Stability problems on minimal Lagrangian submanifolds

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Let \((M, \omega, g)\) be a Kähler manifold of complex dimension \(n\), Kähler form \(\omega\) and metric \(g\). We will denote by \(J\) the complex structure of \(M\). An immersion \(\Phi: \Sigma \to M\) of a manifold \(\Sigma\) of dimension \(n\) is called *lagrangian* if \(\Phi^*\omega = 0\). This means that the complex structure \(J\) defines a bundle isomorphism

\[
J: T\Sigma \to N\Sigma,
\]

from the tangent bundle to the normal bundle to \(\Sigma\). If \(H\) is the mean curvature vector of \(\Phi\), then we can define a 1-form on \(\Sigma\) by \(\sigma_H = H \lrcorner \omega\). Then it is known that

\[
d\sigma_H = \Phi^*\text{Ric},
\]

where Ric denotes the Ricci 2-form on \(M\). In particular, if \(M\) is Einstein, and as \(\Sigma\) is lagrangian, we obtain that \(\sigma_H\) is a closed 1-form on \(\Sigma\) and it defines a cohomology class in \(H^1_{dR}(\Sigma)\).

A *variation* of \(\Phi\) is a smooth map

\[
F: (-\varepsilon, \varepsilon) \times \Sigma \to M
\]
satisfying

1. Each map \(\Phi_t = F(t, -) : \Sigma \to M\) is an immersion,

2. \(\Phi_0 = \Phi\).

Let \(X = F_*(\frac{\partial}{\partial t})\) denote the *variation vector field*. If \(dv_t\) denotes the volume form of the induced metric on \(\Sigma\) by \(\Phi_t\), then we consider the *volume functional* \(V : (-\varepsilon, \varepsilon) \to \mathbb{R}\) given by

\[
V(t) = \int_{\Sigma} dv_t.
\]

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Then, it is well known that

\[ V'(0) = -n \int_{\Sigma} g(X, H) \, dv_0, \]

\( H \) being the mean curvature vector of \( \Phi \). \( \Phi \) is called a minimal immersion if its volume is critical for arbitrary compactly supported variations, i.e. \( H = 0 \).

A variation \( F \) of \( \Phi \) is called a Lagrangian variation if for each \( t \in (-\varepsilon, \varepsilon) \), \( \Phi_t : \Sigma \to M \) is a lagrangian immersion. The lagrangian immersion \( \Phi \) is called lagrangian stationary if its volume is critical for arbitrary compactly supported Lagrangian variations.

**Proposition 1** (Schoen-Wolfson, 2001) Let \( M \) be a Kähler-Einstein manifold and \( \Phi : \Sigma \to M \) a Lagrangian immersion. Then \( \Phi \) is lagrangian stationary if and only if \( \Phi \) is minimal.

Among the lagrangian variations, are particulary importants the hamiltonian variations. A variation with variation vector field \( X \) is called hamiltonian, if the 1-form on \( \Sigma \)

\[ \alpha_X := \Phi^*(X \lrcorner \omega) \]

is exact, i.e. there exists a function \( f \in C^\infty(\Sigma) \) such that \( df = \alpha_X \). The lagrangian immersion \( \Phi \) is called Hamiltonian stationary (Hamiltonian minimal) if its volume is critical for arbitrary compactly supported hamiltonian variations. This means that

\[ 0 = \int_{\Sigma} g(X, H) = \int_{\Sigma} g(df, \sigma_H) = \int_{\Sigma} g(f, \delta \sigma_H), \]

for all \( f \in C^\infty_0(\Sigma) \), where \( \delta \) is the adjoint operator of the differential \( d \). Hence \( \Phi \) is Hamiltonian stationary if and only if \( \delta \sigma_H = 0 \). This condition is equivalent to \( \text{div} \, JH = 0 \), where \( \text{div} \) stands for the divergence operator. If \( M \) is Kähler-Einstein, \( \sigma_H \) is also closed, and then \( \sigma_H \) is a harmonic 1-form.

Let \( M \) be a Calabi-Yau manifold, i.e., a Kähler manifold of complex dimension \( n \) whose holonomy group is a subgroup of \( SU(n) \). Then there exists a non-trivial parallel section of the canonical bundle of \( M \), i.e. a parallel complex volume form \( \Omega \) on \( M \). It is known that \( \Theta_\theta := \Re(e^{-i\theta}\Omega), \theta \in [0, 2\pi[ \) is a 1-parameter family of calibrations on \( M \), called the special lagrangian calibration with phase \( \theta \). An immersion \( \Phi : \Sigma \to M \) calibrated for the calibration \( \Theta_\theta \), i.e. such that \( \Phi^*(\Theta_\theta) \) is a volume form on \( \Sigma \), is called special Lagrangian with phase \( \theta \).
Proposition 2 (Harvey-Lawson, (1982)). Let $\Phi: \Sigma \to M$ an immersion of an $n$-dimensional orientable manifold $\Sigma$ in a Calabi-Yau manifold $M$ of complex dimension $n$. Then $\Phi$ is a minimal lagrangian immersion if and only if $\Phi$ is a special lagrangian immersion with phase $\theta$ for some $\theta \in [0, 2\pi]$.

Now we are going to consider the second variation of the volume functional and from now on we assume that the lagrangian submanifold $\Sigma$ is compact. Suppose that $F$ is a variation of a minimal Lagrangian immersion $\Phi : \Sigma \to M$ whose variational vector field $\xi$ is normal to $\Sigma$. Then it is well-known that

$$V''(0) = - \int_{\Sigma} g(L\xi, \xi) \, dv_0$$

where $L = \Delta^\perp + B + R : \Gamma(N\Sigma) \to \Gamma(N\Sigma)$ is the Jacobi operator given by:

$$\Delta^\perp = \sum_{i=1}^{n} \left\{ \nabla^\perp_{e_i} \nabla^\perp_{e_i} - \nabla^\perp_{\nabla^\perp_{e_i} e_i} \right\}, \quad B(\xi) = \sum_{i=1}^{n} \sigma(A_\xi e_i, e_i),$$

$$R(\xi) = \sum_{i=1}^{n} \bar{R}(\xi, e_i)e_i^\perp,$$

being $\nabla^\perp$ the normal connection, $\sigma$ the second fundamental form of $\Phi$, $A_\xi$ the Weingarten endomorphism associated to $\xi$, $\bar{R}$ the curvature operator of $M$ and $\{e_1, \ldots, e_n\}$ an orthonormal reference tangent to $\Sigma$, where $\perp$ stands for normal component.

Let $Q(\xi) = - \int_{M} \langle L\xi, \xi \rangle \, dv_0$ be the quadratic form associated to the Jacobi operator $L$. We will represent by $\text{Ind}(\phi)$ and $\text{Nul}(\phi)$ the index and nullity of the quadratic form $Q$ which are respectively the number of negative eigenvalues of $L$ and the multiplicity of zero as an eigenvalue of $L$. We say that $\Phi$ is stable if $\text{Ind}(\Phi) = 0$.

As $\Phi$ is Lagrangian, we consider the identification

$$\Gamma(N\Sigma) \equiv \Omega^1(\Sigma)$$

$$\xi \equiv \alpha,$$

where $\alpha = \Phi^*(\xi \omega)$, and in general $\Omega^p(\Sigma)$ denotes the space of $p$-forms on $\Sigma$. As consequence the Jacobi operator becomes in a operator $L : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ given by (Oh,1990)

$$L(\alpha) = \Delta \alpha + S(\alpha)$$
where $\Delta = \delta d + d \delta$ is the Laplacian operator acting on 1-forms and $S(\alpha)(V) = Ricc(\alpha, V)$, for any vector field $V$ tangent to $\Sigma$.

The Hodge decomposition

$$\Omega^1(\Sigma) = \mathcal{H}(\Sigma) \oplus dC^\infty(\Sigma) \oplus \delta \Omega^2(\Sigma),$$

allows to write in a unique way any 1-form $\alpha$ as $\alpha = \alpha_0 + df + \delta \beta$, with $\alpha_0$ a harmonic 1-form, $f$ a real function and $\beta$ a 2-form on $\Sigma$. The space $\mathcal{H}(\Sigma)$ of harmonic 1-forms is the kernel of $\Delta$, and its dimension is $\beta_1(\Sigma)$, the first Betti number of $\Sigma$. In the general case

**Theorem 1** (Harvey-Lawson (1982), Oh (1990), Takeuchi (1984)). Let $\Phi : \Sigma \to M$ be a minimal Lagrangian immersion of a compact manifold $\Sigma$.

1. If the first Chern class of $M$ is negative, then $\Phi$ is stable and $\text{Nul}(\Phi) = 0$.

2. If the first Chern class of $M$ is nonpositive, then $\Phi$ is stable.

3. If $M$ is a Calabi-Yau manifold, then $\Phi$ is volume minimizing in its homology class.

4. If the first Chern class of $M$ is positive and $\Phi$ is stable, then $H^1_{dR}(\Sigma) = 0$.

5. If $M$ is a Hermitian symmetric space of compact type and $\Sigma$ is a compact totally geodesic lagrangian submanifold embedded in $M$, then $\Sigma$ is stable if and only if $\Sigma$ is simply-connected.

Now we are going to study stability problems when the ambient space is the complex projective space. Let $\mathbb{CP}^n$ be the complex projective space endowed with the Fubibi-Study metric of constant holomorphic sectional curvature $4$. Then $\mathbb{CP}^n$ is a Kähler-Einstein manifold of complex dimension $n$ with $Ricc = 2(n + 1)g$. Hence the Jacobi operator $L$ of a minimal Lagrangian immersion $\Phi : \Sigma \to \mathbb{CP}^n$ is given by

$$L : \Omega^1(\Sigma) \to \Omega^1(\Sigma), \quad L = \Delta + 2(n + 1)I_d.$$

So $\text{Ind}(\Phi)$ is the number of eigenvalues of $\Delta$ (counted with multiplicity) less than $2(n + 1)$, and $\text{Nul}(\Phi)$ is the multiplicity of $2(n + 1)$ as a eigenvalue of $\Delta$. As $\Delta$ commutes with $d$ and $\delta$, the positive eigenvalues of $\Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$
are the positive eigenvalues of $\Delta : C^\infty(\Sigma) \to C^\infty(\Sigma)$ joint with the positive eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$. Hence

$$\text{Ind}(\Phi) = \beta_1(\Sigma) + \text{Ind}_0(\Sigma) + \text{Ind}_2(\Sigma),$$

where $\text{Ind}_0\Sigma$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : C^\infty(\Sigma) \to C^\infty(\Sigma)$ less than $2(n+1)$ and $\text{Ind}_2\Sigma$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ less than $2(n+1)$. As the variations associated to 1-forms in $dC^\infty(\Sigma)$ are Hamiltonian ones, we will call to $\text{Ind}_0(\Sigma)$ the Hamiltonian index of $\Phi$. It is clear that $\Phi$ is Hamiltonian stable if $\text{Ind}_0(\Sigma) = 0$.

Considering the standard minimal embedding $\mathbb{C}P^n \subset S^{n(n+2)-1}$ of the complex projective space into the $(n(n+2)-1)$-dimensional sphere of constant curvature $\frac{2(n+1)}{n}$, then $\Phi : \Sigma \to S^{n(n+2)-1}$ defines a minimal immersion too. In particular $2(n+1)$ is an eigenvalue of $\Delta : C^\infty(\Sigma) \to C^\infty(\Sigma)$ and the coordinates of the position vector $\Phi : \Sigma \to \mathbb{R}^{n(n+2)}$ are eigenfunctions for this eigenvalue. Hence we can say that $\Phi$ is Hamiltonian stable iff the first eigenvalue of $\Delta : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is $2(n+1)$. Using this fact, it is possible to estimate the nullity of a such minimal Lagrangian submanifold. In fact

**Theorem 2** (Montiel, Urbano (2006), Urbano (2007)). Let $\Phi : \Sigma \to \mathbb{C}P^n$ be a minimal Lagrangian immersion of a compact manifold $\Sigma$. Then

1. $\text{Nul}\Phi \geq \frac{n(n+3)}{2}$,

and the equality holds if and only if $\Phi$ is either the totally geodesic Lagrangian immersion of the unit sphere or the totally geodesic Lagrangian embedding of the real projective space.

2. If $n = 2$ and $\Phi$ is not totally geodesic, then $\text{Nul}\Phi \geq 6$,

and the equality holds if and only if $\Phi$ is the Lagrangian Clifford torus.

With respect to the index we have the following general result.

**Theorem 3** (Lawson (19), Urbano (1993)) Let $\Phi : \Sigma \to \mathbb{C}P^n$ be a minimal Lagrangian immersion of a compact manifold $\Sigma$. Then the first eigenvalue $\mu_1$ of $\Delta : \mathcal{H}(\Sigma) \oplus \delta\Omega^2(\Sigma) \to \mathcal{H}(\Sigma) \oplus \delta\Omega^2(\Sigma)$ satisfies

$$\mu_1 \leq 2(n-1),$$
and the equality holds if and only if $\Phi$ is either the totally geodesic Lagrangian immersion of the unit sphere or the totally geodesic Lagrangian embedding of the real projective space.

In particular $\Phi$ is always unstable.

In order to understand the next result, we need to give some preliminaries. If $\Pi : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration, then the canonical bundle $K_{\mathbb{CP}^n}$ can be identified with the bundle $S^{2n+1}/\mathbb{Z}_{n+1}$. Using symplectic topology, it is easy to prove that if $\Phi : \Sigma \to \mathbb{CP}^n$ is a minimal Lagrangian immersion, then $\Phi$ has a horizontal lift (also called Legendrian) to $S^{2n+1}/\mathbb{Z}_{n+1}$ if $\Sigma$ is orientable and a horizontal lift to $S^{2n+1}/\mathbb{Z}_{2(n+1)}$ if $\Sigma$ is non-orientable. These mean that a $(n+1) : 1$-covering of $\Sigma$ admits a Legendrian lift to $S^{2n+1}$ when $\Sigma$ is orientable and that a $(2n+2) : 1$-covering of $\Sigma$ admits a Legendrian lift to $S^{2n+1}$ when $\Sigma$ is non-orientable. In this setting, we define the level of $\Phi$ as the smallest number $q|2(n+1)$ such that $\Phi$ admits a Legendrian lift to $S^{2n+1}/\mathbb{Z}_q$. In particular the level of $\Phi$ is 1 when $\Phi$ admits a Legendrian lift to $S^{2n+1}$ and the level is 2 when $\Phi$ admits a Legendrian lift to $\mathbb{RP}^{2n+1} = S^{2n+1}/\mathbb{Z}_2$.

**Theorem 4 (Montiel, Urbano (2006))** Let $\Phi : \Sigma \to \mathbb{CP}^n$ be a minimal Lagrangian immersion of a compact manifold $\Sigma$. Then,

1. If the level of $\Phi$ is 1, then
   $$\text{Ind}_0(\Phi) \geq n + 1,$$
   and the equality holds if and only if $\Phi$ is the totally geodesic Lagrangian immersion of the unit $n$-sphere. If $\Phi$ is not totally geodesic, then $\text{Ind}_0(\Phi) \geq 2(n+1)$. Also
   $$\text{Ind}(\Phi) \geq \frac{(n+1)(n+2)}{2},$$
   and the equality holds if and only if $\Phi$ is the totally geodesic Lagrangian immersion of the unit $n$-sphere.

2. If the level of $\Phi$ is 2, then
   $$\text{Ind}(\Phi) \geq \frac{n(n+1)}{2},$$
   and the equality holds if and only if $\Phi$ is the totally geodesic Lagrangian embedding of the $n$-dimensional real projective space.
It is known that if $\Phi : \Sigma \to \mathbb{C}P^n$ is a minimal Lagrangian immersion of a compact manifold and $H^1(\Sigma,\mathbb{Z}_{2(n+1)}) = 0$ then the level of $\Sigma$ is 1. So, using the above result, if $\text{Ind}_0 \Phi < n + 1$ then $H^1(\Sigma,\mathbb{Z}_{2(n+1)}) \neq 0$, and in particular $H_1(\Sigma,\mathbb{Z}) \neq 0$.

Among the Lagrangian submanifolds with parallel second fundamental form of $\mathbb{C}P^n$ obtained by Naitoh and Takeuchi [NT], Amarzaya and Ohnita determined those which are Hamiltonian stable, proving the following result.

**Theorem 5** The compact minimal Lagrangian submanifolds embedded in $\mathbb{C}P^n$ of the following list:

1. $SU(p)/\mathbb{Z}_p$, $n = p^2 - 1$,
2. $SU(p)/SO(p)/\mathbb{Z}_p$, $n = (p - 1)(p + 2)/2$,
3. $SU(2p)/Sp(p)/\mathbb{Z}_{2p}$, $n = (p - 1)(2p + 1)$,
4. $E_6/F_4/\mathbb{Z}_3$, $n = 26$,

are Hamiltonian stables.

Now we consider the particular case $n = 2$, in which we can get better results.

If $\Phi : \Sigma \to \mathbb{C}P^2$ is a minimal Lagrangian immersion of a compact orientable surface $\Sigma$, then the star operator $\star : C^\infty(\Sigma) \to \Omega^2(\Sigma)$ tell us that the eigenvalues of $\Delta$ acting on $C^\infty(\Sigma)$ or on $\Omega^2(\Sigma)$ are the same, and so $\text{Ind}_0(\Sigma) = \text{Ind}_2(\Sigma)$. Hence, if $g$ is the genus of $\Sigma$, we have

$$\text{Ind}(\Sigma) = 2g + 2\text{Ind}_0(\Sigma).$$

If $\Phi : \Sigma \to \mathbb{C}P^2$ is a minimal Lagrangian immersion of a compact nonorientable surface $\Sigma$, we consider $\Phi \circ \pi : \tilde{\Sigma} \to \mathbb{C}P^2$ the corresponding minimal Lagrangian immersion of its 2:1 orientable covering $\tilde{\Sigma}$). If $\tau : \Sigma \to \Sigma$ is the change of sheet involution, then the spaces of forms on $\tilde{\Sigma}$ can be decompose in the following way:

$$\Omega^i(\tilde{\Sigma}) = \Omega^i_+(\tilde{\Sigma}) \oplus \Omega^i_-(\tilde{\Sigma}), \quad i = 0, 1, 2,$$

where

$$\Omega^i_{\pm}(\tilde{\Sigma}) = \{\alpha \in \Omega^i(\tilde{\Sigma}) / \tau^*\alpha = \pm \alpha\}.$$
Also the space of harmonic 1-forms on $\tilde{\Sigma}$ decomposes into two subspaces $H(\tilde{\Sigma}) = H^+(\tilde{\Sigma}) \oplus H^-(\tilde{\Sigma})$, where again $H^\pm(\tilde{\Sigma}) = \{ \alpha \in H(\tilde{\Sigma}) / \tau^* \alpha = \pm \alpha \}$.

In this way we obtain

$$\Omega^1(\tilde{\Sigma}) = H^\pm(\tilde{\Sigma}) \oplus d\Omega^0(\tilde{\Sigma}) \oplus \delta \Omega^2(\tilde{\Sigma}).$$

As $\pi \circ \tau = \pi$, the map $\alpha \in \Omega^i(\Sigma) \mapsto \pi^* \alpha \in \Omega^i(\tilde{\Sigma})$ allows to identify $H(\Sigma) \equiv H^+(\tilde{\Sigma})$ and $\Omega^i(\Sigma) \equiv \Omega^i(\tilde{\Sigma})$, $i = 0, 1, 2$. Also, as $\Sigma$ is nonorientable, $\star \tau^* = -\tau^* \star$, and so $\star$ identifies $\Omega^0(\tilde{\Sigma}) \equiv \Omega^2(\tilde{\Sigma})$. Hence we obtain the identification

$$\Omega^2(\Sigma) \equiv \Omega^0(\tilde{\Sigma})$$

$$\alpha \equiv f$$

where $\pi^* \alpha = f \omega_0$, being $\omega_0$ the volume 2-form on $\tilde{\Sigma}$.

As $\Sigma$ is nonorientable, the eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ are positives, and hence, taking into account the above remarks, $\text{Ind}_2(\Sigma)$ is the number of eigenvalues (counted with multiplicity) of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\tilde{\Sigma})$ less than 6. Also, as $\text{Ind}_0(\Sigma)$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : \Omega^0(\Sigma) \to \Omega^2(\tilde{\Sigma})$ less than 6, we obtain that

(1) \hspace{1cm} \text{Ind}_0(\Sigma) + \text{Ind}_1(\Sigma) = \text{Ind}_0(\tilde{\Sigma}),

and hence from (3)

$$2\text{Ind}(\Sigma) = \text{Ind}(\tilde{\Sigma}).$$

**Theorem 6** (Urbano (1993), (2007)) Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact orientable surface of genus $g$.

1. If $\Phi$ is Hamiltonian stable and $g \leq 4$, then $\Phi$ is an embedding and $\Phi(\Sigma)$ is the Clifford torus.

2. $\text{Ind}(\Phi) \geq 2$ and the equality holds if and only if $\Phi$ is an embedding and $\Phi(\Sigma)$ is the Clifford torus.

**Theorem 7** (Urbano (2007)) Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact nonorientable surface with Euler characteristic $\chi(\Sigma)$.

1. If $\Phi$ is Hamiltonian stable and $\chi(\Sigma) \geq -1$, then $\Phi$ is an embedding and $\Phi(\Sigma)$ is the totally geodesic real projective plane.
2. \( \text{Ind}(\Sigma) \geq 3 \) and the equality holds if and only if \( \Phi \) is an embedding and \( \Phi(\Sigma) \) is the totally geodesic real projective plane.

Now we consider as Kähler-Einstein surface the complex quadric \( S^2 \times S^2 \) which is, joint to \( \mathbb{CP}^2 \), the only compact Hermitian symmetric space of complex dimension 2. In [CU], Castro and Urbano have studied in depth its minimal Lagrangian surfaces, with special emphasis in its stability.

**Theorem 8** Let \( \Phi : \Sigma \to S^2 \times S^2 \) be a minimal Lagrangian immersion of a compact surface \( \Sigma \). Then

1. If \( \Sigma \) is stable, then \( \Phi(\Sigma) \) is the totally geodesic Lagrangian sphere \( M_0 \).
2. If \( \Sigma \) is Hamiltonian stable and \( \Sigma \) is orientable with genus \( g \leq 2 \), then \( \Phi \) is an embedding and \( \Phi(\Sigma) \) is either the totally geodesic sphere \( M_0 \) or the totally geodesic torus \( T \).
3. If \( \Sigma \) is a Hamiltonian stable Klein bottle, then \( \Phi \) is an embedding and \( \Phi(\Sigma) \) is the Klein bottle \( B \).
4. If \( \Sigma \) is unstable, then \( \text{Ind}(\Sigma) \geq 2 \) and the equality holds if and only if \( \Phi \) is an embedding and \( \Phi(\Sigma) \) is the totally geodesic torus \( T \).

**References**


