A dynamical characterization of planar symmetries

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1 Introduction

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving, real analytic diffeomorphism of the plane, having a fixed point $x_* = f(x_*)$. This point is Lyapunov stable if, given any neighborhood $\mathcal{U} = \mathcal{U}(x_*)$, it is possible to find another, $\mathcal{V} = \mathcal{V}(x_*)$, such that $f^n(\mathcal{V}) \subset \mathcal{U}$ for each $n \geq 0$. This definition corresponds to stability for the future ($n \geq 0$), equivalent in this setting to perpetual stability ($n \in \mathbb{Z}$). The study of the dynamics around those fixed points is a classical topic in Hamiltonian dynamics, see [2, 12]. Traditionally the term area-preserving has been reserved for maps satisfying $\det f'(x) \equiv 1$, where $f'(x)$ is the Jacobian matrix. Under this condition also orientation is preserved and it is well known that many different dynamics around the stable fixed point can appear. In this paper it will be shown that the situation is very different when the area is preserved but the orientation is reversed. In that case the only possible dynamics is that of a symmetry, say $S(x_1, x_2) = (x_1, -x_2)$.

Theorem 1 *Let f be a real analytic diffeomorphism of $\mathbb{R}^2$ satisfying*

$$\det f'(x) = -1, \quad \textit{for each } x \in \mathbb{R}^2.$$

In addition assume that f has a stable fixed point. Then there exists a homeomorphism ψ of $\mathbb{R}^2$ such that $f = \psi \circ S \circ \psi^{-1}$.

Throughout the paper it is understood that a homeomorphism (or diffeomorphism) of a space X is, in particular, an onto map. After the proof

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of the theorem we will present examples showing that the result cannot be extended to higher dimensions or to the non-analytic case.

The study of the dynamics of orientation-reversing maps has received some attention in the last years. We refer to the papers by Bonino [3] and Ruiz del Portal and Salazar [11]. The results in [11] describe the dynamics of orientation-reversing maps around a stable fixed point that is isolated in $\mathrm{Fix}(f^2)$. Isolated stable fixed points cannot appear when the map is orientation-reversing and area-preserving.

I thank F. R. Ruiz del Portal for reading a preliminary version of this paper and informing me on reference [6].

2 A fixed point theorem on generalized disks

A compact subset of the plane, $\Delta \subset \mathbb{R}^2$, will be called a *generalized disk* if its interior is homeomorphic to the open disk,

$$\mathrm{int}(\Delta) \cong \{x \in \mathbb{R}^2 : \ ||x|| < 1\}.$$

Examples of generalized disks are the square $\Delta = [0,1] \times [0,1]$ or the region encircled by the topologist's sine curve,

$$\Delta = \{(x,y) \in \mathbb{R}^2 : \ x \in]0,1], \ -2 \leq y \leq \sin \frac{1}{x}\} \cup (\{0\} \times [-2,1]).$$

Proposition 2 *Let Δ be a generalized disk and let h be an orientation-reversing, area-preserving homeomorphism of Δ. Then h has a fixed point lying at the boundary $\partial\Delta$.*

The property of preservation of area means that $\mu(h(B)) = \mu(B)$ for each Borel set B contained in Δ. For the measure μ we can consider Lebesgue measure or, more generally, any regular measure such that $\mu(\Delta) < \infty$ and the measure of every open set is positive.

Alpern constructed in [1] examples of maps in the conditions of the previous proposition and such that all fixed points lie on the boundary. An example of Alpern's construction can be obtained using the flow $\{\phi_t\}$ associated to the Hamiltonian system $\dot{q} = H_p$, $\dot{p} = -H_q$ with

$$H(q,p) = q^2(1-q)^2(1-p)^2(1+p)^2 p.$$

All points on the boundary of $\Delta = [0,1] \times [-1,1]$ are equilibria and so the set Δ is invariant under ϕ_t. The phase portrait on the interior of Δ is composed by two symmetric centers separated by a heteroclinic orbit $p = 0$ travelling from $(0,0)$ to $(1,0)$. For a fixed $T>0$ the map $h=S\circ\phi_T$ has no fixed points in $\mathrm{int}(\Delta)$.

Proposition 2 is a direct consequence of the main result in [6], at least when h can be extended to a homeomorphism of the whole plane. I will present a short proof of the proposition based on the theory of prime ends. An exposition of this theory can be found in [10] and the classical paper [4] explains how to use it in fixed point theory. For the notation we follow [9]. Let U be a proper subset of $\mathbb{R}^2$ that is open and simply connected. The set of prime ends $\mathbb{P}=\mathbb{P}(U)$ is homeomorphic to $\mathbb{S}^1$ and can be added to U so that the disjoint union $U^*=U\cup\mathbb{P}$ is homeomorphic to the unit disk $\mathbb{D}=\{x\in\mathbb{R}^2:\ ||x||\leq 1\}$. Given a homeomorphism h of $\overline{U}$ there appear an induced homeomorphism h^* of U^* so that h and h^* coincide on U. Generally speaking, the existence of a prime end in $\mathbb{P}$ fixed under h^* does not imply the existence of a fixed point of h lying on ∂U, the boundary of U. This is discussed in [4]. The situation is different when h is area-preserving. We present a consequence of lemma 11 in [4].

Lemma 3 *Assume that U is bounded and h is an area-preserving homeomorphism of $\overline{U}$ such that h^* has a fixed prime end $p\in\mathbb{P}$. Then every point in $\Pi(p)$ is fixed under h.*

Recall that $\Pi(p)$, the set of principal points associated to p, is a non-empty continuum contained in ∂U.

Proof of proposition 2. The previous discussions on prime ends can be applied to $U=\mathrm{int}(\Delta)$ and h, understood as a homeomorphism of $\overline{U}\subset\Delta$. The map h^* defines a homeomorphism of the pair $(U^*,\mathbb{P})$. The space U^* can be thought as an orientable manifold with boundary. The maps h and h^* coincide on the open set U and so h^* is an orientation-reversing homeomorphism of U^*. In consequence the restriction to the boundary, denoted by $h^*:\mathbb{P}\to\mathbb{P}$, is also orientation-reversing. Orientation reversing homeomorphisms of $\mathbb{S}^1\cong\mathbb{P}$ have exactly two fixed points. This can be applied to deduce that h^* has two fixed prime ends. The proof of the proposition is now a consequence of lemma 3. Notice that the boundary of $\overline{U}$ is contained in the boundary of Δ.

3 Proof of the main result and examples

We will obtain the result by combining the above fixed point theorem with several known results. They are presented in three lemmas.

Lemma 4 *Assume that $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ and $h : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a one-to-one and continuous map having a Lyapunov stable fixed point $x_* = h(x_*)$ in $\mathcal{U}$. In addition assume that h is area-preserving. Then there exists a sequence of generalized disks (Δ_n) contained in $\mathcal{U}$ and satisfying*

$$x_* \in \mathrm{int}(\Delta_n), \quad \Delta_{n+1} \subset \mathrm{int}(\Delta_n), \quad \bigcap_n \Delta_n = \{x_*\}, \quad h(\Delta_n) = \Delta_n.$$

The proof of this result can be found in [12]. See also [7] for some additional details.

Lemma 5 *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a real analytic one-to-one map satisfying*

$$\det g'(x) = 1 \quad for\ each\ x \in \mathbb{R}^2$$

and assume that g has a stable fixed point $x_* = g(x_*)$. The either g is the identity or x_* is isolated in the set of fixed points $\mathrm{Fix}(g)$.

This is the main result in [7]. See also [8] for an alternative proof.

Lemma 6 *Assume that h is an orientation reversing homeomorphism of $\mathbb{R}^2$ satisfying $h^2 = h \circ h = identity$. Then there exists a homeomorphism ψ of $\mathbb{R}^2$ such that $h = \psi \circ S \circ \psi^{-1}$.*

This is a particular case of a well-known result due to Kerékjártó. A proof can be found in [5].

We are ready for the proof of Theorem 1. In view of Lemma 4 we can find a sequence of generalized disks Δ_n that are invariant under f. From proposition 2 we deduce that f has a fixed point on the boundary of each Δ_n. These fixed points will accumulate on x_* and we conclude that x_* is not isolated in $\mathrm{Fix}(f)$. The map $g = f \circ f$ is in the conditions of lemma 5. Since $\mathrm{Fix}(f) \subset \mathrm{Fix}(g)$, x_* is not isolated in $\mathrm{Fix}(g)$. This implies that g is the identity. The proof is finished by an application of lemma 6.

Theorem 1 has not an analogue in higher dimension. To show this it is sufficient to consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \theta \text{ not commensurable with } 2\pi.$$

It satisfies $\det A = -1$ but the origin is stable and isolated in $\mathrm{Fix}(f)$.

To prove the need of the analyticity we define $f = S \circ M$ where M is a C^∞ diffeomorphism of $\mathbb{R}^2$ satisfying the properties below,

1. $\det M'(x) = 1$ if $x \in \mathbb{R}^2$
2. there exists a sequence $R_n \downarrow 0$ such that the disks $||x|| \leq R_n$ are invariant under M
3. $M(0, R_n) = (0, R_n)$ and $M(0, -R_n) \neq (0, -R_n)$.

Then the origin is a stable fixed point for f because the disks $||x|| \leq R_n$ are also invariant under f. In contrast to the main result of the paper we observe that $\det f' = -1$ and f is not conjugate to S. To prove the last assertion notice that

$$f^2(0, R_n) = S \circ M(0, -R_n) \neq (0, R_n)$$

and so f^2 is not the identity.

A method to construct M is as follows. Consider a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

1. $\varphi^{(n)}(0) = 0$, $n = 0, 1, 2, \ldots$
2. φ and φ' are bounded
3. there exists a sequence $r_n \downarrow 0$ with $\varphi(r_n) = 0$ and $\varphi'(r_n) \neq 0$.

Next consider the Hamiltonian system in the cylinder $(\theta, r) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$,

$$\dot{r} = -H_\theta, \;\; \dot{\theta} = H_r, \;\; H(\theta, r) = \varphi(r)(1 - \sin\theta).$$

The circumference $r = r_n$ is invariant under the flow, it contains the equilibrium $\theta = \frac{\pi}{2}$, $r = r_n$ and a homoclinic orbit. The symplectic change of variables

$$x_1 = \sqrt{2r}\cos\theta, \;\; x_2 = \sqrt{2r}\sin\theta$$

transforms the above system in

$$\dot{x}_1 = -K_{x_2}, \;\; \dot{x}_2 = K_{x_1}, \;\; K(x_1, x_2) = \varphi(\frac{x_1^2 + x_2^2}{2})(1 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}}).$$

The function K is in $C^\infty(\mathbb{R}^2)$ with $K(0, 0) = 0$ and so the associated flow $\{\phi_t\}$ is globally defined. The boundedness of φ and φ' plays a role here. The map M can be defined as ϕ_t for any $t > 0$. Notice that $R_n = \sqrt{2r_n}$.

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