On the motion of an oscillator with a periodically time-varying mass

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Abstract. The stability of the motion of an oscillator with a periodically time-varying mass is under consideration. The key idea is that an adequate change of variables leads to a newtonian equation, where classical stability techniques can be applied: Floquet theory for the linear oscillator, KAM method in the nonlinear case. To illustrate this general idea, first we have generalized the results of [9] to the forced case; second, for a weakly forced Duffing's oscillator with variable mass, the stability in the nonlinear sense is proved by showing that the first twist coefficient is not zero.
Keywords: time-varying mass oscillator, stability, twist

1 Introduction

The study of the dynamics of a single-degree of freedom oscillator with variable mass is an important question for a variety of problems of Mechanical Engineering and Astronomy. In Celestial Mechanics, the problem of a variable mass is closely connected with the planar oscillations of a satellite describing an elliptic orbit around its mass center [2, 30] (see also [21] and the references therein). In Solid and Fluid Mechanics, the general principle of conservation of mass may be violated in many different mechanical systems in which material is expelled or captured by some mechanism. The review [10] presents a complete bibliography with references to applied problems in different areas like Biomechanics, Robotics, conveyor systems, fluid-structure interaction problems and many others situations. In particular, the rain-wind induced vibrations of cables [9] is a

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relevant example, where the formation of a water rivulet along the upper part of the cable may have a resonant effect. When the mass flow incoming on the cable is different from the mass flow shaken off then the mass of raindrops attached to the oscillator varies in time. According with the deduction made in [10, p.152], this model can be described by the scalar differential equation

$$M\ddot{y} = \dot{M}(w - \dot{y}) - ky + F, \tag{1}$$

where $M = M(t)$ is the periodic total mass of the oscillator, $y(t)$ is the displacement of center mass measured form its rest, $w(t)$ is the mean velocity at which the mass flow is hitting or leaving the oscillator, k is the positive stiffness coefficient of the linear restoring force and $F = F(t, y, \dot{y})$ is an external force. The history of equation (1) goes back to 1890 with Painleé and Seeliger. A nice relate of this history can be found in [10].

When the function M is a periodic piecewise-constant function, the stability properties of this model have been studied in [9] for the linear unforced case ($F = 0$ and $w = 0$) obtaining stability diagrams in the relevant physical parameters. We are also interested in the nonlinear and forced case. The key idea of this paper is that by means a suitable change of variables, the equation under study can be transformed in a Newtonian periodic equation of the form $x'' = f(t, x)$. This is the aim of Section 2. With this approach, Section 3 recovers and generalizes the result obtained in [9]. On the other hand, regarding the nonlinear oscillator (Section 4), we can achieve a rich theory for the dynamics of newtonian equations developed in recent years. Specially interesting are the results revealing the type of dynamics arising near of an equilibrium or a periodic response of the system by means the KAM theory: existence of sub-harmonic with periods tending to infinity, conditionally periodic motions, Smale's horseshoes, and chaotic movements. This behavior appears generically in absence of resonance and when one introduces a nonlinear external force in the above oscillator. Regarding the stability theory there are interesting results which are non local in the sense of non proximity to integrable situations. As a particular case result, we will see that the stability diagrams made in [9] remain almost unalterable when the nonlinear force in (1) is of Duffing type (a case yet studied numerically in [8]), in others words we shall show that the linear stability implies the stability in the nonlinear sense.

2 The reduction to a newtonian equation.

From now on, we assume that our model (1) exhibits a periodic dependence on time with minimal period $T > 0$. Moreover, $M(t)$ is assumed to be positive an piecewise continuous. Under such conditions, the equation (1) can be rewritten as a newtonian equation through a nonlinear scale of time.

Let us take a primitive function $\tau = \tau(t)$ of $1/M(t)$ in the interval $[0, T]$. More precisely the change is given by

$$\tau = \tau(t) = \int_0^t 1/M(s)ds \tag{2}$$

As $d\tau/dt$ is always positive then this transformation is a continuous biyection having an inverse $t = t(\tau)$ with the same property. So

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt}$$

and

$$\frac{dM}{dt} = \frac{dM}{d\tau}\frac{d\tau}{dt}.$$

Introducing these relations in (1) we arrive to the equation

$$\frac{d^2y}{d\tau^2} + a(\tau)y = -w^*(\tau)\frac{dM}{d\tau} + f(\tau, y) \tag{3}$$

where

$$a(\tau) = kM(t(\tau)), \quad w^*(\tau) = w(t(\tau)), \quad f(\tau, y) = M(t(\tau)))F(t(\tau), y) \tag{4}$$

are periodic functions with the new period

$$T^* = \tau(T) = \int_0^T 1/M(s)ds. \tag{5}$$

Clearly, the equations (1) and (3) are equivalent and the dynamical properties of the solutions remain unchanged.

On the other hand, we notice that if the function M is only piece-wise continuous (as in [9]), the term $\frac{dM}{d\tau}$ in (3) produces at least locally a Dirac delta concentrated at the discontinuity points, so the equation should be considered in the sense of distributions (see [29] for the terminology and a general exposition). To avoid such a formal procedure we will translate the reference system in such a way that (3) becomes a standard differential equation.

Let $H(\tau)$, $H(\tau + T^*) = H(\tau)$ be a second primitive of $-w^*(\tau)\frac{dM}{d\tau}$, that is,

$$H''(\tau) = -w^*\frac{dM}{d\tau},$$

in the sense of distributions. Of course, H is a continuous function. Then, the T^*-periodic change of variables

$$y = x + H(\tau) \tag{6}$$

transforms the equation (3) into the equation

$$x'' + a(\tau)x = -a(\tau)H(\tau) + f(\tau, x + H(\tau)). \tag{7}$$

Note that when $w = 0$ the equation (7) coincides with (3) so this change is only necessary when the mean velocity is not null.

In conclusion, the qualitative properties of the solutions of the original equation (1) are equivalent those of equation (7). For convenience, we will return to the more comfortable notation of t for the independent variable and hence in the following sections we will study the equation

$$x'' + a(t)x = -a(t)H(t) + f(t, x + H(t)). \tag{8}$$

3 The linear oscillator.

In this section we consider the case $F = 0$. By the construction done in Section 2, the dynamics is equivalent to the complete Hill's equation

$$x'' + a(t)x = -a(t)H(t) \tag{9}$$

and the standard theory for periodic linear equations applies (see for instance the monograph [16]). We will resume the most relevant facts in two separate subsections.

3.1 Free oscillations.

If $w = 0$, then $H \equiv 0$ and (9) is the classical Hill's equation

$$x'' + a(t)x = 0, \tag{10}$$

with $a(t+T^*) = a(t)$ for all t. The stability analysis of this equation is a classical topic [6, 16]. To explain it perhaps will result more convenient represent the Hill's equation like a periodic system

$$x' = y, \qquad y' = -a(t)y \tag{11}$$

or equivalently

$$X' = A(t)X, \;\; A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}$$

with $X = (x, y)^T$. Let $\Phi(t)$ be the fundamental matrix of the above system such that $\Phi(0) = I$, where I denotes the identity matrix. The matrix $\Phi(T^*)$ is called the *monodromy matrix* of the equation. It is a well known fact that the stability properties of the Hill's equation depends on the relative position in the complex plane of the eigenvalues of the monodromy matrix, also called the *characteristic multipliers.* This is resumed by the discriminant function $\Delta = tr\Phi(T^*)$ in the following way: the equation is stable (all solutions are bounded) if $|\Delta| < 2$, unstable (all non-trivial solutions are unbounded) if $|\Delta| > 2$.

For the cases $\Delta = \pm 2$ the multipliers are repeated $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$ and the equation will be stable only if the monodromy matrix is diagonal. In such cases all solutions are periodic with the same period T^* ($\lambda_{1,2} = 1$) or $2T^*$ ($\lambda_{1,2} = -1$).

For a general Hill's equation, there are many classical criteria for stability, most of them are collected in the classical book of Cesari [6]. A more recent L^p-norm stability criterion is proved in [11].

In the case of a piecewise constant coefficient, the equation can be explicitly integrated. As a particular example, let us analyze the case studied in [9]. In the related literature it is customary to separate the mass $M(t) = M_0 - m(t) > 0$ in a time invariant part M_0 and a time-varying part $m(t)$. The equation studied in [9] is

$$((M_0 - m(t))x')' + kx = 0 \tag{12}$$

where

$$m(t) = \begin{cases} m & \text{if } 0 < t < T_0 \\ 0 & \text{if } T_0 < t < T. \end{cases} \tag{13}$$

Therefore, $m(t)$ is a piecewise constant function and $m > 0$ is interpreted as the mass which is added at time T_0.

The change given by (2) can be given explicitly as

$$\tau = \tau(t) = \begin{cases} t/(M_0 - m) & \text{if } 0 < t \leq T_0 \\ (t - T_0)/M_0 + T_0/(M_0 - m) & \text{if } T_0 < t < T. \end{cases} \tag{14}$$

With this change the equation (12) becomes

$$x''(\tau) + k(M_0 - m(t(\tau)))x = 0. \tag{15}$$

Let us rewrite this equation as

$$x''(\tau) + kM_0(1 - h(\tau))x = 0 \tag{16}$$

where

$$h(\tau) = \begin{cases} \epsilon & \text{if } 0 < \tau < \tau_0 \\ 0 & \text{if } \tau_0 < \tau < T^*, \end{cases} \tag{17}$$

where $\epsilon = m/M_0$, $T^* = (T - T_0)/M_0 + T_0/(M_0 - m)$ is the new period and $\tau_0 = \tau(T_0) = T_0/(M_0 - m)$ is the new "switching time". A Hill's equation with a piecewise constant coefficient is known as Meissner's equation in the specialized literature. The monodromy matrix $\Phi(T^*)$ is obtained by direct integration as follows. The fundamental matrix associated to system $x'' + kM_0(1 - \epsilon)x = 0$ with initial condition $X(0) = I$ is

$$\Phi_1(t) = \begin{pmatrix} \cos \alpha t & \frac{1}{\alpha} \sin \alpha t \\ -\alpha \sin \alpha t & \cos \alpha t \end{pmatrix},$$

where $\alpha = \sqrt{kM_0(1 - \epsilon)}$. This is the flow of the system until τ_0. At this time, the equation changes to $x'' + kM_0 x = 0$, whose fundamental matrix with initial condition $X(\tau_0) = I$ is

$$\Phi_2(t) = \begin{pmatrix} \cos \beta(t - \tau_0) & \frac{1}{\beta} \sin \beta(t - \tau_0) \\ -\beta \sin \beta(t - \tau_0) & \cos \beta(t - \tau_0) \end{pmatrix},$$

where $\beta = \sqrt{kM_0}$. Finally, the monodromy matrix of equation (16) is just

$$\Phi(T^*) = \Phi_2(T^*)\Phi_1(\tau_0).$$

After some elemental computations we get the discriminant

$$\Delta = tr\Phi(T^*) = 2 \cos \alpha\tau_0 \cos \beta(T^* - \tau_0) - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) \sin \alpha\tau_0 \sin \beta(T^* - \tau_0).$$

The discriminant results a function in the original variables k, T, T_0, M_0, m of the form

$$\begin{aligned}\Delta(T, T_0, M_0, m) = \quad & 2\cos\sqrt{\tfrac{k}{M_0-m}}T_0\cos\sqrt{\tfrac{k}{M_0}}(T-T_0) \\ -\frac{2M_0-m}{\sqrt{M_0(M_0-m)}} \quad & \sin\sqrt{\tfrac{k}{M_0-m}}T_0\sin\sqrt{\tfrac{k}{M_0}}(T-T_0)\end{aligned} \tag{18}$$

In this context, the example analyzed in [9] is equivalent to take $k = M_0$. Then, our equation (12) is exactly equation (4) of the cited paper after dividing by M_0. The discriminant can be written as a function of T, T_0 and the relative mass $\epsilon = m/M_0$ as follows

$$\Delta(T, T_0, \epsilon) = 2\cos\frac{T_0}{\sqrt{1-\epsilon}}\cos(T-T_0) - \frac{2-\epsilon}{\sqrt{1-\epsilon}}\sin\frac{T_0}{\sqrt{1-\epsilon}}\sin(T-T_0) \tag{19}$$

This formula is in consonance with the one deduced in [9]. Accordingly, different diagrams of stability for our oscillator can be easily plotted. If we fix the parameter ϵ we call a vector (T, T_0) for which (10) is stable and $|\Delta(T, T_0)| = 2$, a coexistence vector. One can see curves (in the plane of parameters T, T_0) joining two consecutive coexistence vectors together to its interior, these ones are classically called instability pockets because in this zone the equation results unstable. The global geometry of the instability pockets has been studied for various authors [3, 4, 7] and nowadays has a renovated interest. The physical relevance of these coexistence parameters vectors relies on its experimental observability because it seems that they are robust by perturbations, in spite of its degenerate character. Starting from one of them one can move to four different zones, two of them are stable and the others are unstable. The coexistence vectors have been explicitly computed in [9]. As an example, we have drawn in Fig. 1 the stability diagram in the plane $T - T_0$ for $\epsilon = 0.75$. Many other possibilities are at hand, for instance fixing T_0 (Fig. 2) or T (Fig. 3).

3.2 Forced oscillations.

The mean velocity $w(t)$ at which the mass flow is hitting or leaving the oscillator acts as a external force. If $w \neq 0$, then the dynamics of the model is determined by the complete second order linear equation

$$x'' + a(t)x = -a(t)H(t). \tag{20}$$

By the Fredholm's alternative, if the homogeneous part has no T^*-periodic solutions, the equation (20) has a unique T^*-periodic solution, and the general solution of (20) is the sum of this particular solution with the general solution of the homogeneous equation. Therefore, if Δ is the discriminant of the Hill's equation $x'' + a(t)x = 0$, the complete equation (20) has a unique T^*-periodic solution if $|\Delta| \neq 2$, and this solution is stable if $|\Delta| < 2$ and unstable if $|\Delta| > 2$. In other words, the information of the stability diagrams drawn in the previous subsection for the unforced case is directly translated to the forced case.

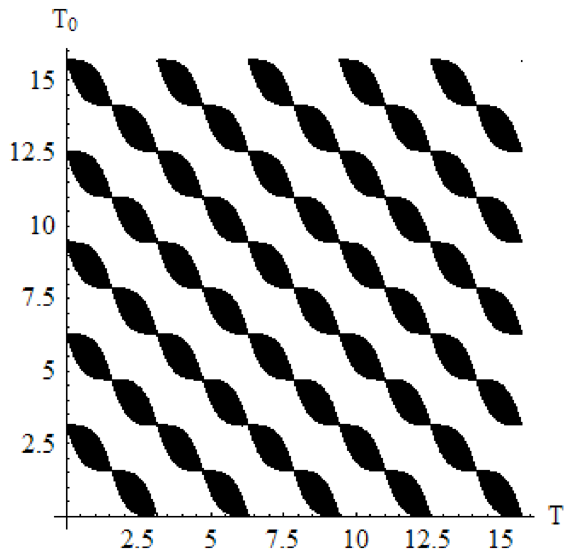

Figure 1: Stability regions for the unforced linear oscillator in the plane $T - T_0$ for $\epsilon = 0.75$. Note that only the case $T_0 < T$ is physically relevant

4 The nonlinear oscillator.

Usually a linear equation is just a rough approximation to the real model, that includes strong nonlinear effects. As a typical nonlinear model, we have selected the Duffing oscillator with a periodically time-varying mass

$$(M(t)y')' + ky(t) + \gamma y(t)^3 = f(t), \tag{21}$$

where $M(t) > 0$ and $f(t)$ are T-periodic functions. This model has been studied form the numerical point of view in [8]. Here f represents a prescribed loading, and the others terms include linear and cubic elastic restoring forces with elastic stiffness parameters $k > 0$ and $\gamma \neq 0$ respectively.

Contrary to the case of a linear equation, in a typical nonlinear equation bounded and unbounded oscillations may coexist, so the stability analysis of a given periodic solution is local. Next we shall show how to get a T-periodic solution for this system at least when f is small enough. So we will focus our attention in the following equation

$$(M(t)y')' + ky(t) + \gamma y(t)^3 = \delta p(t), \tag{22}$$

where p is a T-periodic an continuous loading fixed and δ is a small real parameter. Notice that this model is included in (1) taking $w = 0$ and $F = \delta p(t) - \gamma y(t)^3$.

The scaling $t = t(\tau)$ given by (2) produces in this case the following newtonian equation equivalent to (22)

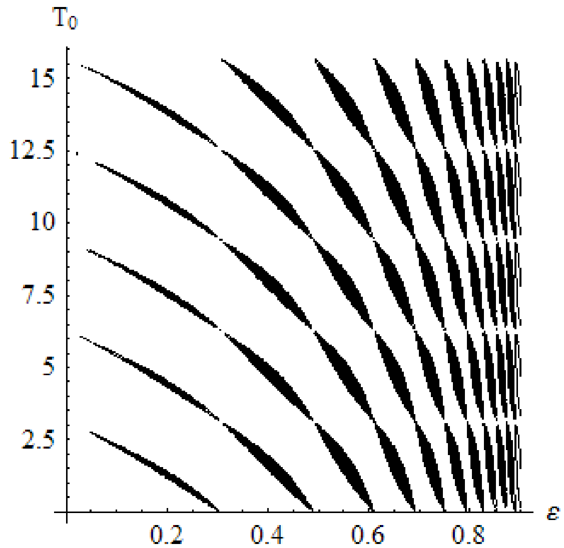

Figure 2: Stability regions for the unforced linear oscillator in the plane $\epsilon - T_0$ for $T = 5\pi$.

$$y'' + a(\tau)y + c(\tau)y^3 = \delta p^*(\tau), \tag{23}$$

where

$$a(\tau) = kM(t(\tau)), \quad c(\tau) = \gamma M(t(\tau)), \quad p^*(\tau) = M(\tau)p(t(\tau)). \tag{24}$$

Note that the equation (23) is also periodic with new period $T^* = \int_0^T 1/M(s)ds$.

Notice that when $\delta = 0$ then one has the equilibrium $y \equiv 0$ for the unforced equation

$$y'' + a(\tau)y + c(\tau)y^3 = 0. \tag{25}$$

The linearized equation around of $y \equiv 0$ is the Hill's equation

$$x'' + a(\tau)x = 0. \tag{26}$$

This equation (see for instance [16]) has characteristic multipliers $\lambda_1, \ \lambda_2 \in \mathbb{C}$ satisfying

$$\lambda_1 \lambda_2 = 1.$$

In a classical terminology, it is said that (26) (or $x = 0$) is elliptic if $\lambda_2 = \overline{\lambda_1} \in \mathbb{C} \setminus \{\pm 1\}$, parabolic if $\lambda_{1,2} = \pm 1$ and hyperbolic if $\lambda_{1,2}\mathbb{R} \setminus \{\pm 1\}$ respectively. Given $n \in \mathbb{N}$ we say that (26) is *n-resonant* if it is elliptic and the Floquet multipliers satisfy $\lambda_i^n = 1$. We say that (26) is *strongly resonant* if it is n-resonant for $n = 3$ or 4.

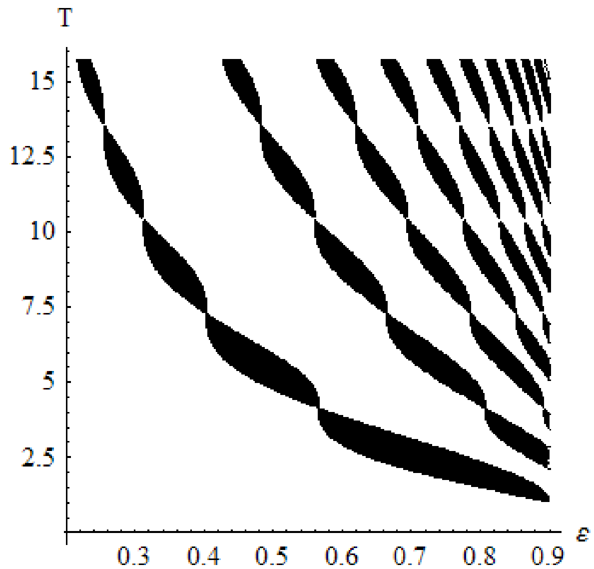

Figure 3: Stability regions for the unforced linear oscillator in the plane $\epsilon - T$ for $T_0 = 1$.

Next we shall introduce a basic approach to continue the equilibrium $y \equiv 0$ in a small periodic solution when δ is small. This approach goes back to H. Poincaré [26].

4.1 Existence of elliptic periodic solutions

Let $\varphi = \varphi(\tau; y_0, v_0, \delta)$ be the unique solution of (23) such that

$$\varphi(0) = y_0, \quad \varphi'(0) = v_0,$$

for any initial conditions $(y_0, v_0) \in \mathbb{R}^2$. It is a well known fact that $\varphi(\tau; y_0, v_0, \delta)$ is T^*-periodic if and only if it verifies

$$\begin{aligned} \varphi(T^*; y_0, v_0, \delta) - y_0 &= 0, \\ \varphi'(T^*; y_0, v_0, \delta) - v_0 &= 0. \end{aligned} \tag{27}$$

In the next result, we will see how the solutions (y_0, v_0, δ) of system (27) yield us to the initial conditions of the periodic solution when δ is small enough. The proof is inspired by chapter IV of [14] and [13].

Proposition 1 *Assume that (26) is elliptic. Then, there exist smooth functions $x = x(\delta), y = y(\delta)$, $\delta \in (-\epsilon_0, \epsilon_0)$ for certain $\epsilon_0 > 0$, such that the solution of (23) $\varphi_\delta(\tau) = \varphi(\tau; x(\delta), v(\delta))$ is T^*-periodic. In consequence the equation (22) has at least a T-periodic solution $y_\delta(t)$. Moreover, if (26) is not strongly resonant then φ_δ do.*

Proof. The jacobian matrix of the system (27) respect to (y_0, v_0) at $(0,0,0)$ is of the form $M - I$ where I is the identity matrix and M is a *monodromy matrix* for the linearized equation (26)(see section 3.1). Like 1 is not a characteristic multiplier of M then $M - I$ is invertible. Thus we can apply the implicit function theorem near to $(0,0,0)$ for reaching the conclusion.

The variational equation for φ_δ is

$$y'' + (a(\tau) + 3c(\tau)\varphi_\delta^2(\tau))y = 0.$$

Then φ_δ will be elliptic and not strongly resonant when the above equation will do. The Floquet multipliers of the above equation $\lambda_i = \lambda_i(\delta)$, $i = 1, 2$, are continuous function of δ because the continuity of the equation respect to the parameter δ. As $\lambda_1(0)$, $\lambda_2(0)$ coincide with the Floquet multipliers of (26) ($\varphi_0 \equiv 0$), a standard continuity argument finishes the proof. $\square$

4.2 Twist coefficient and stability

To study the stability properties of the solution y_δ obtained in the above proposition, the classical approach of the first Lyapunov method (linearization) is not enough because the stability in a conservative system depends strongly on nonlinear terms. Also, generally it is difficult to identify a Lyapunov's function, because the dynamics near to a stable solution in a hamiltonian system is too complicated.

An alternative approach is provided in the 60's by the so called KAM theory and the Twist Theorem ([27], [17], [1]). From this point of view, the nonlinear terms of the Taylor's expansion around a given periodic solution are taken into account to decide the kind of dynamic rising around such solution. The basic idea consists in to express the system in suitable geometrical coordinates as a perturbation of a canonical system which is integrable and therefore possesses invariant tori near to the periodic solution. These invariant tori are persistent under perturbations and produce jails or barriers for the flux trapping the orbits inside.

More recently these ideas have taken a renewal interest from Ortega's works ([19, 20, 18, 21, 11, 31, 12]) and provide us some stability criteria based on the third approximation [23, 24, 25]

$$y'' + a(\tau)y + b(\tau)y^2 + c(\tau)y^3 = 0. \tag{28}$$

When $b(t) \equiv 0$, we have the following remarkable result ([23]).

Theorem 1 *Let $a(\tau)$ and $c(\tau)$ periodic and real piece-wise continuous function. We assume that $c(\tau)$ doesn't changes sign and is not identically null. Assume that the Hill's equation*

$$y'' + a(\tau)y = 0 \tag{29}$$

is stable (for instance elliptic). Then the equilibrium $y \equiv 0$ is stable in the sense of Lyapunov for the nonlinear equation (28)*. In particular if (29) is elliptic and not strongly resonant then the first twist coefficient $\beta \neq 0$ (see definitions bellow).*

The Poincaré mapping $\mathcal{P}_\delta$ associated to (23) is defined near the origin by

$$\mathcal{P}_\delta(x,y) = \varphi(T^*; x, y, \delta), \tag{30}$$

where $\varphi(\tau; x, y, \delta),\ \delta \in]-\epsilon_0, \epsilon_0[$ is the unique solution of (23) with initial data $\varphi(0) = x,\ \varphi'(0) = y$. Note that $\mathcal{P}_0(0,0) = (0,0)$ because $y \equiv 0$ is the equilibrium of the unforced equation (25). More generally,

$$\mathcal{P}_\delta(x(\delta), y(\delta)) = (x(\delta), y(\delta)),$$

where $x(\delta)$ and $y(\delta)$ are the initial conditions of φ_δ, the unique periodic solution given by the proposition 1. Therefore the stability of φ_δ is equivalent to the stability of $(x(\delta), y(\delta))$ as fixed point of $\mathcal{P}_\delta$.

Other elementary property of the Poincaré map states that $\mathcal{P}'_\delta(x(\delta), y(\delta))$ is a monodromy matrix for the variational equation along φ_δ

$$y'' + (a(\tau) + 3c(\tau)\varphi_\delta^2(\tau))y = 0. \tag{31}$$

Then its Floquet multipliers $\lambda_{1,2}(\delta)$ are continuous function of δ. From Proposition 1 if (26) is elliptic and not strongly resonant then φ_δ do. So, the Birkhoff's normal form Theorem provides a canonical change of variables $z = \Phi_\delta(\xi),\ z = (x,y)$, such that $\mathcal{P}_\delta$ adopts in the new coordinates the following form

$$\mathcal{P}_\delta^*(\xi) = (\Phi_\delta^{-1} \circ \mathcal{P}_\delta \circ \Phi_\delta)(\xi) = R[\theta(\delta) + \beta(\delta)|\xi|^2](\xi) + O_4, \tag{32}$$

where $R[\omega]$ denotes the rigid rotation of angle ω, $\lambda_{1,2} = e^{\pm i\theta(\delta)}$ ($\theta(\delta)$ is a continous selection of the argument), and O_4 indicates a term that is $O(|\xi|^4)$ when $\xi \to 0$. The coefficient $\beta(\delta)$ is called the *(first) twist coefficient* and plays a central role in the stability theory. A remarkable property of the twist coefficient is its invariance under symplectic changes of variables ([25, 15]).

From the Twist Theorem it follows that if $\beta(\delta) \neq 0$ then the fixed point $(x(\delta), y(\delta))$ is stable (see Chapter 3 of [27]). On the other hand, the changes of variables Φ_δ can be selected continuous in δ. Like $\mathcal{P}_\delta$ depends continuously of δ too, it can be proved that $\beta(\delta)$ *is a continuous function provided that (31) is not strongly resonant* (see [28] and [15] for details).

At this point, we are ready to prove the main result of this section.

Theorem 2 *Assume that*

$$(M(t)y')' + ky(t) = 0,$$

is elliptic and not strongly resonant (i.e. (26) is elliptic and not strongly resonant). Then the equation (22) has at least a T-periodic solution $y_\delta(t)$ for small δ, which is stable in the nonlinear sense.

Proof. From Proposition 1 there is at least a T-periodic solution y_δ for small δ, i.e., a continuation of the equilibrium $y \equiv 0$. From the discussion above it is sufficient to prove the stability for the corresponding elliptic and not strongly resonant periodic solution φ_δ. In consequence the twist coefficient $\beta(\delta)$ is well defined for small δ. Note that all the hypotheses of Theorem 1 hold for the unforced equation (25), so one gets that $\beta(0) \neq 0$. The conclusion is reached by the continuity of $\beta(\delta)$. □

We come back to the two-valued variable mass $M(t) = M_0 - m(t)$ where $m(t)$ is defined by (13). Specifically, we are interested in the model studied in [9], equivalent to take $k = M_0$ in Section 3.1 . In such a case the linear stability is analyzed by mean of discriminant function $\Delta(T, T_0, \epsilon)$ given by (19). Remember that ϵ is the (adding) mass relative parameter and T_0 is the switching time. We have the following result.

Corollary 1 *Assume that the discriminant function defined by (19) satisfies*

$$0 < |\Delta(T, T_0, \epsilon)| < 2, \quad \Delta \neq -1.$$

Then for δ small enough, the nonlinear equation (22) *with $M(t) = M_0 - m(t)$ defined by (13), has a T-periodic solution $y_\delta(t)$ which is stable in the nonlinear sense.*

Note that $\Delta = 0$ (resp. $\Delta = -1$) corresponds to the fourth (resp. third) order resonance. By excluding these values, the linear part is elliptic and not strongly resonant and the previous theorem applies directly. In the stability diagrams presented in Section 3, $\Delta = 0$ (resp. $\Delta = -1$) is a curve inside each stability region.

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