All periodic minimizers are unstable

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Abstract. We consider the periodic action functional associated to some lagrangian verifying Legendre convexity condition and show that all minimizers are unstable.

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1. Introduction

Consider the action functional

$$\mathcal{A} : \mathcal{C}^1(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}\,, \qquad \mathcal{A}[x] := \int_0^1 L(t, x(t), x'(t))dt\,, \tag{1}$$

where the lagrangian $L : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$, $L = L(t, x, p)$, has class $\mathcal{C}^{0,2}$ and verifies the classical Legendre condition

$$L_{pp} > 0 \;\text{ on }\; (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2\,. \tag{2}$$

The critical points of $\mathcal{A}$ are well-known to coincide with the $1-$periodic solutions of the associated Euler-Lagrange equation

$$\frac{d}{dt} L_p(t, x(t), x'(t)) = L_x(t, x(t), x'(t))\,. \tag{3}$$

A particular role is played by local periodic minimizers, which we shall simply call *periodic minimizers* in what follows. Precisely, with this expression we refer to those critical points $x_* \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$ which minimize the action functional $\mathcal{A}$ on some small neighborhood $\mathcal{N}_r$, defined by

$$\mathcal{N}_r = \Big\{ x \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z}) : |x(t) - x_*(t)| < r,\; |x'(t) - x_*'(t)| < r \text{ for any } t \in \mathbb{R}/\mathbb{Z} \Big\}\,. \tag{4}$$

In [1], Carathéodory showed that, in case the periodic minimizer x_* is *non-degenerate*, meaning that the quadratic form $\mathcal{A}''[x_*]$ is positive definite, then, x_* is hyperbolic, and, in particular, (by Lyapunov First Method), unstable. It opened

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the question on the necessity of the nondegeneracy condition. Because it turns out that when the local minimizer x_* is degenerate -i.e., the quadratic form $\mathcal{A}''[x_*]$ is only semidefinite-, then it is parabolic, and Lyapunov First Method does not apply. However, Dancer and Ortega [2] showed that any stable isolated periodic solution of (3) has fixed point index 1 and cannot be a minimizer, thus extending Carathéodory's Theorem on the instability of the nondegenerate case to the (possibly degenerate) isolated case. More elementary proofs appeared subsequently [5, 6].

In a different approach to the problem, Ortega [4] showed that fixed points of area-preserving, analytic mappings on the plane other than the identity, are either unstable or isolated. It means that, if the lagrangian L is *analytic* on the state variables x, p, then *all* local minimizers are unstable. The goal of this paper is to show a similar result without the analytic assumption:

Theorem 1.1. *Let the lagrangian* $L : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$ *have class* $\mathcal{C}^{0,2}$ *and verify (2). Then, every (local) periodic minimizer is unstable as a solution of the Euler-Lagrange equation (3).*

A remark on the meaning of the word *stability.* Indeed, for Hamilitonian systems -or area-preserving maps-, past and future Lyapunov stability are equivalent concepts, and equivalent to the existence of a basis of invariant neighborhoods of the periodic solution or fixed point under consideration. The resulting 'perpetual' notion, and its logical negation, are respectively referred to as *stability* and *instability* in this work.

A few words also to comment the $\mathcal{C}^{0,2}$ regularity and the Legendre condition (2) which we assume on the lagrangian L. They guarantee, (see Lemma 2.1 of [6]), the existence and uniqueness of solution for initial value problems associated to (3), giving sense to the problem of stability. Observe, for instance, that the Newtonian lagrangian $L(t, x, p) = p^2 - V(t, x)$ verifies them both provided only that the force $f(t, x) = -V_x(t, x)$ has class $\mathcal{C}^{0,1}$ on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$.

Before finishing this Introduction, I want to express my gratitude to R. Ortega, who first introduced me to the problem of the instability of minimizers, and did not lose his patience after listening to several mistaken proofs which came along before this one. My thanks also to R. Ortega and A. Chenciner for suggesting to me the study of the Aubry-Mather theory.

2. Some aspects of the Aubry-Mather theory

Theorem 1.1 applies to general $\mathcal{C}^{0,2}$ lagrangians $L : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$ verifying the Legendre condition (2), and to local minimizers x_*. However, our proof will be constructed under some further global assumptions which we pass to describe:

(i) L is 1-periodic, not only on time, but also on the variable x. With symbols,

$$L(t, x + 1, p) = L(t, x, p) , \qquad (t, x, p) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 .$$

(ii) L_x is bounded on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$.

(iii) There are constants $0 < m < M$ such that $m \leq L_{pp} \leq M$.
(iv) x_* is not just a local, but a *global* minimizer of the action functional $\mathcal{A}$ under periodic boundary conditions.

This, in appearance, more restrictive framework, will not mean a loss of generality, since

Lemma 2.1. *Let L and x_* be under the framework of Theorem* 1.1*. Then, there is a second lagrangian $\tilde{L} : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$, which coincides with L outside some small neighborhood of x_*, and verifies assumptions (i,ii,iii,iv).*

To maintain the tempo of our exposition, we postpone the proof of Lemma 2.1 to Section 4. Meanwhile, *throughout Sections* 2 *and* 3*, we shall always assume that $L : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^{0,2}$ lagrangian verifying (i,ii,iii,iv)* above.

Observe that these assumptions are *not enough* to make our lagrangian L fit under the framework of the Aubry-Mather theory as described in Chapter 2 of [3], mainly because we do not assume L to be a $\mathcal{C}^2$ function with respect to time. Nevertheless, there are still many well-known notions and results from this theory which can be translated, with no or little variation, into our framework. Some of these will play an important role in our proof, and we summarize them in the remaining of this Section.

And our starting point will be the existence of minima for the action functional under given Dirichlet boundary conditions. Indeed, assumptions *(i)* and *(iii)*, together with a simple integration argument, imply the existence of some constant $K > 0$ such that

$$-K + mp^2/4 \leq L(t, x, p) \leq K + Mp^2 \,, \qquad (t, x, p) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \,. \tag{5}$$

In particular, for any real numbers a, b, u, v with $a < b$, the action functional

$$x \mapsto \int_a^b L(t, x(t), x'(t))dt \,,$$

is well defined, and coercive on the affine Sobolev space

$$H^1_{u,v}(a, b) := \Big\{ x \in H^1(a, b) : x(a) = u, x(b) = v \Big\} \,.$$

Well-known arguments, based on the boundedness of L_x (assumption *(ii)*) and the Legendre condition (2), show that this action functional is also weak lower-semicontinuous, so that it attains its global minimum on $H^1_{u,v}(a, b)$ (see Theorem 2.2.1 of [3]). It is usual to denote by $\mathcal{M}[a, b]$ to the set of those $H^1(a, b)$ functions x which minimize the action functional on $H^1_{x(a),x(b)}(a, b)$; they are said to be *minimal on* $[a, b]$, and turn out to be $\mathcal{C}^1$ *extremals*, i.e., solutions of the Euler-Lagrange equation (3). We summarize the discussions above in following result:

Lemma 2.2. *$\mathcal{M}[a, b] \neq \emptyset$. Moreover, for any $u, v \in \mathbb{R}$ there exists some $x \in \mathcal{M}[a, b]$ with $x(a) = u$ and $x(b) = v$.*

Observe that, in general, this minimal x may not be unique, i.e., two elements $x, y \in \mathcal{M}[a,b]$ may cross at both endpoints a, b of the interval. This is, however, the only possibility for two crossing points between minimal functions:

Lemma 2.3. *Let $x, y \in \mathcal{M}[a,b]$ with $x \not\equiv y$, and $t_1 < t_2 \in [a,b]$ be given. If $x(t_i) = y(t_i)$ for $i = 1, 2$, then $t_1 = a$ and $t_2 = b$.*

The proof of this result follows closely that of Theorem 1.3.4 of [3], so that we shall not repeat it in this paper. We turn now our attention to the matter of compactness for sets of minimals. For reasons which will be clear in Section 3, it will be convenient to consider sequences of minimals which are not necessarily defined on the same interval. Thus, let the sequences of real numbers $a_n \to a$ and $b_n \to b$ with $a < b$ be given. Assume that, for each $n \in \mathbb{N}$, we are given some continuous function $x_n : [a_n, b_n] \to \mathbb{R}$. The sequence $\{x_n\}_n$ is said to be *uniformly bounded* if there exists some constant $K > 0$ such that $|x_n(t)| \leq K$ on $[a_n, b_n]$ for each $n \in \mathbb{N}$. It is called *equicontinuous* if for each $\epsilon > 0$ there exists some $\delta > 0$ such that $|x_n(t) - x_n(s)| \leq \epsilon$ for any $t, s \in [a_n, b_n]$ and any $n \in \mathbb{N}$.

Lemma 2.4. *Assume that $x_n \in \mathcal{M}[a_n, b_n]$ for any $n \in \mathbb{N}$ and that the sequence $\{x_n\}$ is uniformly bounded. Then, $\{x'_n\}_n$ is uniformly bounded and equicontinuous.*

This result is related to Lemma 2.3.4 of [3]; however, it does not exactly follow from it, mainly because of the lower regularity of our lagrangians with respect to time, and we give an alternative proof in Section 4. When combined with Ascoli-Arzela Lemma, it yields the following consequence:

Corollary 2.5. *Under the assumptions of Lemma 2.4, there exists some partial subsequence $\{x_{n_k}\}_k$ which converges to some element $x \in \mathcal{M}[a,b]$ in the following sense: for any converging sequence $t_k \to t$ with $t_k \in [a_{n_k}, b_{n_k}]$, one has that*

$$\lim_{k\to\infty} x_{n_k}(t_k) = x(t)\,, \qquad \lim_{k\to\infty} x'_{n_k}(t_k) = x'(t)\,.$$

This concept of convergence might sound strange at first glance, but in case $[a_n, b_n] \equiv [a,b]$ is a constant interval, it becomes the classical $\mathcal{C}^1[a,b]$ convergence. Observe that, under the framework of Corollary 2.5,

$$\lim_{k\to\infty} \int_{a_{n_k}}^{b_{n_k}} L(t, x_{n_k}(t), x'_{n_k}(t))dt = \int_a^b L(t, x(t), x'(t))dt\,. \tag{6}$$

The notion of minimality may be easily extended to $\mathcal{C}^1$ functions on non-compact intervals $I \subset \mathbb{R}$. Namely, the extremal x is called *minimal on I* if it is defined, and minimal, on any compact subinterval $[a,b] \subset I$. In case the extremal x is minimal on the whole real line $\mathbb{R}$, then it is simply said to be a *global minimal*. For instance, *global periodic minimizers*, i.e., periodic functions $x \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$ where the action functional $\mathcal{A}$ attains its global minimum, provide a first example of global minimals (Theorem 2.3.2 of [3]); they will be henceforth called *periodic minimals*. Notice that in the usual definition of periodic minimal the period can be any positive integer. This makes no difference since it can be proved that global minimizers of any period n are indeed of period 1.

The set of all periodic minimals of period 1 will be denoted as $\mathcal{M}_1$. Lemma 2.3 implies that any two periodic minimals cannot intersect at all (see also Theorem 2.3.1 of [3]), so that $\mathcal{M}_1$ is naturally ordered. This order is *total*, i.e., given $x, y \in \mathcal{M}_1$ with $x \neq y$, we have either $x < y$ of $x > y$. This gives sense to the following definition: the periodic minimals $x \neq y$ are called *neighboring* if there are no periodic minimals between them. When this happens, it is possible to show that there are heteroclinic orbits linking x and y (Theorem 2.6.2 of [3]), and, in particular, they are both unstable. This result in fact implies the instability of all periodic minimals x_* which are isolated in $\mathcal{M}_1$, since for them it is always possible to find other periodic minimals $x_- < x_* < x_+$ such that both $\{x_-, x_*\}$ and $\{x_*, x_+\}$ are neighboring.

However, this argumentation cannot be directly translated to nonisolated minimizers x_*, because, in this case, x_* could be *not* the limit of any asymptotic extremal. To check this statement, it suffices to consider the example provided by the Galilean lagrangian $L(t, x, p) = p^2/2$; observe that each constant $x_* \equiv C$ is a periodic minimal, but has no associated asymptotic solutions.

3. Nonisolated periodic minimals and quasi-asymptotic sequences

The main result of this Section will be the following:

Proposition 3.1. *Let $x_* \in \mathcal{M}_1$ be not isolated. Then, there exists some constant $c \neq x_*(0)$, and a sequence $\{z_n : [0, n] \to \mathbb{R}\}_n$ of minimals, such that*

$$z_n(0) = x_*(0), \; z_n(n) = c \text{ for any } n, \qquad \lim_{n \to +\infty} z_n'(0) = x_*'(0).$$

Observe that the existence of such a sequence forces the instability of x_*. Thus, the combination of Proposition 3.1 with the comments at the end of Section 2 implies Theorem 1.1, at least when the lagrangian L verifies assumptions *(i,ii,iii,iv)*.

It will be convenient to prepare our proof of Proposition 3.1 by means of a lemma. Assume that $x < y$ are two (not necessarily neighboring) periodic minimals with

$$\int_0^1 L(t, x(t), x'(t))dt = 0 = \int_0^1 L(t, y(t), y'(t))dt\,.$$

Using Lemma 2.2 we choose, for each positive number $a > 0$, some function $z_a \in \mathcal{M}[0, a]$ with $z_a(0) = x(0)$ and $z_a(a) = y(a)$. Lemma 2.3 implies that z_a cannot intersect x or y on $]0, a[$, and thus, must verify

$$x(t) < z_a(t) < y(t)\,, \qquad t \in]0, a[\,. \tag{7}$$

Consider now the function $\Gamma :]0,+\infty[\to \mathbb{R}$ defined by $\Gamma(a) := \int_0^a L(t, z_a(t), z_a'(t))dt$. The minimality property of z_a implies that

$$\Gamma(a) = \min \left\{ \int_0^a L(t, z(t), z'(t))dt : z \in H^1_{x(0),y(a)}(0,a) \right\},$$

for each $a > 0$. In particular, $\Gamma(a)$ does not depend on the particular choice of the minimal z_a, even in case several possibilities coexist for a same value of a. The following lemma collects some relevant properties of the function Γ.

Lemma 3.2. *The following hold:*

1. $\Gamma :]0,+\infty[\to \mathbb{R}$ *is continuous.*
2. $\Gamma(a+1) < \Gamma(a)$ *for any* $a > 0$.
3. *There exists some constant* $C > 0$, *(depending on the periodic minimals* x, y *and the lagrangian* L, *but not on* a*), such that*

$$\Gamma(a+2) < \Gamma(a) - C, \qquad\qquad 0 < a < 1.$$

4. Γ *is bounded from below.*

Proof. 1. Let $a_n \to a > 0$ be given. For each n, (7) implies that $z_n := z_{a_n}$ lies between the periodic minimals x and y, and thus, this sequence is uniformly bounded. Corollary 2.5 then states that there exists some subsequence $\{z_{n_k}\}$ which converges, in the sense described there, to some element $z_* \in \mathcal{M}[0,a]$. Now, in view of (6),

$$\Gamma(a_{n_k}) = \int_0^{a_{n_k}} L(t, z_{n_k}(t), z_{n_k}'(t))dt \to \int_0^a L(t, z_*(t), z_*'(t))dt = \Gamma(a),$$

as $k \to +\infty$.

2. Choose some point $a > 0$. The curve $\beta : [0, a+1] \to \mathbb{R}$ defined by

$$\beta(t) := \begin{cases} z_a(t) & \text{if } 0 \le t \le a, \\ y(t) & \text{if } a \le t \le a+1, \end{cases}$$

is only piecewise $\mathcal{C}^1$, so that it is not minimal on $[0, a+1]$, and we deduce that

$$\int_0^{a+1} L(t, z_{a+1}(t), z_{a+1}'(t))dt < \int_0^{a+1} L(t, \beta(t), \beta'(t))dt = \int_0^a L(t, z_a(t), z_a'(t))dt,$$

since, by assumption, $\int_a^{a+1} L(t, y(t), y'(t))dt = \int_0^1 L(t, y(t), y'(t))dt = 0$.

3. We consider the mapping $h : [0,1] \to \mathbb{R}$ defined by

$$h(a) := \Gamma(a+1) - \Gamma(a+2).$$

It follows from *(i)* that h is continuous, while *(ii)* implies that h is positive. It implies the existence of some constant $C > 0$ such that $h(a) > C$ for any $a \in [0,1]$, and we deduce that:

$$\Gamma(a+2) = \Gamma(a+1) - h(a) < \Gamma(a+1) - C < \Gamma(a) - C, \qquad 0 < a < 1.$$

4. We define

$$K := \int_0^1 |L(t, y(t), y'(t))|\, dt + \int_0^1 |L(t, r(t), r'(t))|\, dt\,,$$

where $r(t) := (1-t)y(0)+tx(0)$ denotes the straight line joining the points $(0, y(0))$ and $(1, x(0))$. Now, given $a > 0$, choose some integer n with $n - 1 < a \leq n$, and define the piecewise $\mathcal{C}^1$ function $\beta : [0, n+1] \to \mathbb{R}$ by the rule

$$\beta(t) := \begin{cases} z_a(t) & \text{if } 0 \leq t \leq a\,, \\ y(t) & \text{if } a \leq t \leq n\,, \\ r(t-n) & \text{if } n \leq t \leq n+1\,, \end{cases}$$

Observe that $\beta(0) = x(0)$ and $\beta(n+1) = x(0) = x(n+1)$. The minimality property of x yields

$$\int_0^{n+1} L(t, \beta(t), \beta'(t))dt \geq \int_0^{n+1} L(t, x(t), x'(t))dt = (n+1) \int_0^1 L(t, x(t), x'(t))dt = 0\,,$$

implying that $\Gamma(a) = \displaystyle\int_0^a L(t, z_a(t), z_a'(t))dt \geq -K$. □

At this moment we are ready to complete the proof of Proposition 3.1:

Proof of Proposition 3.1. By assumption, x_* is not isolated in $\mathcal{M}_1$, implying the existence of some sequence $\{x_d\}_d \to x_*$ of periodic minimals. Since $\mathcal{M}_1$ is totally ordered, this sequence may be chosen ordered, and after using, if necessary, the change of variables $\tilde{x} = x_*(t)-x$, it is not restrictive to assume that $x_d > x_{d+1} > x_*$ for any $d \in \mathbb{N}$. We shall also assume that $\int_0^1 L(t, x_*(t), x_*'(t))dt = 0$, since it can be easily achieved by adding a constant to the lagrangian L. And, the sequence $\{x_d\}_d$ being made of periodic minimals, we deduce that

$$\int_0^1 L(t, x_d(t), x_d'(t))dt = \int_0^1 L(t, x_*(t), x_*'(t))dt = 0 \quad \text{for any } d \in \mathbb{N}\,.$$

Next, we define $c := x_1(0) > x_*(0)$, and we choose, for each $n \in \mathbb{N}$, some $z_n \in \mathcal{M}[0, n]$ such that $z_n(0) = x_*(0)$ and $z_n(n) = c$.

Observe that the sequence $\{z_n\}$ is related with the discussions preceding Lemma 3.2. Indeed, letting $x := x_*$ and $y := x_1$ we see that, by (7),

$$x_* \leq z_n \leq x_1\,, \qquad n \in \mathbb{N}\,, \tag{8}$$

and, in particular, $x_*'(0) \leq z_n'(0)$ (actually, the inequality must be strict because of the uniqueness of solutions of initial value problems). Observe that, in order to complete the proof of Proposition 3.1, we only have to show that

$$z_n'(0) \to x_*'(0) \text{ as } n \to +\infty\,.$$

Thus, we use a contradictions argument and assume that the statement above did not hold; in view of (8), it means the existence of some number $\epsilon > 0$ and some subsequence $\{z_{n_k}\}_k$ such that

$$z'_{n_k}(0) \geq x'_*(0) + \epsilon\,, \qquad\qquad k \in \mathbb{N}\,.$$

When combined the uniform continuity of $\{z'_{n_k}\}$ on the constant interval $[a_k, b_k] \equiv [0, 1]$, (ensured by Lemma 2.4), it implies that, for some $0 < \delta < 1$,

$$z'_{n_k}(t) \geq x'_*(t) + \epsilon/2\,, \qquad\qquad t \in [0, \delta]\,, \quad k \in \mathbb{N}\,.$$

Remember now that the sequence $\{x_d\}_d$ was assumed to be decreasing and uniformly converging to x_*; it means that, if $d_0 \in \mathbb{N}$ is fixed big enough, then every z_{n_k} crosses x_{d_0} at some time $a_k \in]0, \delta[\subset]0, 1[$. We use now item *3.* of Lemma 3.2, applied to $x = x_*$ and $y = x_{d_0}$, to obtain the existence of some constant $C > 0$ (not depending on k), and a sequence $\{w_k : [0, a_k + 2] \to \mathbb{R}\}_k$ of minimals, such that

$$\int_0^{a_k+2} L(t, w_k(t), w'_k(t))dt < \int_0^{a_k} L(t, z_{n_k}(t), z'_{n_k}(t))dt - C\,, \qquad k \in \mathbb{N}\,. \quad (9)$$

Consider now the sequence of piecewise $\mathcal{C}^1$ functions $\{\varphi_k : [0, n_k + 2] \to \mathbb{R}\}_k$ defined by

$$\varphi_k(t) := \begin{cases} w_k(t) & \text{if } t \in [0, a_k + 2]\,, \\ z_{n_k}(t-2) & \text{if } t \in [a_k + 2, n_k + 2]\,, \end{cases}$$

and observe that, for any $k \in \mathbb{N}$,

$$\int_0^{n_k+2} L(t, z_{n_k+2}(t), z'_{n_k+2}(t))dt \leq \int_0^{n_k+2} L(t, \varphi_k(t), \varphi'_k(t))dt <$$
$$< \int_0^{n_k} L(t, z_{n_k}(t), z'_{n_k}(t))dt - C\,,$$

the first inequality being a direct consequence of the minimality of z_{n_k+2}, while the second one follows from (9). It means that the sequence of real numbers

$$\Gamma(n) := \int_0^n L(t, z_n(t), z'_n(t))dt\,, \qquad\qquad n \in \mathbb{N}\,,$$

verifies $\Gamma(n_k + 2) \leq \Gamma(n_k) - C$ for any $k \in \mathbb{N}$. However, item *2.* of Lemma 3.2, (applied this time for $x = x_*$ and $y = x_1$), implies that $\{\Gamma(n)\}_n$ is decreasing, and, consequently, $\Gamma(n) \to -\infty$, contradicting item *4.* of the same Lemma. It completes the proof. □

4. Two miscellaneous results from Section 2

This last Section of the paper is devoted to prove Lemmas 2.4 and 2.1, which were stated without proof in Section 2. We shall start with Lemma 2.4, whose role has been important to guarantee the compactness of bounded sequences of minimals given by Corollary 2.5. Moreover, it has been used again in the proof of Proposition 3.1. Observe that, in case the lagrangian L has class $\mathcal{C}^2$ in all three variables t, x, p, then it can be seen as a consequence of Lemma 2.3.4 of [3]:

Proof of Lemma 2.4. Choose some constant $K > 0$ such that $b_n - a_n \geq 1/K$ and $|x_n| \leq K$ on $[a_n, b_n]$ for any $n \in \mathbb{N}$. We remember that the extremals x_n must solve the Euler-Lagrange equation (3), which we rewrite in Moser's integrated form (see Theorem 1.1.1 of [3]):

$$y_n(t) := L_p(t, x_n(t), x_n'(t)) = L_p(t_n, x_n(t_n), x_n'(t_n)) + \int_{t_n}^{t} L_x(s, x_n(s), x_n'(s))ds\,, \tag{10}$$

for any $t \in [a_n, b_n]$. Here, t_n might be any point of $[a_n, b_n]$, but we choose it verifying $|x_n'(t_n)| \leq 2K^2$; the existence of such a point t_n is provided by Lagrange's Mean Value Theorem and the definition of K. We denote

$$K_1 := \sup\{|L_p(t, x, p)| : t \in \mathbb{R}/\mathbb{Z},\ |x| \leq K,\ |p| \leq 2K^2\}\,, \qquad K_2 := \sup_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}^2} |L_x|\,,$$

(remember that L_x was assumed to be bounded in *(ii)*). Taking absolute values in both sides of (10), we obtain

$$|y_n(t)| \leq K_3 := K_1 + K_2/K\,, \qquad t \in [a_n, b_n] \cap [t_n - 1/K, t_n + 1/K]\,.$$

Observe now that this argument can be repeated for every $t_n \in [a_n, b_n]$ with $|x'(t_n)| \leq 2K^2$, and there is at least one possible choice on any subinterval of length $1/K$ on $[a_n, b_n]$, so that

$$|y_n(t)| \leq K_3\,, \qquad t \in [a_n, b_n]\,. \tag{11}$$

On the other hand, assumption *(iii)* implies that for any fixed t, x, the function $L_p(t, x, \cdot)$ is an homeomorphism from the real line to itself. We denote by $S(t, x, \cdot)$ to its inverse, i.e.

$$L_p(t, x, S(t, x, y)) = y\,,\ \ S(t, x, L_p(t, x, p)) = p\,, \qquad t, x \in \mathbb{R}/\mathbb{Z},\ \ y, p \in \mathbb{R}\,.$$

In this way, S turns out to be a continuous mapping on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$, and the definition of y_n in (10) may be rewritten as

$$x_n'(t) = S(t, x_n(t), y_n(t))\,, \qquad t \in [a_n, b_n]\,, \tag{12}$$

an equality which, when combined with (11) yields

$$|x_n'(t)| \leq K' := \max_D |S|\,, \qquad t \in [a_n, b_n]\,,\ n \in \mathbb{N}\,,$$

for $D := (\mathbb{R}/\mathbb{Z}) \times [-K, K] \times [-K_3, K_3]$. This proves the the uniform boundedness of the sequence $\{x_n'\}$.

So far, the continuity of S was used only to say that it is bounded on the compact set D. We choose now some $\epsilon > 0$ and use the uniform continuity of S on D to find some $\delta_1 > 0$ such that for any $(t_1, x_1, y_1), (t_2, x_2, y_2) \in D$ with $|t_2 - t_1| < \delta_1$, $|x_2 - x_1| < \delta_1$, and $|y_2 - y_1| \leq \delta_1$, one has $|S(t_2, x_2, y_2) - S(t_1, x_1, y_1)| \leq \epsilon$.

We define $\delta := \min\{\delta_1, \delta_1/K', \delta_1/K_2\}$. Thus, if $t_1, t_2 \in [a_n, b_n]$ for some $n \in \mathbb{N}$ and verify $|t_2 - t_1| \leq \delta$, one checks that $|t_2 - t_1| < \delta_1$, $|x_n(t_2) - x_n(t_1)| < \delta_1$, and (by (10)), $|y_n(t_2) - y_n(t_1)| \leq \delta_1$. When combined with (12), this implies that $|x_n'(t_2) - x_n'(t_1)| < \epsilon$, showing the equicontinuity of $\{x_n'\}$. □

To end the paper, we undertake the proof of Lemma 2.1 on the possibility of transforming the (local) periodic minimizer x_* into a *global* minimizer by modifying the lagrangian L outside some neighborhood of x_*. Observe firstly that the problem is not completely trivial, since x_* is assumed to locally minimize the action functional $\mathcal{A}$, while what we are allowed to modify is the lagrangian L. And before going into the details, we observe that it is not restrictive to assume that $x_* \equiv 0$; otherwise, it would suffice to replace L by the translated lagrangian $(t,x,p) \mapsto L(t, x + x_*(t), p + x_*'(t))$. Also, there is no loss of generality in assuming that $\int_0^1 L(t,0,0)dt = 0$; this argument was already carried out at the beginning of the proof of Proposition 3.1. The extremal $x_* \equiv 0$ being a local minimizer for the periodic action functional $\mathcal{A}$, there must exist some number $r > 0$ such that $\mathcal{A}[x] \geq 0$ for any $x \in \mathcal{N}_r$ (defined as in (4)). For simplicity, we shall assume that $r = 1$, i.e.,

$$\int_0^1 L(t, x(t), x'(t))dt \geq 0 \text{ for any } x \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z}) \text{ with } |x(t)|, |x'(t)| < 1 \text{ on } \mathbb{R}/\mathbb{Z}\,. \tag{13}$$

Choose now $\mathcal{C}^2$ functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ verifying, respectively,

$$\varphi(s) = 1 \text{ if } |s| \leq 1/2\,, \qquad \varphi(s) = 0 \text{ if } |s| \geq 1\,, \qquad 0 \leq \varphi \leq 1\,,$$

and

$$\psi(p) = 0 \text{ if } |p| \leq 1/2\,, \qquad \psi(p) = p^2 \text{ if } |p| \geq 1\,, \qquad \psi'' \geq 0\,,$$

and define, for each $n \in \mathbb{N}$, the lagrangian $L_n : (\mathbb{R}/\mathbb{Z}) \times [-1/2, 1/2] \times \mathbb{R} \to \mathbb{R}$ as follows:

$$L_n(t,x,p) := \varphi(3x)\varphi(3p)L(t,x,p) + (1 - \varphi(3nx))\varphi(3p)(n + p^2) + \psi(3np),\ |x| \leq 1/2\,. \tag{14}$$

Observe that, if $1/3 \leq |x| \leq 1/2$, then $L_n(t,x,p) = \varphi(3p)(n + p^2) + \psi(3np)$ does not depend on x. This allows us to extend L_n to the whole space $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2$ by periodicity, to get a $\mathcal{C}^{0,2}$ lagrangian $L_n : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$ verifying (14) and assumption *(i)* of Lemma 2.1. We want to show that, if n is big enough, then $\tilde{L} := L_n$ actually verifies all conditions of the just mentioned Lemma 2.1. With this purpose, it will be convenient to summarize first some properties of these lagrangians:

Lemma 4.1. *The following hold:*

(a) $L_n(t,x,p) = L(t,x,p)$ *if* $|x|, |p| \leq \dfrac{1}{6n}$.

(b) $L_n(t,x,p) \geq L(t,x,p)$ *if* $|x|, |p| \leq \dfrac{1}{6}$.

(c) $L_n(t,x,p) = 9n^2p^2$ *if* $|p| \geq \dfrac{1}{3}$.

(d) *There exists some constant $K > 0$ (not depending on t, x, p or n), such that*

$$L_n(t,x,p) \geq 9n^2p^2 - K\,, \qquad (t,x,p) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2\,.$$

(e) *There exists some constant $K > 0$ (not depending on t, x, p or n), such that*

$$L_n(t,x,p) \geq n - K\,, \qquad \frac{1}{3n} \leq |x| \leq \frac{1}{2}, \ |p| \leq \frac{1}{6}\,.$$

Proof. Statements *(a),(b),(c)* follow directly from the definitions. Also *(e)* is immediate; it suffices to take $K := 1 + \max_{(\mathbb{R}/\mathbb{Z})\times[-1/3,1/3]^2} |L|$. The same value of K is suitable for *(d)*, since $\psi(p) \geq p^2 - 1$ for any $p \in \mathbb{R}$; this inequality can be easily deduced from the set of properties under which ψ was chosen. □

There is a last property of the lagrangians L_n which deserves some attention. It states that all of them verify the Legendre convexity condition (2) when n is big:

Lemma 4.2. $(L_n)_{pp} > 0$ *if n is big enough.*

Proof. We distinguish several cases depending on the value of p. For instance, if $|p| \leq 1/6$, the convexity of ψ means that

$$(L_n)_{pp}(t,x,p) \geq \varphi(3x)L_{pp}(t,x,p) + 2(1 - \varphi(3nx)) > 0\,,$$

for any $n \in \mathbb{N}$. On the other hand, if $|p| \geq 1/3$, item *(c)* of Lemma 4.1 implies $(L_n)_{pp}(t,x,p) = 18n^2 > 0$. Thus, it only remains to check what happens for $1/6 < |p| < 1/3$, and direct derivation on (14) leads us to the inequality

$$(L_n)_{pp}(t,x,p) \geq 18n^2 - 9\Big(\max_{\mathbb{R}} |\varphi''|\Big)n - K'\,, \qquad \frac{1}{6} \leq |p| \leq \frac{1}{3}\,, \ n \geq 2\,.$$

where $K' > 0$ is some constant (not depending on t, x, p or n). It implies that, for big values of n, $(L_n)_{pp}(t,x,p) > 0$ on the double stripe $1/6 \leq |p| \leq 1/3$, and concludes the proof. □

Proof of Lemma 2.1. Observe, to start, that all lagrangians L_n verify assumptions *(i)* (by definition), and *(ii)* (as a consequence of *(i)* and part *(c)* of Lemma 4.1). On the other hand, *(iii)* follows, for big values of n, from the combination of Lemma 4.2, assumption *(i)*, and part *(c)* of Lemma 4.1. Moreover, item *(a)* of Lemma 4.1 may be read by saying that each L_n coincides with L on some neighborhood of $x_* \equiv 0$.

Thus, it only remains to show that, if n is big enough, then $x_* \equiv 0$ globally minimizes the periodic action functional

$$\mathcal{A}_n[x] := \int_0^1 L_n(t, x(t), x'(t))dt\,, \qquad x \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})\,,$$

associated to the lagrangian L_n. With this aim, we choose, for each $n \in \mathbb{N}$, some *global* minimizer $x_n \in \mathcal{C}^1(\mathbb{R}/\mathbb{Z})$ for $\mathcal{A}_n$. The periodicity of L_n on the variable x means that it is not restrictive to assume

$$-\frac{1}{2} \leq x_n(0) < \frac{1}{2}\,, \qquad n \in \mathbb{N}\,. \tag{15}$$

Now, remembering part *(d)* of Lemma 4.1,

$$\mathcal{A}_n[x_n] \geq 9n^2 \int_0^1 x_n'(t)^2 dt - K\,, \qquad\qquad n \in \mathbb{N}\,,$$

for some constant $K > 0$. But, since x_n is a minimizer,

$$\mathcal{A}_n[x_n] \leq \mathcal{A}_n[x_*] = 0 \text{ for any } n \in \mathbb{N}\,, \tag{16}$$

and we deduce that $\{x_n'\} \to 0$ in $L^2(\mathbb{R}/\mathbb{Z})$. In particular

$$x_n(t) - x_n(0) \to 0 \text{ as } n \to \infty \text{ uniformly with respect to } t \in \mathbb{R}/\mathbb{Z}\,. \tag{17}$$

At this point, items *(d)* and *(e)* of Lemma 4.1, together with the periodicity on x of L_n, mean that, for $n \geq 4$,

$$L_n(t,x,p) \geq n - K\,, \qquad\qquad \frac{1}{3n} \leq |x| \leq 1 - \frac{1}{3n}\,,$$

which, when combined with (15), (16) and (17), implies that $x_n(0) \to 0$, and then,

$$x_n(t) \to 0 \text{ as } n \to \infty \text{ uniformly with respect to } t \in \mathbb{R}/\mathbb{Z}\,. \tag{18}$$

On the other hand, remembering part *(c)* of Lemma 4.1, we observe that, wherever $|x_n'| > 1/3$ the Euler-Lagrange equation of L_n read $x_n'' = 0$, making it impossible for x_n to be periodic. This means that $|x_n'| \leq 1/3$ for each n, and we combine this fact with (18) to deduce that, if n is big enough,

$$|x_n(t)| \leq 1/6\,,\ |x_n'(t)| \leq 1/3\,, \qquad\qquad t \in \mathbb{R}/\mathbb{Z}\,. \tag{19}$$

However, items *(b)* and *(d)* of Lemma 4.1 imply that, for big values of n, $L_n(t,x,p) \geq L(t,x,p)$ on $(\mathbb{R}/\mathbb{Z}) \times [-1/6, 1/6] \times [-1/3, 1/3]$. Consequently,

$$\mathcal{A}_n[x_n] = \int_0^1 L_n(t, x_n(t), x_n'(t))dt \geq \int_0^1 L(t, x_n(t), x_n'(t))dt \geq 0\,,$$

last inequality following from (19) and (13). In view of (16) we deduce that $\mathcal{A}_n[x_n] = 0 = \mathcal{A}_n[x_*]$, meaning that x_* is also a global minimizer for $\mathcal{A}_n$ if n is big enough. The proof is complete.

□

References

[1] Carathéodory, C. *Calculus of variations and partial differential equations of the first order.* Chelsea, New York, 1989.

[2] Dancer, E. N.; Ortega, R. *The index of Lyapunov stable fixed points in two dimensions.* J. Dynam. Differential Equations 6 (1994), no. 4, 631–637.

[3] Moser, J. *Selected chapters in the calculus of variations.* Lecture notes by Oliver Knill. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003.

[4] Ortega, R. *The number of stable periodic solutions of time-dependent Hamiltonian systems with one degree of freedom.* Ergodic Theory Dynam. Systems 18 (1998), no. 4, 1007–1018.

[5] Ortega, R. *Instability of periodic solutions obtained by minimization*, 'The first 60 years of Nonlinear Analysis of Jean Mawhin', 189-197. World Scientific, 2004.

[6] Ureña, A.J., *Isolated periodic minima are unstable.* Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 6, 877–889.

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