(i) every left $\mathfrak{C}$–comodule is semisimple and ${}^{\mathfrak{C}}\mathcal{M}$ is abelian;*
*(ii) every right $\mathfrak{C}$–comodule is semisimple and $\mathcal{M}^{\mathfrak{C}}$ is is abelian;*
*(iii) $\mathfrak{C}$ is semisimple as a left $\mathfrak{C}$–comodule and $\mathfrak{C}_A$ is flat;*
*(iv) $\mathfrak{C}$ is semisimple as a right $\mathfrak{C}$–comodule and ${}_A\mathfrak{C}$ is flat;*
*(v) $\mathfrak{C}$ is semisimple as a right $\mathfrak{C}^*$–module and $\mathfrak{C}_A$ is projective;*
*(vi) $\mathfrak{C}$ is semisimple as a left ${}^*\mathfrak{C}$–module and ${}_A\mathfrak{C}$ is projective.*

**Proof:** $(i) \Rightarrow (iii)$ Every monomorphism splits in the semisimple category ${}^{\mathfrak{C}}\mathcal{M}$. Thus $U$ : ${}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ preserves monomorphisms, whence it is exact. Therefore, $\mathfrak{C}_A$ is flat.
$(iii) \Rightarrow (iv)$ $\mathfrak{C}_A$ flat $\Rightarrow$ ${}^{\mathfrak{C}}\mathcal{M}$ Grothendieck and $\mathfrak{C} \otimes_A -$ : ${}_A\mathcal{M} \to {}^{\mathfrak{C}}\mathcal{M}$ is exact.
Thus, $U \dashv \mathfrak{C} \otimes_A - \Rightarrow U$ : ${}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ preserves projectives.
If $M \in {}^{\mathfrak{C}}\mathcal{M}$, then

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \uparrow \\ 0 & \longrightarrow & M & \longrightarrow & \mathfrak{C} \otimes_A M \\ & & & & \uparrow \\ & & & & A^{(I)} \otimes_A \mathfrak{C} \cong \mathfrak{C}^{(I)} \end{array},$$

and $M$ is semisimple.
Thus, every object in ${}^{\mathfrak{C}}\mathcal{M}$ is projective, in particular ${}_{\mathfrak{C}}\mathfrak{C}$ is projective. Hence, ${}_A\mathfrak{C}$ is projective.
Finally, $\mathfrak{C} \in {}^{\mathfrak{C}}\mathcal{M}$ semisimple $\Rightarrow$ ${}_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})}\mathfrak{C}$ is a semisimple module. Since ${}^*\mathfrak{C} \cong \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$, we get that ${}_{*\mathfrak{C}}\mathfrak{C}$ is semisimple.
(vi) $\Rightarrow$ (ii) If ${}_A\mathfrak{C}$ is projective, then $\mathcal{M}^{\mathfrak{C}} \sim Rat({}^{*\mathfrak{C}}\mathcal{M})$.
Thus, $\mathcal{M}^{\mathfrak{C}}$ is abelian. Moreover, ${}_{*\mathfrak{C}}\mathfrak{C}$ subgenerates $Rat({}^{*\mathfrak{C}}\mathcal{M})$ and, thus, this category is semisimple.

$\mathfrak{I} \subseteq \mathfrak{C}$ subbicomodule $\equiv$
$\Delta(\mathfrak{I}) \subseteq Ker(\mathfrak{C} \otimes_A \mathfrak{C} \to \mathfrak{C}/I \otimes_A \mathfrak{C}/I)$

$\mathfrak{C}$ simple $\equiv$ every subbicomodule is trivial

**Theorem.** *The $A$–coring $\mathfrak{C}$ is semisimple if and only if $\mathfrak{C} = \oplus_{\lambda \in \Lambda} \mathfrak{C}_\lambda$ for $\mathfrak{C}_\lambda$ simple semiartinian $A$–corings with ${}_A\mathfrak{C}_\lambda$, $\mathfrak{C}_{\lambda A}$ proyective for every $\lambda$. This decomposition is unique.*

semiartinian object $\equiv$ every proper factor contains a simple subobject.

**Theorem.** *Assume* ${}_A\mathfrak{C}$ *y* $\mathfrak{C}_A$ *projective. The following are equivalent*

*(i)* $\mathfrak{C}$ *is a simple semiartinian* $A$*–coring;*

*(ii)* $\mathfrak{C}$ *is simple with nonzero left socle;*

*(iii)* $\mathfrak{C}$ *is semisimple with a unique type of simple left comodule;*

*(iv)* $\mathfrak{C}$ *is simple with nonzero right socle;*

*(v)* $\mathfrak{C}$ *is semisimple with a unique type of simple right comodule;*

*(vi)* $\mathfrak{C} \cong \Sigma^* \otimes_D \Sigma$*, where* ${}_D\Sigma_A$ *is a bimodule with* $\Sigma_A$ *finitely generated and projective, and* $D$ *a division ring.*

**Corollary. (Wedderburn's Theorem)** *Let* $C$ *be a coalgebra over a field* $K$*. Then* $C$ *is simple if and only if* $C \cong \Sigma^* \otimes_D \Sigma$ *for a finite-dimensional vector space* $\Sigma_K$ *and a division ring* $D \subseteq \mathrm{End}(\Sigma_K)$*. Moreover,* ${}^*C \cong \mathrm{End}({}_D\Sigma)$*.*

## Comatrix Corings

Let ${}_B\Sigma_A$ be a $B-A$–bimodule; assume $\Sigma_A$ is finitely generated and projective. Consider $\Sigma^* = \mathrm{Hom}_A(\Sigma, A_A)$ canonically as an $A-B$–bimodule.

Pick $\{e_i^*, e_i\} \subseteq \Sigma^* \times \Sigma$ a dual basis.

Bimodule: $\Sigma^* \otimes_B \Sigma$, $a(\varphi \otimes u)a' = a\varphi \otimes ua'$.

Comultiplication:

$$\Sigma^* \otimes_B \Sigma \xrightarrow{\Delta} \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \Sigma$$

$$\varphi \otimes_B u \longmapsto \textstyle\sum_i \varphi \otimes_B e_i \otimes_A e_i^* \otimes_B u$$

Counity:

$$\Sigma^* \otimes_B \Sigma \xrightarrow{\epsilon} A, \qquad \varphi \otimes_B u \longmapsto \varphi(u)$$

The name of "comatrix" comes from

$$^*(\Sigma^* \otimes_B \Sigma) \cong \mathrm{End}({}_B\Sigma),$$

as rings.

## Examples of comatrix corings

**Sweedler's canonical coring.** Let $B \subseteq A$ a ring extension. Put $\Sigma = {}_B A_A$. Then $\Sigma^* \otimes_B \Sigma \cong A \otimes_B A$ is the usual Sweedler's canonical $A$–coring.

**Dual coring.** Let $A \subseteq B$ a ring extension. Assume $B_A$ finitely generated and projective. Take $\Sigma = {}_B B_A$; then $\Sigma^* \otimes_B \Sigma = B^* \otimes_B B \cong B^*$, and the $A$–coring structure is dual to the multiplication of $B$.

**Comatrix coalgebras.** Let $A = B = K$ be a commutative field, $\Sigma$ a finite dimensional vector space. Then $\Sigma^* \otimes_K \Sigma$ is the usual comatrix coalgebra.

## The structure theorem.

**Theorem.** *Let $A$ be any ring. An $A$–coring $\mathfrak{C}$ is semisimple if and there is a family $\Lambda$ of finitely generated projective right $A$–modules, and a division ring $D_\Sigma \subseteq \mathrm{End}(\Sigma_A)$ for each $\Sigma \in \Lambda$ such that*

$$\mathfrak{C} \cong \bigoplus_{\Sigma \in \Lambda} \Sigma^* \otimes_{D_\Sigma} \Sigma$$

*Moreover, if $\Lambda'$ is another such a family, then there is a bijective map $\Phi : \Lambda \to \Lambda'$, and a isomorphism of right $A$–modules $g_\Sigma : \Sigma \to \Phi(\Sigma)$ for every $\Sigma \in \Lambda$ such that $D_{\Phi(\Sigma)} = g_\Sigma D_\Sigma g_\Sigma^{-1}$.*