An example of biseparable extension which is not Frobenius

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Based on a joint work with F.J. Lobillo, G. Navarro and P. Sánchez-Hernández.

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- In the paper

S. Caenepeel and L. Kadison.
Are biseparable extensions Frobenius?
K-Theory, 24(4):361–383, 2001.

it is explained how deep connections between separable and Frobenius extensions were found from the very beginning.

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- Split extensions are naturally considered since separability and splitting can be viewed as particular cases of the notion of separable module introduced by Sugano in 1971.
- Biseparable extensions are therefore considered because they contain both notions of separable and split extensions under the same module theoretic approach.
- Biseparable extensions are finitely generated and projective, hence the example they provide is not a counter example of their main question: "Are biseparable extensions Frobenius?"

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We found this example during our investigation on duality for convolutional error correcting codes with a cyclic structure.

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Let also denote by $\sigma : A \to A$ an algebra $\mathbb{F}$-automorphism and $\delta : A \to A$ a σ-derivation on A, i.e.

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

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$$S = A[x; \sigma, \delta],$$

the Ore extension of A whose product is subject to the rule

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We give conditions on σ and δ in order to get that $R \subseteq S$ inherits the corresponding properties (separable, split, Frobenius) from $\mathbb{F} \subseteq A$. A precise construction of A, σ and δ will lead to the counterexample.

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Given $a \in A$, $n \geq 0$, we denote by $N_i^n(a)$ the coefficients in A determined by

$$x^n a = \sum_i N_i^n(a) x^i. \quad (1)$$

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For an Ore polynomial $\sum_{i=0}^n g_i x^i \in S$, we have

$$\left(\sum_{i=0}^n g_i x^i \right) a = \sum_{i=0}^n \left(\sum_{k=i}^n g_k N_i^k(a) \right) x^i. \tag{2}$$

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Consider $\mathbb{F}$-linear operators $N_i^n : A \to A$. Then

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The ring extension $R \subseteq S$ makes S free of finite rank both as a left as a right R–module. More precisely,

Lemma 2

Let $\{a_1, \dots, a_r\}$ be an $\mathbb{F}$-basis of A. The following statements hold.

1. *$\{a_1, \dots, a_r\}$ is a right basis of S over R.*
2. *$\{a_1, \dots, a_r\}$ is a left basis of S over R.*

The main construction

Set $S^* = \hom_R(S_R, R)$, which is a right S–module with

$$(\varphi s)(s') = \varphi(ss'), \qquad (\varphi \in S^*, s, s' \in S)$$

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There exists a bijective correspondence between the following sets.

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To define a right S–linear map $\alpha_\varepsilon : S \to S^*$ we need just to specify $\alpha_\varepsilon(1) \in S^*$.

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Note that, for every $f, g \in S$, one has

$$\alpha_\varepsilon(f)(g) = \alpha_\varepsilon(1)(fg) = \alpha_\varepsilon(fg)(1).$$

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Then, for every $b \in A$, we get from (2) that

$$0 = \alpha_\varepsilon(f)(b) = \alpha_\varepsilon(fb)(1) \overset{(2)}{=} \sum_{i=0}^{n} \varepsilon \left(\sum_{k=i}^{n} f_k N_i^k(b) \right) x^i.$$

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Since σ is an automorphism, $\langle f_n, b\rangle_\varepsilon = \varepsilon(f_n b) = 0$ for all $b \in A$, which contraducts the non-degeneracy of $\langle -, -\rangle_\varepsilon$.

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Thus α_ε is injective.

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Let us show that $x^n a_i^* \in \operatorname{Im} \alpha_\epsilon$ for all $n \geq 0$ and $1 \leq i \leq r$, which yields the result (recall that $R = \mathbb{F}[x]$).

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For any $m \geq 0$, since $\{\sigma^m(a_1), \dots, \sigma^m(a_r)\}$ is an $\mathbb{F}$-basis of A, and $\langle -, - \rangle_\varepsilon$ is non-degenerate, there exist $b_1^{(m)}, \dots, b_r^{(m)} \in A$ such that

$$\varepsilon \left(b_i^{(m)} \sigma^m(a_j) \right) = \delta_{ij} \tag{4}$$

for all $1 \leq i, j \leq r$.

The main construction, IV

For each $1 \leq i \leq r$, set

$$g^{(i)} = \sum_{k=0}^{n} g_k^{(i)} x^k \in S,$$

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$$g_m^{(i)} = -\sum_{\ell=1}^{r} b_\ell^{(m)} \left(\sum_{k=m+1}^{n} \varepsilon \left(g_k^{(i)} N_m^k(a_\ell) \right) \right). \tag{5}$$

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Then, by (4), for all $1 \leq i, j \leq r$,

$$\varepsilon \left(g_n^{(i)} \sigma^n(a_j) \right) = \varepsilon \left(b_i^{(n)} \sigma^n(a_j) \right) = \delta_{ij} \tag{6}$$

and

The main construction, V

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\begin{aligned}
\varepsilon\left(g_m^{(i)} N_m^m(a_j)\right) &= \varepsilon\left(g_m^{(i)} \sigma^m(a_j)\right) \\
&\overset{(5)}{=} \varepsilon\left(-\sum_{\ell=1}^{r} b_\ell^{(m)} \left(\sum_{k=m+1}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_\ell)\right)\right) \sigma^m(a_j)\right) \\
&= -\sum_{\ell=1}^{r} \sum_{k=m+1}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_\ell)\right) \varepsilon\left(b_\ell^{(m)} \sigma^m(a_j)\right) \\
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$$\begin{aligned}\varepsilon\left(g_m^{(i)} N_m^m(a_j)\right) &= \varepsilon\left(g_m^{(i)} \sigma^m(a_j)\right)\\ &\overset{(5)}{=} \varepsilon\left(-\sum_{\ell=1}^{r} b_\ell^{(m)}\left(\sum_{k=m+1}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_\ell)\right)\right)\sigma^m(a_j)\right)\\ &= -\sum_{\ell=1}^{r}\sum_{k=m+1}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_\ell)\right)\varepsilon\left(b_\ell^{(m)}\sigma^m(a_j)\right)\\ &\overset{(6)}{=} -\sum_{k=m+1}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_j)\right).\end{aligned}$$

Hence

$$\sum_{k=m}^{n} \varepsilon\left(g_k^{(i)} N_m^k(a_j)\right) = 0 \tag{7}$$

for $1 \le i, j \le r$, $0 \le m \le n-1$.

The main construction, VI

Now,

$$\alpha_\varepsilon(g^{(i)})(a_j) = \alpha_\varepsilon(g^{(i)}a_j)(1) \overset{(2)}{=} \sum_{m=0}^{n} \varepsilon\left(\sum_{k=m}^{n} g_k^{(i)} N_m^k(a_j)\right) x^m \overset{(7),(6)}{=} x^n a_i^*(a_j),$$

for all $j = 1, \ldots, r$.

The main construction, VI

Now,

$$\begin{aligned}\alpha_\varepsilon(g^{(i)})(a_j) &= \alpha_\varepsilon(g^{(i)}a_j)(1) \\ &\overset{(2)}{=} \sum_{m=0}^{n} \varepsilon\left(\sum_{k=m}^{n} g_k^{(i)} N_m^k(a_j)\right) x^m \\ &\overset{(7),(6)}{=} x^n a_i^*(a_j),\end{aligned}$$

for all $j = 1, \ldots, r$.

So $x^n a_i^* = \alpha_\varepsilon(g^{(i)}) \in \operatorname{Im} \alpha_\varepsilon$, as required.

The main construction, VII

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$$\delta_{ij} = a_i^*(a_j) = \alpha(g_i)(a_j) = \alpha\left(\sum_{k=0}^{n_i} g_{ij} x^k\right)(a_j) = \sum_{k=0}^{n_i} \alpha(g_{ij} x^k)(a_j)$$
$$= \sum_{k=0}^{n_i} \alpha(g_{ik})(x^k a_j) = \sum_{k=0}^{n_i} \alpha(g_{ik})\left(\sum_{m=0}^{k} N_m^k(a_j) x^m\right) = \sum_{k=0}^{n_i} \sum_{m=0}^{k} \alpha(g_{ik})(N_m^k(a_j)) x^m.$$

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Therefore, the $\mathbb{F}$–linear map α satisfies that $\alpha(g_i)(a_j) = \delta_{ij}$, for the $\mathbb{F}$–bases $\{g_1, \dots, g_r\}$ and $\{a_1, \dots, a_r\}$ of A.

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Therefore, ε_α is a well defined Frobenius functional on A.

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And these two maps are right R–linear, so that the following computation, for $a \in A$, suffices:

$$\alpha_{\varepsilon_\alpha}(1)(a) = \varepsilon_\alpha(a) = \alpha(a)(1) = \alpha(1)(a).$$

Frobenius and semi Frobenius

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Definition 4

A unital ring extension $C \subseteq B$ is said to be right (rest. left) semi Frobenius if B_C (resp. ${}_CB$) is finitely generated and projective and $B_B \cong B_B^*$ (resp. ${}_BB \cong {}_B{}^*B$). (Duals w.r.t. C).

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Theorem 3 gives:

Theorem 5

With $R = \mathbb{F}[x]$ and $S = A[x; \sigma, \delta]$, the following statements are equivalent:

1. *A is a Frobenius $\mathbb{F}$-algebra,*
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Proof.

Apply Theorem 3 plus the well known identity $S^{op} = A^{op}[x; \sigma^{-1}, -\delta\sigma^{-1}]$. □

¿Symmetry?

By Theorem 5, $\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ is left semi Frobenius if and only if it is right semi Frobenius.

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Problem: ¿Is the notion of a semi Frobenius ring extension left-right symmetric?

¿When is $\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ Frobenius?

Recall that the ring extension $R \subseteq S$ is Frobenius if there exists an isomorphism of bimodules ${}_R S^*_S \cong {}_R S_S$.

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Theorem 6

With $R = \mathbb{F}[x]$ and $S = A[x;\sigma,\delta]$. There exists a bijective correspondence between the sets of

1. *$R - S$-isomorphisms from S to S^*.*
2. *Frobenius functionals $\varepsilon : A \to \mathbb{F}$ satisfying $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = 0$.*

Proof.

Let $\alpha : S_R \to S_S^*$ be an isomorphism corresponding to a Frobenius functional $\varepsilon : A \to \mathbb{F}$ under the bijection stated in Theorem 3.

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Both $\alpha(x)$ and $x\alpha(1)$ are right R–linear maps, so, they are equal if and only if $\alpha(x)(a) = (x\alpha(1))(a)$ for every $a \in A$.

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Thus, from the computations

$$\alpha(x)(a) = \alpha(xa)(1) = \alpha(\sigma(a)x + \delta(a))(1) = \varepsilon(\sigma(a))x + \varepsilon(\delta(a)),$$

$$(x\alpha(1))(a) = x\alpha(1)(a) = x\alpha(a)(1) = x\varepsilon(a) = \varepsilon(a)x,$$

we get that α is left R–linear if and only if $\varepsilon(\sigma(a)) = \varepsilon(a)$ and $\varepsilon(\delta(a)) = 0$ for every $a \in A$.

□

¿When is $\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ Frobenius?, II

The following is the characterization which will be used to built an example of biseparable extension which is not Frobenius.

Theorem 7

The ring extension $\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ is Frobenius if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}$ verifying $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = 0$.

$\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ split

We keep the notation $R = \mathbb{F}[x]$, $S = A[x;\sigma,\delta]$.

Recall that the extension $R \subseteq S$ is said to be *split* if the inclusion map is a split monomorphism of R–bimodules.

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Proposition 8

Assume that there exists a linear form $\xi : A \to \mathbb{F}$ such that

$$\xi\sigma = \xi, \quad \xi\delta = 0, \text{ and } \xi(1) = 1.$$

Then the ring extension $R \subseteq S$ is split.

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Proof.

Assume ξ as in the statement. Define

$$\pi : S \to R, \qquad \sum_i f_i x^i \mapsto \sum_i \xi(f_i) x^i.$$

A straightforward computation shows that π is R–bilinear. Since $\pi(1) = \xi(1)$, we get that π splits the inclusion $R \subseteq S$. □

$\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ separable, I

Recall that a ring extension $C \subseteq B$ is *separable* if the multiplication map $\mu : B \otimes_C B \to B$ is a split epimorphism of B–bimodules.

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For brevity, we denote by $\sigma^{\otimes}$ and $\delta^{\otimes}$ the maps

$$\begin{aligned} \sigma^{\otimes} : A \otimes_{\mathbb{F}} A &\to A \otimes_{\mathbb{F}} A \\ a_1 \otimes a_2 &\mapsto \sigma(a_1) \otimes \sigma(a_2) \end{aligned}$$

$$\begin{aligned} \delta^{\otimes} : A \otimes_{\mathbb{F}} A &\to A \otimes_{\mathbb{F}} A \\ a_1 \otimes a_2 &\mapsto \sigma(a_1) \otimes \delta(a_2) + \delta(a_1) \otimes a_2 \end{aligned}$$

$\mathbb{F}[x] \subseteq A[x;\sigma,\delta]$ separable, II

Proposition 9

Let A be a separable $\mathbb{F}$–algebra with separability element $p \in A \otimes_{\mathbb{F}} A$. If $\sigma^{\otimes}(p) = p$ and $\delta^{\otimes}(p) = 0$, then $R \subseteq S$ is a separable ring extension.

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Proof.

Consider $p \in S \otimes_R S$ via the map $A \otimes_{\mathbb{F}} A \to S \otimes_R S$ resulting from the embedding $A \otimes_{\mathbb{F}} A \to S \otimes_{\mathbb{F}} S$ followed by the projection $S \otimes_{\mathbb{F}} S \to S \otimes_R S$.

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Write $p = p_1 \otimes_R p_2$ (sum understood).

$$
\begin{aligned}
xp &= xp_1 \otimes_R p_2 = (\sigma(p_1)x + \delta(p_1)) \otimes_R p_2 = \sigma(p_1) \otimes_R xp_2 + \delta(p_1) \otimes_R p_2 \\
&= \sigma(p_1) \otimes (\sigma(p_2)x + \delta(p_2)) + \delta(p_1) \otimes_R p_2 = \sigma(p_2) \otimes_R \sigma(p_1)x + \sigma(p_1) \otimes_R \delta(p_2) + \delta(p_1) \otimes_R p_2 \\
&= \sigma^{\otimes}(p)x + \delta^{\otimes}(p) = px
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□

The Example, I

Set $A = \mathcal{M}_2(\mathbb{F}_8)$, the ring of 2×2 matrices over $\mathbb{F}_8$, the field with eight elements.

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Let σ be the $\mathbb{F}_2$–algebra automorphism of A defined by

$$\sigma \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} {x_0}^2 & {x_1}^2 \\ {x_2}^2 & {x_3}^2 \end{pmatrix} \text{ for every } \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \in A. \tag{8}$$

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$$\sigma \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} {x_0}^2 & {x_1}^2 \\ {x_2}^2 & {x_3}^2 \end{pmatrix} \text{ for every } \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \in A. \tag{8}$$

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$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Hence, an $\mathbb{F}_2$-basis of A is given by $\mathcal{B} = \{a^{2^i} e_j \text{ with } 0 \leq i \leq 2 \text{ and } 0 \leq j \leq 3\}$.

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so that ε is determined by four values $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_2$ such that $\varepsilon(a^{2^i} e_j) = \gamma_j$ for $0 \leq i \leq 2$ and $0 \leq j \leq 3$.

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Therefore, $\xi\delta = 0$. By Proposition 8, the ring extension $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta]$ is split.

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- If $\gamma_1 = 1$, then $\varepsilon\delta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = 1$,
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Note that the kernel of ξ contains the left ideal

$$J = \left\{ \begin{pmatrix} 0 & c_2 \\ 0 & c_3 \end{pmatrix} \mid c_2, c_3 \in \mathbb{F}8 \right\},$$

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so that there is no Frobenius functional $\varepsilon : A \to \mathbb{F}_2$ verifying $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = 0$. By Theorem 7, the extension $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta]$ is not Frobenius.

The Example, IV

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$$+ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix}.$$

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$$+ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix}.$$

This is a separability element of the extension $\mathbb{F}_2 \subseteq A$, since it is the "composition" of the separability element $a \otimes a + a^2 \otimes a^2 + a^4 \otimes a^4$ of the extension $\mathbb{F}_2 \subseteq \mathbb{F}_8$, and the separability element $e_0 \otimes e_0 + e_2 \otimes e_3$ of the extension $\mathbb{F}_8 \subseteq A$.

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$$+ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix}.$$

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Since the Frobenius automorphism of $\mathbb{F}_8$ induces a permutation on $\{a, a^2, a^4\}$, it follows that

$$\sigma^{\otimes}(p) = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$
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$$= p.$$

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$$\delta^{\otimes}\left(\begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a^{2^{i+1}} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix},$$

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$$\delta^{\otimes}\left(\begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a^{2^{i+1}} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^{2^i} & 0 \\ 0 & 0 \end{pmatrix},$$

for $0 \leq i \leq 2$. Hence

$$\begin{aligned} \delta^{\otimes}(p) &= \delta^{\otimes}\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) + \delta^{\otimes}\left(\begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \right) \\ &+ \delta^{\otimes}\left(\begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \right) + \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \right) \\ &+ \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \right) + \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \right) \\ &= \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \right) \\ &+ \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \right) + \delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \right) \end{aligned} \tag{10}$$

The Example, VI

Moreover, by (9) and (8) again,

$$\delta^{\otimes}\left(\begin{pmatrix} 0 & 0 \\ a^{2^i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^{2^i} \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a^{2^{i+1}} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^{2^{i+1}+1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^{2^i+1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^{2^i} \\ 0 & 0 \end{pmatrix},$$

so we can follow the computations in (10) to get

$$\begin{aligned} \delta^{\otimes}(p) = {} & \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^5 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^5 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{11}$$

where we used that $a^7 = 1$.

The Example, VII

The identities $a^3 = a + a^4$ and $a^5 = a^2 + a^4$ in $\mathbb{F}_8$ allow us to expand (11) in order obtain

$$\begin{aligned}
\delta^{\otimes}(p) &= \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a+a^4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2+a^4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a+a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2+a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} = 0.
\end{aligned} \tag{12}$$

The Example, VIII

Conclusion: Therefore, $\mathbb{F}_2[x] \subseteq A[x;\sigma,\delta]$ is separable. Hence $\mathbb{F}_2[x] \subseteq A[x;\sigma,\delta]$ is a biseparable extension which is not Frobenius.

The Example, VIII

Conclusion: Therefore, $\mathbb{F}_2[x] \subseteq A[x;\sigma,\delta]$ is separable. Hence $\mathbb{F}_2[x] \subseteq A[x;\sigma,\delta]$ is a biseparable extension which is not Frobenius. It is semi Frobenius, since A is a Frobenius $\mathbb{F}_2$ algebra.

Nor of second kind, I

Let us recall the notion of a Frobenius extension of second kind, introduced by Nakayama and Tsuzuku in 1960.

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There is a structure of left C-module on C given by $a \cdot_{\kappa} b = \kappa(a)b$ for each $a, b \in C$.

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The $C - B$-bimodule structure on B^{*_κ} $\mathrm{Hom}(B_C, {}_\kappa C_C)$ is then given by $(afb)(c) = a \cdot_\kappa f(b'c) = \kappa(a)f(bc)$ for any $f \in B^{*_\kappa}$, $a \in C$ and $b, c \in B$.

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Definition 10

The ring extension $C \subseteq B$ is said to be a κ-*Frobenius extension*, or a *Frobenius extension of second kind*, if B is a finitely generated projective right C-module, and there exists a $C - B$-isomorphism from B to $B^{*\kappa}$.

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Nor of second kind; II

We keep denote $\mathbb{F}[x] = R \subseteq S = A[x;\sigma,\delta]$.

Proposition 11

Let $\kappa : R \to R$ be an automorphism with $\kappa(x) = mx + n$ for some $m, n \in \mathbb{F}$ with $m \neq 0$. There exists a bijection between the sets of

1. *$R - S$-isomorphisms $\alpha : S \to S^{*_\kappa}$.*
2. *Frobenius functionals $\varepsilon : A \to \mathbb{F}$ verifying $\varepsilon\sigma = m\varepsilon$ and $\varepsilon\delta = n\varepsilon$.*

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Proof.

By Theorem 3, there exists a left S-isomorphism $\beta : S \to S^{*_\kappa}$ if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}$. Now, analogously to the proof of Theorem 7,

$$\kappa(x)\beta(1)(a) = m\varepsilon(a)x + n\varepsilon(a).$$

and

$$\beta(x)(a) = \beta(1)(xa) = \beta(1)(\sigma(a)x + \delta(a)) = \varepsilon(\sigma(a))x + \varepsilon(\delta(a))$$

for every $a \in A$. Hence, β is left R-linear if and only if $\varepsilon\sigma = m\varepsilon$ and $\varepsilon\delta = n\varepsilon$. □

Nor of second kind, III

¿Are biseparable extensions Frobenius extensions of second kind? The answer is again negative.

We will show a counterexample with the help of the following

Theorem 12

$R \subseteq S$ is a Frobenius extension of second kind if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}$ and $m, n \in \mathbb{F}$ with $m \neq 0$ such that $\varepsilon\sigma = m\varepsilon$ and $\varepsilon\delta = n\varepsilon$.

Example 13 (Biseparable extensions are not necessarily Frobenius of second kind)

The same example $S = \mathcal{M}_2(\mathbb{F}_8)[x;\sigma,\delta]$ also provides an example of a biseparable extension which is not Frobenius of second kind. By Theorem 12, $\mathbb{F}_2[x] \subseteq A[x;\sigma,\delta]$ is Frobenius of second kind if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}_2$ verifying $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = \varepsilon$ (since we know that $R \subseteq S$ is not Frobenius). As reasoned before, ε is determined by four values $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_2$ such that $\varepsilon(a^{2^i} e_j) = \gamma_j$ for $i = 0, 1, 2$ and $j = 0, 1, 2, 3$. Now,

- If $\gamma_0 = 1$, then $0 = \varepsilon\delta \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 1$,
- If $\gamma_1 = 1$, then $0 = \varepsilon\delta \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = 1$,
- If $\gamma_2 = 1$, then $0 = \varepsilon\delta \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} = 1$,
- If $\gamma_3 = 1$, then $0 = \varepsilon\delta \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} \neq \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} = 1$,

so that $\varepsilon\delta = \varepsilon$ if and only if $\varepsilon = 0$. Thus, $R \subseteq S$ is not Frobenius of second kind.

Reformulation of the problem

Problem: ¿Are biseparable extensions left and right semi Frobenius?

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