Retracts, fixed point index and differential equations *

Rafael Ortega

Departamento de Matemática Aplicada
Facultad de Ciencias
Universidad de Granada, 18071 Granada, Spain
rortega@ugr.es

Dedicated to E. Norman Dancer

1 Introduction

Many problems in the theory of differential equations were initially treated with analytic techniques and later evolved towards more topological approaches. Perhaps the most paradigmatic case is found in the study of nonlinear boundary value problems. The classical proofs based on successive approximations or in the implicit function theorem were soon replaced by the use of fixed points and degree theory. The modern point of view is already found in the famous paper by Leray and Schauder [15]. The same process has been experienced by other branches of differential equations. The next pages are an attempt to illustrate this evolution in two concrete problems.

First we will discuss the existence of asymptotic solutions. These are non-trivial solutions tending to the origin as time increases to infinity and they appear in systems of differential equations having the trivial solution. Asymptotic solutions have been studied since Poincaré's times. The classical method for proving their existence consists in the reduction of the problem to an integral equation. Once this equation has been found one uses the method of successive approximations or the contraction principle. This analytical method leads to the Principle of Linearization and to the Stable Manifold

*Supported by D.G.I. MTM2005-03483, Ministerio de Educación y Ciencia, Spain

Theorem for autonomous equations (see [14] and [25]). Wazewski applied the theory of retracts and developed an alternative method for constructing asymptotic solutions in his paper [28]. We will illustrate Wazewski's ideas in a concrete situation, and later we will discuss the connections with the analytical approach. In the process we will find that other tools such as topological degree and global continuation are also applicable to this problem.

The second part of the paper will deal with the stability properties of closed orbits in Hamiltonian systems. This is a question of particular relevance in Celestial Mechanics and again we must refer to Poincaré for the origins of the problem. He mainly considered systems with two degrees of freedom and was lead to the study of the fixed points of area-preserving maps in the plane. This line of research was continued by Birkhoff but the stability problem was not solved until the appearance of the KAM method, which was fully developed in the sixties (see [2, 27]). In contrast, results on instability were already obtained by Levi-Civita in 1901. The interesting paper [19] contains the linearization principle due to Lyapunov and more delicate instability criteria depending on higher order terms. We will present a topological approach to the problem of instability treated by Levi-Civita. The main tools will be the fixed point index and the notion of translation arc. These arcs are very useful in the study of planar homeomorphisms and they already appeared in Brouwer's work. It would have been desirable to present also a topological approach to the problems of stability solved by KAM techniques. However these problems seems to be of a rather analytic character and I do not know if this could be possible.

I would like to thank Rogerio Martins for his careful reading of a preliminary version of this paper. His comments and questions have influenced the final form of the manuscript.

2 Asymptotic solutions and Wazewski's principle

Given a continuous vector field $X : [0,\infty[\times\mathbb{R}^d \to \mathbb{R}^d$, we can consider the system of differential equations

$$\dot{x} = X(t,x), \quad t \geq 0, \; x \in \mathbb{R}^d. \tag{1}$$

It will be always assumed that there is uniqueness for the associated initial value problem.

A solution $x(t)$ is called *asymptotic* to $x \equiv 0$ if it is well defined in $[0,\infty[$

and it satisfies

$$x(t) \to 0 \;\text{ as } t \to +\infty.$$

To make the discussion more visual we assume from now on that $d = 3$. The space $\mathbb{R}^3$ is decomposed as $\mathbb{R}^2 \times \mathbb{R}$, with coordinates $\xi \in \mathbb{R}^2$ and $z \in \mathbb{R}$. The notations $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ refer to the dot product and the euclidean norm in $\mathbb{R}^2$. The system (1) with $d = 3$ is rewritten as

$$\dot{\xi} = F(t, \xi, z), \;\; \dot{z} = G(t, \xi, z) \tag{2}$$

and we can imagine a cylindrical region shrinking to the origin as time goes to infinity. More precisely, we consider the set

$$\Omega_t = \{(\xi, z) : \; ||\xi|| < \varphi(t), \; |z| < \psi(t)\}$$

where $\varphi, \psi : [0, \infty[\to \mathbb{R}$ are C^1 decreasing functions with $\varphi(t) \to 0$, $\psi(t) \to 0$ as $t \to +\infty$. The flow associated to (2) will enter into the cylinder through the faces $z = \psi(t)$ and $z = -\psi(t)$ and will exit through the lateral boundary $||\xi|| = \varphi(t)$. In this setting our intuition says that some orbit should remain inside the cylinder forever. For if all the orbits would escape from the moving cylinder, they would do it through the lateral boundary. This process would define a retraction of the solid cylinder onto its lateral boundary, which is impossible. Once we know of the existence of orbits lying in Ω_t for each t we observe that they must produce asymptotic solutions. This is so because the cylinder is shrinking to the origin.

Precise statements can be produced from the above discussions. To this end it is convenient to consider the set

$$\Omega = \{(t, \xi, z) \in [0, \infty[\times \mathbb{R}^2 \times \mathbb{R} : \; (\xi, z) \in \Omega_t\},$$

which can also be described by the inequalities

$$V_i(t, \xi, z) < 0 \quad i = 1, 2, 3,$$

where

$$V_1(t, \xi, z) = z - \psi(t), \;\; V_2(t, \xi, z) = -z - \psi(t), \;\; V_3(t, \xi, z) = ||\xi||^2 - \varphi(t)^2.$$

The boundary of Ω relative to $[0, \infty[\times \mathbb{R}^2 \times \mathbb{R}$ will be denoted by Γ and decomposed in five parts

$$\Gamma = \mathcal{E} \cup \mathcal{I}_+ \cup \mathcal{I}_- \cup \mathcal{B}_+ \cup \mathcal{B}_-$$

with

$$\mathcal{E}: \quad ||\xi|| = \varphi(t), \ |z| < \psi(t)$$
$$\mathcal{I}_\pm: \quad ||\xi|| < \varphi(t), \ z = \pm\psi(t)$$
$$\mathcal{B}_\pm: \quad ||\xi|| = \varphi(t), \ z = \pm\psi(t).$$

The Lyapunov derivative of a function $V = V(t, \xi, z)$ along the solutions of (2) is defined as

$$\dot{V} := \frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial \xi}, F \rangle + \frac{\partial V}{\partial z} G.$$

We assume

$$\dot{V}_1 < 0 \ \text{ on } \mathcal{I}_+ \cup \mathcal{B}_+, \quad \dot{V}_2 < 0 \ \text{ on } \mathcal{I}_- \cup \mathcal{B}_-, \quad \dot{V}_3 > 0 \ \text{ on } \mathcal{E} \cup \mathcal{B}_+ \cup \mathcal{B}_-.$$

This assumption is modelled after the previous qualitative discussion. For later discussions we reformulate it in the next result.

Theorem 1 *Assume that $\varphi(t)$ and $\psi(t)$ are admissible functions*[1] *and that the two conditions below hold,*

$$\dot{\psi}(t) > \max\{G(t, \xi, \psi(t)), -G(t, \xi, -\psi(t))\} \ \text{ if } t \geq 0, \ ||\xi|| \leq \varphi(t), \tag{3}$$

$$\langle F(t, \xi, z), \xi \rangle > \varphi(t)\dot{\varphi}(t) \ \text{ if } t \geq 0, \ ||\xi|| = \varphi(t), \ |z| \leq \psi(t). \tag{4}$$

Then for each $z_0 \in [-\psi(0), \psi(0)]$ there exists at least one asymptotic solution of (2) satisfying

$$||\xi(0)|| < \varphi(0), \quad z(0) = z_0.$$

Proof. First of all we need to be more precise about the behavior of the flow on Γ. A point $(\tau, \xi_0, z_0) \in \Gamma$ is called of *strict entry* if the solution of (2) with initial conditions $\xi(\tau) = \xi_0$, $z(\tau) = z_0$ satisfies $(t, \xi(t), z(t)) \in \Omega$ if $t \in]\tau, \tau + \epsilon]$ and $(t, \xi(t), z(t)) \notin \overline{\Omega}$ if $t \in [\tau - \epsilon, \tau[$ for some $\epsilon > 0$. Notice that the second condition is empty if $\tau = 0$. A point of *strict exit* is defined in the same way excepting that the roles of the past and the future are exchanged. Finally we say that the point is of *exterior sliding* if $(t, \xi(t), z(t)) \notin \overline{\Omega}$ if $t \in [\tau - \epsilon, \tau + \epsilon], t \neq \tau$. As the reader probably suspects, the points on $\mathcal{I}_+ \cup \mathcal{I}_-$ are of strict entry, while those on $\mathcal{E}$ are of strict exit. Finally the points in $\mathcal{B}_+ \cup \mathcal{B}_-$ are of exterior sliding. This is the effect of the assumptions (3) and (4). For more information on this point the reader is referred to Section X.3 of the book [16].

[1] By an admissible function we understand a C^1 function $\varphi : [0, \infty[\to \mathbb{R}$ which is strictly decreasing and tends to 0 as $t \to +\infty$.

After classifying the points in Γ we are ready to complete the proof. We shall proceed by contradiction and assume that, for some $z_\star \in [-\psi(0), \psi(0)]$, there are no asymptotic solutions satisfying

$$||\xi(0)|| \leq \varphi(0), \quad z(0) = z_\star.$$

We will consider the disk

$$D = \{p \in \mathbb{R}^2 : \ ||p|| \leq \varphi(0)\}$$

and denote by $(\xi(t,p), z(t,p))$ the solution of (2) satisfying

$$\xi(0,p) = p, \ \ z(0,p) = z_\star.$$

Let $\tau = \tau(p) \geq 0$ be the first instant when the graph of the solution touches the exit set, that is

$$\tau(p) := \min\{t \in [0, \infty[: \ (t, \xi(t,p), z(t,p)) \in \mathcal{E} \cup \mathcal{B}_+ \cup \mathcal{B}_-\}.$$

It is important to notice that this definition is meaningful thanks to the contradiction argument. Indeed, we are assuming that for $||p|| < \varphi(0)$ the solution is not asymptotic and so it must escape from Ω. This escape must occur through $\mathcal{E}$ and so $\tau(p) \in]0, \infty[$. The previous discussion also includes the extreme case $z_\star = \pm\psi(0)$ since the points $(0, p, \pm\psi(0))$ are of strict entry. Finally we observe that if $||p|| = \varphi(0)$ then $\tau(p) = 0$. Once we know that τ is well defined we are going to prove that it is continuous as a function from D to $\mathbb{R}$. The continuous dependence with respect to initial conditions says that the solution is continuous as a function of (t,p) and this implies that $\tau = \tau(p)$ is lower semi-continuous. For a general differential equation it would not be possible to prove the other semi-continuity but we have a convenient classification of the points of Γ at our disposal. The point $(\tau(p), \xi(\tau(p), p), z(\tau(p), p))$ is of strict exit or of exterior sliding and this is sufficient to prove the upper semi-continuity of τ. More details on this argument can be found in Wazewski's paper [28] or in Chapter II of Conley's memoir [8]. The points of strict exit or exterior sliding lie on $\mathcal{E} \cup \mathcal{B}_+ \cup \mathcal{B}_-$ and so $||\xi(\tau(p), p)|| = \varphi(\tau(p))$. This observation allows us to consider the map

$$p \in D \mapsto \frac{\varphi(0)}{\varphi(\tau(p))}\xi(\tau(p), p) \in \partial D.$$

Since $\tau(p) = 0$ when $p \in \partial D$ this map would be a retraction of the disk onto its boundary. This is of course impossible and we have arrived at the searched contradiction.

In the paper [28], Wazewski says that the first attempts to develop his method employed homology and Kronecker index. Finally he found more convenient to attach his method to the notion of retract. In an exercise of history-fiction we will prove the previous result using degree (or Kronecker index).

Another proof of Theorem 1. This time the proof will be direct and $z_\star$ will be an arbitrary number in $[-\psi(0), \psi(0)]$. The notations for the disk D and the solution $(\xi(t,p), z(t,p))$ are preserved. Consider the function $\hat{\tau} : D \to [0, \infty]$ defined by

$$\hat{\tau}(p) := \sup\{s \geq 0 : \ (t, \xi(t,p), z(t,p)) \in \overline{\Omega} \ \text{ for each } t \in [0,s]\}$$

and observe that for those solutions getting out of $\overline{\Omega}$ this is equivalent to the definition of $\tau(p)$ in the previous proof. Once again the classification of the points on Γ has been used. The novelty is that we admit the possibility of solutions remaining in $\overline{\Omega}$ forever. In such cases $\hat{\tau}(p) = \infty$. We claim that $\hat{\tau}$ is continuous. Of course this is clear for those points with $\hat{\tau}(p) = \tau(p) < \infty$. For points with $\hat{\tau}(p) = \infty$ we proceed by contradiction. If p_n were a sequence in D converging to p with $\hat{\tau}(p_n) \to T < \infty$, then $(T, \xi(T,p), z(T,p))$ would belong to $\mathcal{E} \cup \mathcal{B}_+ \cup \mathcal{B}_-$. Since $\hat{\tau}(p) = \infty > 0$ we infer that $(0, p, z_\star)$ cannot be in the exit set and so $T > 0$. This would lead to the contradictory conclusion $\hat{\tau}(p) \leq T$. Once we know that $\hat{\tau}$ is continuous we consider the map $\Phi : D \to \mathbb{R}^2$ given by

$$\Phi(p) = e^{-\hat{\tau}(p)} \xi(\hat{\tau}(p), p) \ \text{ if } \hat{\tau}(p) < \infty, \quad \Phi(p) = 0 \ \text{ if } \hat{\tau}(p) = \infty.$$

All points on the boundary of D are fixed under Φ and so we can compute the Brouwer degree

$$\deg(\Phi, D) = \deg(id, D) = 1.$$

This implies that Φ has at least one zero contained in the interior of D, but the zeros of Φ correspond to the solutions remaining in $\overline{\Omega}$ and these solutions are asymptotic to the origin.

Indeed the conclusion of the above proof can be sharpened with the Leray-Schauder continuation principle [15, 10]. With the previous notations we interpret $z_\star$ as a parameter varying in the interval $I = [-\psi(0), \psi(0)]$. The map $\Phi = \Phi(p, z_\star)$ now goes from $D \times I$ into $\mathbb{R}^2$. Since $\Phi = id$ on $\partial D \times I$, there are no zeros on this set and $\deg(\Phi(\cdot, z_\star), D) \neq 0$ for any $z_\star$. Then there exists a continuum C contained in $D \times I$, joining the sets $D \times \{-\psi(0)\}$ and $D \times \{\psi(0)\}$, and such that

$$\Phi(p, z_\star) = 0 \quad \text{for each } (p, z_\star) \in C.$$

We can now state the following improvement of Theorem 1,

Assume that the conditions (3) and (4) hold. Then there exists a continuum $C \subset \Omega_0$, joining $z = \psi(0)$ and $z = -\psi(0)$, such that every solution of (2) with $(\xi(0), z(0)) \in C$ is asymptotic to $x \equiv 0$.

There are several topological versions of the stable manifold theorem [26, 18]. They deal with homeomorphisms of the plane having the origin as an isolated invariant set. The previous proofs suggest that Wazewski's method could be useful to obtain related results in higher dimensions. There is another remarkable feature in Wazewski's method, it deals with a general non-autonomous differential equation. This is very close to the topological notion of isotopy.

To finish this section on asymptotic solutions we discuss two examples which help to clarify the connection with more classical analytical results. First we consider a system (2) with the semi-linear form

$$\dot{\xi} = A\xi + R_1(t, \xi, z), \quad \dot{z} = -\lambda z + R_2(t, \xi, z), \tag{5}$$

where A is a 2×2 matrix satisfying

$$\langle A\xi, \xi \rangle \geq 0 \quad \text{for each } \xi \in \mathbb{R}^2$$

and λ is a positive constant. The remainders R_1 and R_2 are small; that is,

$$\lim_{||\xi||+|z| \to 0} \frac{||R_1(t, \xi, z)|| + |R_2(t, \xi, z)|}{||\xi|| + |z|} = 0,$$

uniformly in $t \in [0, \infty[$. In the unperturbed case, $R_i = 0,\ i = 1, 2$, the system is linear and has the asymptotic solutions $\xi(t) = 0$, $z(t) = z_0 e^{-\lambda t}$. The perturbed system (5) has also asymptotic solutions and the reader is referred to Chapter X of Hartman's book [16] for more general results on linearization principles. Next we show how to prove the existence of asymptotic solutions using Theorem 1. Select two numbers μ and ϵ with

$$0 < 2\epsilon < \min\{\lambda - \mu, \mu\}$$

and find a positive δ such that

$$||R_1(t, \xi, z)|| + |R_2(t, \xi, z)| \leq \epsilon(||\xi|| + |z|)$$

if $||\xi|| \leq \delta$, $|z| \leq \delta$. A simple computation shows that the assumptions (3) and (4) hold with $\varphi(t) = \psi(t) = \delta e^{-\mu t}$.

The second example is the system

$$\dot{\xi} = 0, \quad \dot{z} = -2z^3,$$

which is not of the type (5). Theorem 1 can be applied with $\varphi(t) = e^{-t}$ and $\psi(t) = (1+3t)^{-1/2}$.

3 Instability criteria for periodic orbits

We start with a Hamiltonian system of two degrees of freedom. The phase space S is an open subset of $\mathbb{R}^4$ and a generic point in S is denoted by $\xi = (q,p)$ with $q, p \in \mathbb{R}^2$. The equations are

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p) \tag{6}$$

with $H : S \to \mathbb{R}$ smooth (*the Hamiltonian function*). This family of equations has many illustrious members, including the Kepler problem and the circular restricted three body problem.

The function H is a first integral of the system (6) and so the sets

$$M_c = \{(q,p) \in S : \ H(q,p) = c\}, \quad c \in \mathbb{R},$$

are invariant. Typically these sets are 3d manifolds and it seems natural to restrict the flow to them. Let us now assume that γ is a closed orbit of (6). The corresponding periodic solution is not constant and so γ cannot contain critical points of H. This implies that $M_{\overline{c}}$, with $\overline{c} = H(\gamma)$, is a smooth manifold, at least in some neighborhood around γ. The orbit γ is called *isoenergetically stable* if it is orbitally stable (in the future and in the past) with respect to the flow on $M_{\overline{c}}$. This means that for each neighborhood $\mathcal{U}$ of γ there exists another neighborhood $\mathcal{V}$ such that any orbit passing through $\mathcal{V} \cap M_{\overline{c}}$ remains entirely in $\mathcal{U} \cap M_{\overline{c}}$. As an example we consider the Kepler problem, whose Hamiltonian is

$$H(q,p) = \frac{1}{2}||p||^2 - \frac{1}{||q||}, \quad (q,p) \in S = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2.$$

In this case M_c is homeomorphic to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ if $c \geq 0$ and to $\mathbb{R}^2 \times \mathbb{S}^1$ if $c < 0$. The orbit associated to the periodic solution $q(t) = (\cos t, \sin t)$, $p(t) = \dot{q}(t)$ has negative energy with $H(\gamma) = -1/2$. The reader who has some familiarity with Celestial Mechanics can prove that this circular motion is stable, for all the motions in a neighborhood are of elliptic type.

The method of transversal sections reduces the problem of isoenergetic stability to the study of a discrete transformation in the plane. This is done as follows. We fix a point $\xi_\star \in \gamma$ and construct a transversal section $\Sigma \subset M_{\overline{c}}$ passing through $\xi_\star$. By restricting the size of Σ we can always assume that $\Sigma \cap \gamma = \{\xi_\star\}$. The section Σ is diffeomorphic to a disk and, given a point $\xi \in \Sigma$ which is close enough to $\xi_\star$, say $\xi \in \Sigma'$, we know that the orbit passing through ξ must cross Σ in the future. The first of these returns will be denoted by $h(\xi)$. The point $\xi_\star$ is fixed under the map $h : \Sigma' \subset \Sigma \to \Sigma$ and the isoenergetic stability of γ is equivalent to the perpetual stability of $\xi_\star$ as a fixed point of h. For future discussions we mention some properties of h. It is a smooth and one-to-one map which preserves orientation. In addition there exists a measure on Σ which is preserved by h, this measure is obtained as a pull-back of the Lebesgue measure in the plane. More details can be found in sections 22 and 31 of the book by Siegel and Moser [27].

We are ready for a discussion with more topological flavor. We shall work with the open disk

$$D = \{\xi \in \mathbb{R}^2 : \ ||\xi|| < 1\}.$$

The group of homeomorphism of D will be denoted by $\mathcal{H}(D)$. We stress that $H(D) = D$ for each H in $\mathcal{H}(D)$. Let us assume that there is a regular measure on the disk, denoted by μ, which is invariant under H. This means that

$$\mu(H(B)) = \mu(B) \quad \text{for each Borel set } B \subset D.$$

The measure μ satisfies two extra conditions:

- the whole disk D has finite measure
- the measure of any non-empty open set is positive.

We summarize the above conditions by saying that H is in the class $\mathcal{H}(D, \mu)$. The following fixed point theorem can be found in the papers by Montgomery [21] and Bourgin [4].

Theorem 2 *Every orientation preserving map in $\mathcal{H}(D, \mu)$ has a fixed point.*

A similar result is false for open balls in higher dimensions. In [4] Bourgin constructed an orientation preserving homeomorphism of the ball $B = B^{135} = \{x \in \mathbb{R}^{135} : \ ||x|| < 1\}$ which was fixed point free and invariant under an admissible measure. Later Asimov found in [3] analogous examples in B^3

using the Hopf fibration. Even in the plane the theorem fails for orientation reversing maps, as was discovered by Alpern in [1].

Next we improve the theorem in the plane by using the fixed point index. Following [10] the fixed point index will be denoted by $I(f,U)$, where U is an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ is continuous. This notion makes sense when the set of fixed points $\mathrm{Fix}(f)$ is compact. We also recall the connection with the Brouwer degree,

$$I(f,U) = \deg(id - f, U).$$

Theorem 3 *Assume that $H \in \mathcal{H}(D,\mu)$ is orientation preserving. If H is not the identity, $H \neq id$, then there exists a Jordan curve $\Gamma \subset D$ such that*

$$I(H,\hat{\Gamma}) = 1.$$

Here $\hat{\Gamma}$ denotes the bounded component of $\mathbb{R}^2 \setminus \Gamma$.

The reader can find related results in the papers by Medvedev [20] and by Le Calvez [17].

To prove this theorem we shall employ the notion of translation arc, which goes back to Brouwer. An oriented arc $\alpha = \widehat{pq}$ in D is called a *translation arc* for $H \in \mathcal{H}(D)$ if $H(p) = q$ and

$$H(\alpha \setminus \{q\}) \cap (\alpha \setminus \{q\}) = \emptyset.$$

The next result probably explains why this notion is so useful in the study

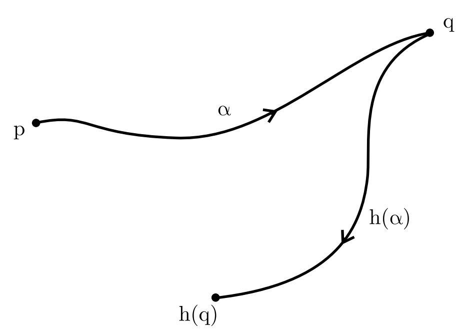

Figure 1: A translation arc. $(H \equiv h)$

of discrete dynamics in the plane.

Lemma 4 (Brouwer). *Assume that $H \in \mathcal{H}(D)$ is orientation preserving and there exists a translation arc α with*

$$H^n(\alpha) \cap \alpha \neq \emptyset \quad \text{for some } n \geq 2.$$

Then there exists a Jordan curve $\Gamma \subset D$ such that

$$I(H, \hat{\Gamma}) = 1.$$

This result has a long history and the proof is delicate. The reader is referred to [5, 11] for more details.

We need a second preliminary result on the existence of translation arcs.

Lemma 5 *Assume that $H \in \mathcal{H}(D)$ and Δ is a compact topological disk contained in D and such that Δ and $H(\Delta)$ lie in the same component of $D \setminus \mathrm{Fix}(H)$. In addition assume that*

$$H(\Delta) \cap \Delta = \emptyset.$$

Then, given points $\xi_1, \ldots, \xi_n \in \Delta$, there exists a translation arc α contained in D and passing through all these points.

This result is obtained by an adaptation of the ideas of Brown in his proof of Lemma 4.1 in [6]. The details will appear in the monograph under construction [24]. After these two lemmas we are ready for the proof.

Proof of Theorem 3. Let $\{U_\lambda\}$ be the family of connected components of $D \setminus \mathrm{Fix}(H)$. These sets are open and invariant under H. The invariance follows from the result by Brown and Kister in [7] since H preserves orientation. We fix one of the components, say U_λ, and a point $\xi_\star \in U_\lambda$. This is possible for any $H \neq id$. The points $\xi_\star$ and $H(\xi_\star)$ are different and so we find a small disk Δ centered at $\xi_\star$ which is contained in U_λ and is such that $H(\Delta) \cap \Delta = \emptyset$. Next we apply Poincaré's recurrence theorem as presented in [22] and find a point ξ_0 in the interior of Δ which is recurrent. For some $n \geq 2$ the iterate $H^n(\xi_0)$ will enter again in the disk Δ. Lemma 5 says that we can find a translation arc passing through ξ_0 and $H^n(\xi_0)$. This last point belongs to $H^n(\alpha) \cap \alpha$ and so we can apply Brouwer's lemma to arrive at the conclusion.

In the next pages we explore the implications of Theorem 3 in stability theory. Assume that $\mathcal{U}$ is an open subset of the plane containing the origin and

$$h : \mathcal{U} \to \mathbb{R}^2, \quad h = h(\xi)$$

is a continuous and one-to-one map having a fixed point at the origin. This point is called *stable in the future* if every neighborhood $\mathcal{V}$ contains another neighborhood $\mathcal{W}$ such that the successive iterates of $\mathcal{W}$ remain in $\mathcal{V}$; that is

$$h^n(\mathcal{W}) \subset \mathcal{V} \quad \text{for each } n \geq 0.$$

The theorem of invariance of the domain implies that h is open and so its inverse is also continuous. This fact allows a parallel definition of *stability in the past.* Finally we say that there is *perpetual stability* when the origin is stable for the future and the past.

The three notions of stability are equivalent for area-preserving maps[2] This fact is well known in Hamiltonian dynamics. The reader can find a proof in Lemma 2.5 of [23]. Next we present a result exploring the implications of the stability in the fixed point index.

Theorem 6 *Assume that h is orientation and area preserving and $\xi = 0$ is stable. Then one of the alternatives below holds,*

(i) *$h = id$ in some neighborhood of the origin*

(ii) *there exists a sequence of Jordan curves $\{\Gamma_n\}$ converging to the origin and such that, for each n,*

$$\Gamma_n \cap \mathrm{Fix}(h) = \emptyset, \quad I(h, \hat{\Gamma}_n) = 1.$$

When the fixed point is isolated this theorem is a consequence of the results in [9]. The novelty is in the case of non-isolated fixed points.

Proof. Let us first recall that the stability of the origin guarantees the existence of a sequence of open neighborhoods $\{\mathcal{U}_n\}$ which are simply connected and satisfy

$$\bigcap_n \mathcal{U}_n = \{0\}, \quad h(\mathcal{U}_n) = \mathcal{U}_n.$$

See section 25 of [27] or [23] for a proof. Each $\mathcal{U}_n$ has finite area and is homeomorphic to D, say $\psi : \mathcal{U}_n \cong D$. The map $H = \psi \circ h \circ \psi^{-1}$ is in the class $\mathcal{H}(D, \mu)$, where μ is obtained as a transport of the Lebesgue measure. Assuming that (i) does not hold it is possible to apply Theorem 3 and find a Jordan curve Γ_n in $\mathcal{U}_n$ such that

$$I(H, \hat{\gamma}_n) = I(h, \hat{\Gamma}_n) = 1$$

[2]For simplicity it is assumed that μ is the Lebesgue measure but it will be clear how to extend the discussions to a large class of measures.

with $\gamma_n = \psi(\Gamma_n)$. This is precisely the second alternative.

Next we present a couple of examples showing that the preservation of orientation and area are essential in the previous result.

Example 1. The symmetry with respect to the x-axis is area-preserving but it reverses orientation. Denote this map by $h_1(x,y) = (x,-y)$. We observe that $\mathrm{Fix}(h_1) = \mathbb{R} \times \{0\}$ and all the fixed points are stable. In contrast to the theorem, $I(h_1, \hat{\Gamma}) = 0$ for any Jordan curve $\Gamma \subset \mathbb{R}^2 \setminus \mathrm{Fix}(h_1)$. Indeed any of these curves lie in one of the half-planes $\{y > 0\}$, $\{y < 0\}$, and these regions do not contain fixed points.

Example 2. The map h_2 is expressed in polar coordinates as

$$(\theta, r) \mapsto (\theta + \sin\theta, r).$$

This time the map preserves the orientation but not the area. The set of fixed points is again the x-axis and the origin is perpetually stable. To check this it is sufficient to notice that all disks centered at the origin are invariant. As in the previous case one can prove that $I(h_2, \hat{\Gamma}) = 0$ for any Jordan curve $\Gamma \subset \mathbb{R}^2 \setminus \mathrm{Fix}(h_2)$.

In a preliminary version of the paper I constructed a more complicated example with similar properties. It was R. Martins who suggested the use of h_2.

We are going to finish the paper with two applications of Theorem 6.

The index and an instability criterion by Levi-Civita. In section 4 of [19] Levi-Civita considered maps of the type

$$h(x,y) = (x + f(x,y), y + x + g(x,y))$$

where f, g were smooth functions defined in a neighborhood of the origin and satisfying

$$f(0,0) = g(0,0) = 0, \quad \nabla f(0,0) = \nabla g(0,0) = 0. \tag{7}$$

Assuming that the Taylor expansion of f was

$$f(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + \cdots$$

he proved that the origin was not stable if $a_{22} \neq 0$.

We are going to present a topological version of this result for the area-preserving case. To this end we assume that h is C^1 and

$$\det h'(x,y) = 1 \quad \text{for each } (x,y).$$

This is sufficient to guarantee that h is in the conditions of Theorem 6. The special structure of the linear part of h allows to reduce the computation of the fixed point index to one-dimensional degree. This is done following ideas from [12]. We first apply the implicit function theorem to solve the equation

$$x + g(x, y) = 0 \tag{8}$$

and obtain x as a function of y, say that $x = \varphi(y)$ is the only solution in the rectangle $R = [-\delta, \delta] \times [-\epsilon, \epsilon]$. Next we define the function

$$\Phi(y) = f(\varphi(y), y),$$

and notice that the fixed points of h in R satisfy $x = \varphi(y)$ and $\Phi(y) = 0$. Given a Jordan curve contained in R and disjoint with $\mathrm{Fix}(h)$, the index can be computed by the formula

$$I(h, \hat{\Gamma}) = -\deg_{\mathbb{R}}(\Phi, \Omega) \tag{9}$$

where $\Omega = \{y \in [-\epsilon, \epsilon] : \ (\varphi(y), y) \in \hat{\Gamma}\}$. We will give a sketch of the proof of this formula later. The index of h will vanish at any Jordan domain if Φ satisfies one of the conditions below,

$$(\mathbf{i}) \quad \Phi(y) \geq 0 \ \text{ for each } y \in [-\epsilon, \epsilon],$$

or

$$(\mathbf{ii}) \quad \Phi(y) \leq 0 \ \text{ for each } y \in [-\epsilon, \epsilon].$$

It can take the values -1 or 0 if Φ satisfies

$$(\mathbf{iii}) \quad y\Phi(y) \geq 0 \ \text{ for each } y \in [-\epsilon, \epsilon].$$

In any of these cases there are no curves of index one and the origin cannot be stable.

We can recover from here the result by Levi-Civita. Assuming that h is C^2 we notice that, from (8) and (7), $\varphi(y) = O(y^2)$ and so

$$\Phi(y) = a_{11}\varphi(y)^2 + 2a_{12}\varphi(y)y + a_{22}y^2 + \cdots = a_{22}y^2 + o(y^2).$$

If $a_{22} \neq 0$ we are in case (i) or (ii). The paper [19] also deals with the case when the linear part of h is the identity and an elegant variation can be found in section 31 of [27]. The topological approach to this case was presented in [9].

It remains to justify the formula (9). Define the map

$$F(x, y) = (\Phi(y), x - \varphi(y)), \quad (x, y) \in R,$$

and notice that, for each $\lambda \in [0, 1]$, the zeros of $\lambda(h - id) + (1 - \lambda)F$ are exactly the fixed points of h. At this point it is convenient to observe that the implicit function theorem implies that $x = \varphi(y)$ is the only solution of

$$\lambda(x + g(x, y)) + (1 - \lambda)(x - \varphi(y)) = 0.$$

By homotopy invariance we must compute the degree of F. To do this we perform the change of variables

$$\xi = x - \varphi(y), \quad \eta = y.$$

The map $\phi(x, y) = (\xi, \eta)$ is a diffeomorphism of R onto its image and so $\deg(F, \hat{\Gamma}) = \deg(F_\star, \hat{\gamma})$, where $F_\star = \psi \circ F \circ \psi^{-1}$ and $\gamma = \psi(\Gamma)$. The new map is $F_\star(\xi, \eta) = (\Phi(\eta), \xi)$ and the reduction to one dimension follows.

Analytic area preserving maps. Assume now that $h : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$, $h = (h_1, h_2)$, is a real analytic map defined on some open set $\mathcal{U}$. The set of fixed points can be described by the equation

$$\mathrm{Fix}(h) : \quad (h_1(x, y) - x)^2 + (h_2(x, y) - y)^2 = 0.$$

When h is not the identity this is a proper analytic subset of the plane. The local structure of these sets is well known (see [13]): they can contain isolated points and points with a finite number of branches emanating from them. In the second case the branches are described by Puiseux series. In view of this we consider a non-isolated fixed point $\xi_\star$ and a small disk $\mathcal{D}$ around it such that $\mathrm{Fix}(h) \cap \mathcal{D}$ is composed by the branches emanating from $\xi_\star$. Moreover we can assume that all the branches touch the boundary of the disk and each component of $\mathcal{D} \setminus \mathrm{Fix}(h)$ is simply connected. From this setting it is clear that if Γ is a Jordan curve in $\mathcal{D}$ without fixed points, then $\hat{\Gamma}$ does not contain fixed points either. In consequence the index $I(h, \hat{\Gamma})$ vanishes. We are lead to a result already obtained in [23].

Corollary 7 *Assume that $h \neq id$ is real analytic and*

$$\det h' \equiv 1.$$

Then every stable fixed point is isolated.

To finish the paper we go back to the Hamiltonian system (6) and assume that the function H is real analytic. Assume also that we are given a closed orbit γ with $\overline{c} = H(\gamma)$ and a transversal section $\Sigma \subset M_{\overline{c}}$ with $\Sigma \cap \gamma = \{\xi_\star\}$. Any closed orbit $\gamma' \subset M_{\overline{c}}$ which is close enough to γ will pass through Σ only a finite number of times. This is a consequence of the transversality of the section. The closed orbit γ' will be called *simple* whenever $\gamma' \cap \Sigma$ is a singleton. This notion is relative to the chosen section Σ but the initial orbit γ is simple just by construction. The previous corollary can be rephrased in the following terms: if γ is isoenergetically stable then there exists a neighborhood $\mathcal{U}$ of γ such that one of the alternatives below holds,

- every orbit contained in $\mathcal{U} \cap M_{\overline{c}}$ is closed and simple
- γ is the only orbit in $\mathcal{U} \cap M_{\overline{c}}$ which is closed and simple.

The circular orbit of the Kepler problem mentioned at the beginning of the section will be in the first situation.

References

[1] S. Alpern, Area-preserving homeomorphisms of the open disk without fixed points, Proc. Amer. Math. Soc. 103 (1988) 624-626.

[2] V.I. Arnold and A. Avez, Ergodic problems of classical Mechanics, Adisson Wesley 1989.

[3] D. Asimov, On volume-preserving homeomorphims of the open n-disk, Houston Math. J. 2 (1976) 1-3.

[4] D.G. Bourgin, Homeomorphims of the open disk, Studia Math. 31 (1968) 433-438.

[5] M. Brown, A new proof of Brouwer's lemma on translation arcs, Houston J. Math. 10 (1984) 35-41.

[6] M. Brown, Homeomorphisms of two-dimensional manifolds, Houston J. Math. 11 (1985) 455-469.

[7] M. Brown and J. Kister, Invariance of the complementary domains of a fixed point set, Proc. Am. Math. Soc. 91 (1984) 503-504.

[8] C. Conley, Isolated invariant sets and the Morse Index, Regional Conferences Series in Mathematics 38, American Mathematical Society 1978.

[9] E.N. Dancer and R. Ortega, The index of Lyapunov stable fixed points in two dimensions, J. Dynam. Diff. Eqns. 6 (1994) 631-637.

[10] A. Granas and J. Dugundji, Fixed point theory, Springer 2003.

[11] L. Guillou, Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré-Birkhoff, Topology 33 (1994) 331-351.

[12] M.A. Krasnosel'skii, A.I. Perov, A.I. Povolotskiy and P.P. Zabreiko, Plane vector fields, Academic Press 1966.

[13] S. Krantz and H. Parks, A primer of real analytic functions, Birkhäuser 2002.

[14] S. Lefschetz, Differential equations: geometric theory. Dover 1977.

[15] J. Leray and J. Schauder, Topologie et équations fonctionnelles, Ann. Sci. École Norm. Sup. 51 (1934) 45-78.

[16] P. Hartman, Ordinary differential equations, John Wiley 1964.

[17] P. Le Calvez, Une nouvelle preuve du théorème de point fixe de Handel, Geom. Topol. 10 (2006)2299-2349.

[18] F. Le Roux, Dynamique des homéomorphismes de surfaces. Versions topologiques des théorèmes de la fleur de Leau-Fatou et de la variété stable, Astérisque, 292 (2004).

[19] T. Levi-Civita, Sopra alcuni criteri di instabilità, Annali di Matematica 5 (1901) 221-307.

[20] V.S. Medvedev, On the index of invariant domains of homeomorphisms of two-dimensional manifolds, Math. USSR Sbornik 62 (1989) 207-221.

[21] D. Montgomery, Measure preserving transformations at fixed points, Bull. Amer. Math. Soc. 51 (1945) 949-953.

[22] V.V. Nemytskii, V.V. Stepanov, Qualitative theory of differential equations, Dover 1989.

[23] R. Ortega, The number of stable periodic solutions of time-dependent Hamiltonian systems with one degree of freedom, Ergod. Th. Dynam. Sys. 18 (1998), 1007-1018.

[24] R. Ortega, Topology of the plane and periodic differential equations, in preparation.

[25] J. Palis, W. de Melo, Geometric theory of dynamical systems, Springer Verlag 1982.

[26] F.R. Ruiz del Portal and J.M. Salazar, A stable/unstable manifold theorem for local homeomorphisms of the plane, Ergodic Theory Dyn. Syst. 25 (2005) 301-317.

[27] C.L. Siegel and J.K. Moser, Lectures on celestial mechanics, Springer 1971.

[28] T. Wazewski, Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires, Ann. Soc. Polon. Math. 20 (1947) 279-313.