Complete orbits for twist maps on the plane: the case of small twist

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Abstract

In this article we consider twist maps that are non-periodic (and hence are defined on the plane rather than on the cylinder) and have small twist at infinity. Under natural assumptions the existence of infinitely many bounded orbits is established, and furthermore it is proved that unbounded orbits follow bounded orbits for long times. An application is given to the Fermi-Ulam ping-pong model with a non-periodic moving wall.

1 Introduction

Twist maps that are non-periodic in the 'angular' variable do frequently arise from mechanical problems where some forcing function is only supposed to be bounded rather than periodic (or, more generally, quasiperiodic or almost periodic). In this work we continue our program [4, 5] to investigate such maps from a general perspective and to prove that under certain natural hypotheses there are infinitely many complete and bounded orbits. While in [4, 5] twist maps were considered whose generating function $h = h(\theta, \theta_1)$ grows like a positive power, $h(\theta, \theta_1) \sim (\theta_1 - \theta)^\kappa$ for some $\kappa > 1$, we now turn our attention to the negative power case where $h(\theta, \theta_1) \sim (\theta_1 - \theta)^{-\kappa}$ for some $\kappa \geq 1$. One main motivation for this is provided by the so-called Fermi-Ulam ping-pong [2] (not to be confused with the Fermi-Pasta-Ulam lattice), where $\kappa = 1$. It describes the motion of a ball that is moving freely, and without experiencing gravity, between two vertical plates (the 'rackets'). On contact the particle is hit by the plates, and the issue is to decide whether or not the resulting motion is of globally bounded energy. From earlier work it is known that all solutions are bounded in energy for a periodically moving plate [6], provided that $p \in C^{4+\varepsilon}(\mathbb{R})$ for the periodic forcing function p; also see [10] and the references therein concerning the regularity assumption. The ping-pong model with a slowly moving wall on a finite but very large time interval is studied in [3].

Usually the argument for boundedness relies on the application of KAM methods to a suitable twist map. It is however not possible to use the same approach also for a general non-periodic p, since in that case the resulting twist map will not be periodic in the 'angular' variable. Taking advantage of a general result, proved as Theorem 2.3 below, we can nevertheless show in Section 3 that for a C^2-bounded forcing p there are infinitely many motions of bounded energy, thereby contributing to the solution of a problem posed by Dolgopyat in [1, Problem 4]. In addition to that more can be said about the unbounded orbits, due to the small twist at infinity. We will prove in Theorem 2.4 that, given $\varepsilon > 0$ and $T \in \mathbb{N}$, an unbounded orbit remains ε-close to one of the bounded orbits that we found for at least T iteration steps. This result shows that these bounded orbits play an important role for the global dynamics. An application to the ping-pong model is given in Section 3, where we also construct (in Theorem 3.4) a forcing function p such that an unbounded complete orbit indeed exists.

Note that here and henceforth we shall apply the following notational convention: an orbit $(\theta_{n+1}, r_{n+1}) = \Phi(\theta_n, r_n)$ of a map Φ is said to be bounded, if $\sup_{n\in\mathbb{Z}} |r_n| < \infty$; otherwise it will be called unbounded. In other words, boundedness will only refer to the r-component of the orbit, since the real variable θ will be non-periodic, and in general we will have $\theta_n \to \infty$ as $n \to \infty$ anyhow.

2 Existence of complete orbits with bounded action

First we are going to describe the setup. The real variable θ will be called an 'angle', although no periodic identification of the type $\theta \equiv \theta + 2\pi$ is assumed. Given a C^1-function $h = h(\theta, \theta')$ on a region $\Omega \subset \mathbb{R}^2$, we shall consider the second order difference equation

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0 \quad \text{for} \quad n \in \mathbb{Z}. \tag{2.1}$$

A second independent real variable r, called the action, will also be used. The above relations can be transformed into a first order system of the type

$$(\theta_{n+1}, r_{n+1}) = \Phi(\theta_n, r_n) \quad \text{for} \quad n \in \mathbb{Z}, \tag{2.2}$$

using the notion of a generating function.

Definition 2.1 *Given a map $\Phi = \Phi(\theta, r) : D \to \mathbb{R}^2$ defined on a domain $D \subset \mathbb{R}^2$, we call a C^1-function $h = h(\theta, \theta') : \Omega \to \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^2$ a generating function for Φ, if given four real numbers θ, θ', r, r',*

$$(\theta, r) \in D \quad \text{and} \quad (\theta', r') = \Phi(\theta, r) \quad \Longleftrightarrow \quad (\theta, \theta') \in \Omega \quad \text{and} \quad \begin{cases} r = \partial_1 h(\theta, \theta') \\ r' = -\partial_2 h(\theta, \theta') \end{cases}. \tag{2.3}$$

The following example shows that the domain Ω plays an important role in the above definition.

Example 2.2 Consider $D = \mathbb{R}\times]0, \infty[$, $\Omega_1 = \{(\theta, \theta') \in \mathbb{R}^2 : \theta < \theta'\}$, and moreover $\Omega_2 = \{(\theta, \theta') \in \mathbb{R}^2 : \theta' < \theta\}$. Define

$$\Phi_1(\theta, r) = \Big(\theta + \sqrt{\frac{2}{r}}, r\Big), \quad \Phi_2(\theta, r) = \Big(\theta - \sqrt{\frac{2}{r}}, r\Big), \quad \text{and} \quad h(\theta, \theta') = \frac{2}{\theta' - \theta}.$$

It can be checked that h on Ω_1 is a generating function for Φ_1 on D, and h on Ω_2 is a generating function for Φ_2 on D.

Our model generating function will be

$$h(\theta,\theta') = \frac{1}{(\theta'-\theta)^\kappa} \tag{2.4}$$

for some $\kappa \geq 1$, defined on $\Omega = \{(\theta,\theta') \in \mathbb{R}^2 : \delta < \theta' - \theta < \Delta\}$, where $\Delta > \delta > 0$. The associated map is

$$\Phi(\theta, r) = (\theta_1, r_1) = \Big(\theta + \Big(\frac{\kappa}{r}\Big)^{\frac{1}{\kappa+1}}, r\Big),$$

defined on $D = \{(\theta, r) : \theta \in \mathbb{R}, \frac{\kappa}{\Delta^{\kappa+1}} < r < \frac{\kappa}{\delta^{\kappa+1}}\}$. Observe that $\partial_r \theta_1$ is negative and vanishes as $r \to \infty$, i.e., Φ has small twist as $r \to \infty$. The orbits of Φ are complete (i.e., defined for all $n \in \mathbb{Z}$) and given by

$$\theta_{n+1} = \theta_n + \Big(\frac{\kappa}{r_0}\Big)^{\frac{1}{\kappa+1}} \quad \text{and} \quad r_n = r_0 \quad \text{for} \quad n \in \mathbb{Z}.$$

Note that all orbits are bounded, in the sense that their action variable is uniformly bounded. Our first result shows that some of the bounded orbits will persist, if we perturb the generating function. This perturbation will be small in the C^0-norm, but not necessarily in the C^1-norm. Since all the assumptions will be imposed on the function h, the result will be stated at the 'Lagrangian level', meaning that it is concerned with the second order difference equation (2.1).

Theorem 2.3 *Let $\Delta > \delta > 0$. Suppose that $h : \Omega = \{(\theta,\theta') \in \mathbb{R}^2 : \delta < \theta' - \theta < \Delta\} \to \mathbb{R}$ is C^1 and such that*

$$\underline{\alpha}(\theta'-\theta)^{-\kappa} \leq h(\theta,\theta') \leq \overline{\alpha}(\theta'-\theta)^{-\kappa}, \quad (\theta,\theta') \in \Omega, \tag{2.5}$$

for some constants $\overline{\alpha} \geq \underline{\alpha} > 0$ so that

$$\overline{\alpha} < \big(1 + 3 \cdot 2^{-(\kappa+3)}\big)\, \underline{\alpha}. \tag{2.6}$$

Then there is a constant $\sigma_{**} \geq 1$ (depending only on $\overline{\alpha}/\underline{\alpha}$) with the following property. If

$$\sigma_{**}\delta < \sigma_{**}^{-1}\Delta,$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ such that $|\theta_0^*| \leq \Delta$, $\delta < \theta_{n+1}^* - \theta_n^* < \Delta$ for $n \in \mathbb{Z}$, and (2.1) is satisfied.

Analogous results for generating functions of the form $h(\theta,\theta') \sim (\theta'-\theta)^\kappa$ for some $\kappa > 1$ were obtained in [4, Thm. 2.1] and [5, Thm. 3.1]. The proof, of variational nature, is similar to the earlier proofs. In the appendix (see Section 4.1) we shall indicate the needed modifications.

For the above model example (2.4) all orbits are bounded, but this is not always the case under the assumptions of the previous theorem. Below in Theorem 3.4 we shall construct a smooth forcing function for the Fermi-Ulam ping-pong model such that there is an unbounded complete orbit. The next result shows that this type of orbit must remain close to a bounded orbit for very large periods of time. Now we have to impose conditions on both h and Φ, so that we state the result at the 'Hamiltonian level', i.e., for (2.2).

Theorem 2.4 *Suppose that $\Phi : (\theta, r) \mapsto (\theta', r')$ is defined and Lipschitz continuous on a domain $D \subset \mathbb{R}^2$. Assume*

$$h : \quad \{(\theta,\theta') \in \mathbb{R}^2 : 0 < \theta' - \theta < \Delta_*\} \to \mathbb{R}$$

to be a generating function for Φ *so that*

$$\underline{\alpha}(\theta' - \theta)^{-\kappa} \le h(\theta, \theta') \le \overline{\alpha}(\theta' - \theta)^{-\kappa}, \quad 0 < \theta' - \theta < \Delta_*, \tag{2.7}$$

where $\kappa \ge 1$*, and* $\overline{\alpha} \ge \underline{\alpha} > 0$ *satisfy* $\overline{\alpha} < (1 + 3 \cdot 2^{-(\kappa+3)})\, \underline{\alpha}$*. Next suppose that there are constants* $a, A, B, K > 0$ *such that*

$$\begin{aligned} a(\theta' - \theta)^{-(\kappa+1)} \le \partial_1 h(\theta, \theta') &\le A(\theta' - \theta)^{-(\kappa+1)}, &(2.8)\\ 0 \le -\partial_2 h(\theta, \theta') &\le B(\theta' - \theta)^{-(\kappa+1)}, &(2.9)\\ |\partial_1 h(\theta, \theta') + \partial_2 h(\theta, \theta')| &\le K(\theta' - \theta), &(2.10)\end{aligned}$$

for all $0 < \theta' - \theta < \Delta_*$*.*

Under the hypotheses stated so far there is a countable family $\mathcal{B}$ *of bounded orbits of* Φ *with the following property. Take* $\varepsilon > 0$*,* $T \in \mathbb{N}$*, and an unbounded forward orbit* $(\theta_n, r_n)_{n \in \mathbb{N}_0}$ *of* Φ*. Then there exists* $(\Theta_n, R_n)_{n \in \mathbb{Z}} \in \mathcal{B}$ *and some* $N \in \mathbb{N}_0$ *such that*

$$|\theta_n - \Theta_n| + |r_n - R_n| \le \varepsilon \quad \textit{for} \quad N \le n \le N + T.$$

Assumptions (2.8) and (2.9) indicate that h is close to the model example $(\theta' - \theta)^{-\kappa}$ also in the C^1-sense. The hypothesis (2.10) is of a different nature. When it is interpreted in terms of the action by use of the generating function relations, it essentially says that r is an adiabatic invariant. By this we mean that $\Delta r = r' - r \to 0$ as $r \to \infty$. More precisely,

$$r' = r + \mathcal{O}(r^{-\frac{1}{\kappa+1}}) \quad \text{as} \quad r \to \infty.$$

The last expansion implies that unbounded orbits cannot grow very fast in the r-variable. For instance, if $\kappa = 1$ then any complete orbit $(\theta_n, r_n)_{n \in \mathbb{Z}}$ satisfies $r_{n+1} \le r_n + C r_n^{-1/2}$, which leads to $r_n = \mathcal{O}(n^{2/3})$ at most.

Next observe that for Theorem 2.4 it is only assumed that $h \in C^1$. If we sharpen this hypothesis to $h \in C^2$, then it is possible to present the previous result from the Lagrangian point of view also, since then the required Lipschitz continuity of Φ can be expressed in terms of h. For, implicit differentiation of (2.3) yields

$$D\Phi = \frac{1}{\partial_{12}^2 h} \begin{pmatrix} -\partial_{11}^2 h & 1 \\ Mh & -\partial_{22} h \end{pmatrix}, \tag{2.11}$$

where $Mh = (\partial_{11}^2 h)(\partial_{22}^2 h) - (\partial_{12}^2 h)^2$ denotes the Monge-Ampère operator. Thus to show that Φ is Lipschitz continuous one can check that the matrix $D\Phi$ is bounded. We will return to this observation later in the proof of Theorem 3.3; see Section 3.4 below. Some connections between the Monge-Ampère operator and generating functions can be found in [8].

Also note that the assumptions of Theorem 2.4 impose some restrictions on the domain D. From (2.3) and (2.8) it follows that

$$\mathbb{R} \times [r^*, \infty[\subset D \subset \mathbb{R} \times [r_*, \infty[, \tag{2.12}$$

where $r^* = A(\Delta_*/2)^{-(\kappa+1)}$ and $r_* = a(\Delta_*)^{-(\kappa+1)}$. Before we go on to the proof of Theorem 2.4 we state two auxiliary results whose verification is postponed to the appendix; see Section 4.2.

Lemma 2.5 *If* $(\theta, r) \in D$ *and* $(\theta', r') = \Phi(\theta, r)$*, then* $(a/B) r' \le r$*.*

Lemma 2.6 *There is* $\gamma_1 \in]0,1[$ *such that for every* $\Delta \in]0, \Delta_*[$ *there exists an orbit* $(\theta_n^\Delta, r_n^\Delta)_{n\in\mathbb{Z}}$ *of* Φ *so that*

$$\gamma_1\Delta \leq \theta_{n+1}^\Delta - \theta_n^\Delta \leq \Delta \quad \textit{and} \quad a\Delta^{-(\kappa+1)} \leq r_n^\Delta \leq A\gamma_1^{-(\kappa+1)}\Delta^{-(\kappa+1)}$$

holds for $n \in \mathbb{Z}$.

Proof of Theorem 2.4: Fix $\varepsilon > 0$, $T \in \mathbb{N}$, and a forward orbit $(\phi_n, \rho_n)_{n\in\mathbb{N}_0}$ of Φ such that $\sup_{n\in\mathbb{N}_0} \rho_n = \infty$, i.e., the original orbit is unbounded. Next observe that $\rho_n \geq r_* > 0$ by (2.12). For every $\Delta \in]0, \Delta_*[$ we now apply Lemma 2.6 to generate $(\theta_n^\Delta, r_n^\Delta)_{n\in\mathbb{Z}}$, and we define the sets

$$\begin{aligned}
\Gamma_+^\Delta &= \{(\theta, r) : \exists\, n \in \mathbb{Z} \text{ such that } |\theta - \theta_n^\Delta| \leq \Delta/2 \text{ and } r \geq r_n^\Delta\}, \\
\Gamma_-^\Delta &= \{(\theta, r) : \exists\, n \in \mathbb{Z} \text{ such that } |\theta - \theta_n^\Delta| \leq \Delta/2 \text{ and } r \leq r_n^\Delta\}.
\end{aligned}$$

Denote

$$\Delta_{**} = \Big(\frac{a}{\rho_0}\Big)^{1/(\kappa+1)}.$$

If $\Delta \in]0, \Delta_{**}[$, then $\rho_0 < a\Delta^{-(\kappa+1)} \leq r_n^\Delta$ for $n \in \mathbb{Z}$ by Lemma 2.6. As a consequence, we must have $(\phi_0, \rho_0) \in \Gamma_-^\Delta$; this is due to the fact that $\gamma\Delta \leq \theta_{n+1}^\Delta - \theta_n^\Delta \leq \Delta$, i.e., the points $(\theta_n^\Delta)_{n\in\mathbb{Z}}$ partition the real line into pieces of length $\leq \Delta$, whence necessarily $|\phi_0 - \theta_n^\Delta| \leq \Delta/2$ for some partition point θ_n^Δ. On the other hand, $\sup_{n\in\mathbb{Z}} r_n^\Delta < \infty$. Accordingly, if $m \in \mathbb{N}_0$ is such that $\rho_m > \sup_{n\in\mathbb{Z}} r_n^\Delta$, then $(\phi_m, \rho_m) \in \Gamma_+^\Delta$ holds. Hence the number

$$N = \max\{n \in \mathbb{N}_0 : \ \forall\, 0 \leq m \leq n \text{ we have } (\phi_m, \rho_m) \in \Gamma_-^\Delta\}$$

is well-defined and finite; it is the longest time such that the orbit starting at (ϕ_0, ρ_0) remains in Γ_-^Δ. Also note that N depends only on Δ and the given orbit. By definition in particular,

$$(\phi_N, \rho_N) \in \Gamma_-^\Delta, \quad (\phi_{N+1}, \rho_{N+1}) \in \Gamma_+^\Delta,$$

which means that for some $t, s \in \mathbb{Z}$,

$$|\phi_N - \theta_t^\Delta| \leq \frac{1}{2}\Delta, \quad \rho_N \leq r_t^\Delta, \quad |\phi_{N+1} - \theta_s^\Delta| \leq \frac{1}{2}\Delta, \quad \rho_{N+1} \geq r_s^\Delta. \tag{2.13}$$

These relations in conjunction with Lemmas 2.5, 2.6 and (2.8) imply that

$$\frac{a^2}{B}\Delta^{-(\kappa+1)} \leq \frac{a}{B} r_s^\Delta \leq \frac{a}{B}\rho_{N+1} \leq \rho_N = \partial_1 h(\phi_N, \phi_{N+1}) \leq A(\phi_{N+1} - \phi_N)^{-(\kappa+1)},$$

and therefore

$$\phi_{N+1} - \phi_N \leq \Big(\frac{AB}{a^2}\Big)^{1/(\kappa+1)}\Delta. \tag{2.14}$$

Hence by (2.13),

$$\begin{aligned}
|\theta_s^\Delta - \theta_t^\Delta| &\leq |\theta_s^\Delta - \phi_{N+1}| + |\phi_{N+1} - \phi_N| + |\phi_N - \theta_t^\Delta| \\
&\leq \frac{1}{2}\Delta + \Big(\frac{AB}{a^2}\Big)^{1/(\kappa+1)}\Delta + \frac{1}{2}\Delta = \gamma_2\Delta
\end{aligned} \tag{2.15}$$

for

$$\gamma_2 = 1 + \Big(\frac{AB}{a^2}\Big)^{1/(\kappa+1)}.$$

Next observe that if $(\theta', r') = \Phi(\theta, r)$, then

$$|r' - r| = |\partial_2 h(\theta, \theta') + \partial_1 h(\theta, \theta')| \le K(\theta' - \theta) \tag{2.16}$$

due to (2.10). Thus if we assume that for instance $t \le s$, then (2.16), (2.15), and (2.14) yield

$$\begin{aligned} |r_t^\Delta - \rho_N| &= r_t^\Delta - \rho_N = (r_t^\Delta - r_s^\Delta) + (r_s^\Delta - \rho_{N+1}) + (\rho_{N+1} - \rho_N) \\ &\le |r_s^\Delta - r_t^\Delta| + |\rho_{N+1} - \rho_N| \\ &\le \sum_{n=t}^{s-1} |r_{n+1}^\Delta - r_n^\Delta| + |\rho_{N+1} - \rho_N| \\ &\le K \sum_{n=t}^{s-1} (\theta_{n+1}^\Delta - \theta_n^\Delta) + K(\phi_{N+1} - \phi_N) \\ &= K(\theta_s^\Delta - \theta_t^\Delta) + K(\phi_{N+1} - \phi_N) \le K\gamma_2\Delta + K\Big(\frac{AB}{a^2}\Big)^{1/(\kappa+1)}\Delta. \end{aligned}$$

This may be summarize as follows. If $\Delta \in]0, \Delta_{**}[$, then there are $N = N(\Delta) \in \mathbb{N}_0$ and $t = t(\Delta) \in \mathbb{Z}$, both depending also on the given orbit, such that

$$|\theta_t^\Delta - \phi_N| + |r_t^\Delta - \rho_N| \le \gamma_3 \Delta,$$

where explicitly

$$\gamma_3 = \frac{1}{2} + K\Big[1 + 2\Big(\frac{AB}{a^2}\Big)^{1/(\kappa+1)}\Big]. \tag{2.17}$$

Now we take

$$\Delta = \min\Big\{\frac{1}{2}\,\Delta_{**}, \frac{\varepsilon}{L^T\gamma_3}\Big\} = \min\Big\{\frac{1}{2}\Big(\frac{a}{\rho_0}\Big)^{1/(\kappa+1)}, \frac{\varepsilon}{L^T\gamma_3}\Big\}$$

and consider the bounded orbit

$$(\Theta_n, R_n)_{n\in\mathbb{Z}} = (\theta_{n-N+t}^\Delta, r_{n-N+t}^\Delta)_{n\in\mathbb{Z}}$$

of Φ. If $N \le n \le N + T$, then

$$\begin{aligned} |\Theta_n - \phi_n| + |R_n - \rho_n| &= |(\theta_{n-N+t}^\Delta, r_{n-N+t}^\Delta) - (\phi_n, \rho_n)| = |\Phi^{n-N}(\theta_t^\Delta, r_t^\Delta) - \Phi^{n-N}(\phi_N, \rho_N)| \\ &\le L^{n-N}\Big(|\theta_t^\Delta - \phi_N| + |r_t^\Delta - \rho_N|\Big) \le L^T\gamma_3\Delta \le \varepsilon, \end{aligned}$$

where $L \ge 1$ is the Lipschitz constant of Φ w.r. to the norm $|(\theta, r)| = |\theta| + |r|$. Hence we can define

$$\mathcal{B} = \Big\{(\theta_{n+k}^\Delta, r_{n+k}^\Delta)_{n\in\mathbb{Z}} : k \in \mathbb{N} \text{ and } \Delta = 1/N \text{ for some } N \in \mathbb{N} \text{ with } N \ge \Delta_{**}^{-1}\Big\}$$

as the desired countable family of orbits with bounded r-component. □

Remark 2.7 For easier reference in future work we record the explicit bound

$$a\Delta^{-(\kappa+1)} \le R_n \le A\gamma_1^{-(\kappa+1)}\Delta^{-(\kappa+1)}, \quad n \in \mathbb{Z},$$

for $\gamma_1 \in]0, 1[$ as in (4.2) below and

$$\Delta = \min\Big\{\frac{1}{2}\Big(\frac{a}{\rho_0}\Big)^{1/(\kappa+1)}, \frac{\varepsilon}{L^T\gamma_3}\Big\},$$

with the Lipschitz constant L as above and γ_3 from (2.17).

3 The non-periodic Fermi-Ulam ping-pong model

3.1 Bounded and unbounded motions

We consider the Fermi-Ulam ping-pong model that is built up from two vertical plates, the left one being fixed at $x = 0$, whereas the right plate serves as a kind of racket and is allowed to move according to some law $x = p(t)$ for a prescribed function p such that $0 < a \leq p(t) \leq b$. The two plates alternately hit a particle that impacts completely elastic and experiences no gravity. The particle is furthermore assumed to travel without being accelerated.

For the mathematical analysis of the model we study the map

$$(t_0, v_0) \mapsto (t_1, v_1)$$

which sends a time $t_0 \in \mathbb{R}$ of impact to the left plate $x = 0$ and the corresponding velocity $v_0 > 0$ immediately after the impact to their successors t_1 and v_1 describing in the same way the subsequent impact to $x = 0$. Denote $\tilde{t} \in]t_0, t_1[$ the time of the particle's impact to the moving plate. Then the relation

$$(\tilde{t} - t_0)v_0 = p(\tilde{t}) \tag{3.1}$$

gives the distance that the particle has to travel to hit the moving plate; it defines $\tilde{t} = \tilde{t}(t_0, v_0)$ implicitly. The elastic impact condition requires $(dz/dt)(\tilde{t}+) = -(dz/dt)(\tilde{t}-)$ for $z(t) = x(t) - p(t)$. The arrival velocity $\tilde{v}$ at the racket is v_0, so that

$$\dot{x}(\tilde{t}+) = -v_0 + 2\dot{p}(\tilde{t}) \tag{3.2}$$

is obtained for the escape velocity. The left plate at $x = 0$ is hit with this velocity at time $t = t_1$ and the distance travelled is

$$(t_1 - \tilde{t})\dot{x}(\tilde{t}+) = -p(\tilde{t}).$$

The velocity after the elastic impact to $x = 0$ is

$$v_1 = -\dot{x}(\tilde{t}+) = v_0 - 2\dot{p}(\tilde{t}).$$

To summarize, we consider the map

$$(t_0, v_0) \mapsto (t_1, v_1), \quad t_1 = \tilde{t} + \frac{p(\tilde{t})}{v_1}, \quad v_1 = v_0 - 2\dot{p}(\tilde{t}), \tag{3.3}$$

where $\tilde{t} = \tilde{t}(t_0, v_0)$ is to be obtained from (3.1).

Remark 3.1 In order to have this map well-defined, it must be ensured that $\tilde{t} > t_0$ is uniquely defined by (3.1) and that there is no further impact to the moving plate before t_1, i.e.,

$$x(t) = p(\tilde{t}) + (t - \tilde{t})\dot{x}(\tilde{t}+) < p(t), \quad t \in]\tilde{t}, t_1[,$$

is needed. Both conditions can be satisfied, if

$$v_0 > \|\dot{p}\|_\infty \quad \text{and} \quad |\dot{x}(\tilde{t}+)| > \|\dot{p}\|_\infty$$

holds. Thus by (3.2) it is sufficient for that to take $v_0 > 3\|\dot{p}\|_\infty$. Thus in principle the map has to be restricted to such v_0, as will be done in the later application; we will return to this point in Remark 3.10 below.

The main results about the ping-pong model are as follows.

Theorem 3.2 *Let p satisfy*

$$p \in C^2(\mathbb{R}), \quad 0 < a \le p(t) \le b \quad (t \in \mathbb{R}), \quad \|\dot{p}\|_\infty + \|\ddot{p}\|_\infty < \infty. \tag{3.4}$$

Then there are infinitely many solutions to the Fermi-Ulam ping-pong model which have bounded velocities. More precisely, for each $j \in \mathbb{N}$ there exists a complete orbit $(t_n^{(j)}, v_n^{(j)})_{n\in\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}} v_n^{(j)} < \infty$ and $\lim_{j\to\infty}(\inf_{n\in\mathbb{Z}} v_n^{(j)}) = \infty$.

Since the particle is confined between the walls, it follows that

$$\sup_{n\in\mathbb{Z}} (t_{n+1}^{(j)} - t_n^{(j)}) < \infty \quad \text{and} \quad \lim_{j\to\infty} [\sup_{n\in\mathbb{Z}} (t_{n+1}^{(j)} - t_n^{(j)})] = 0.$$

Also the shifted orbits $(t_{n+k}^{(j)}, v_{n+k}^{(j)})_{n\in\mathbb{Z}}$ with $k \in \mathbb{Z}$ have the same properties. The family of solutions depending on the integer parameters j and k will be denoted by $\mathcal{B}$. If the hypotheses on p are sharpened, then it can be shown additionally that, in an appropriate system of coordinates, every unbounded orbit has to stay close to one of these bounded orbits for long times. This system of coordinates is related to the fact that the quantity $W = \frac{1}{2}p(t)^2v^2$ is an adiabatic invariant. Indeed, it will turn out that

$$\Delta W = \mathcal{O}(W^{-1/2}) \quad \text{as} \quad W \to \infty.$$

Theorem 3.3 *Let p satisfy*

$$p \in C^3(\mathbb{R}), \quad 0 < a \le p(t) \le b \quad (t \in \mathbb{R}), \quad \|\dot{p}\|_\infty + \|\ddot{p}\|_\infty + \|\dddot{p}\|_\infty < \infty, \tag{3.5}$$

and let $(t_n, v_n)_{n\in\mathbb{N}_0}$ be an unbounded forward orbit of the Fermi-Ulam ping-pong model. Let $\varepsilon > 0$ and $T \in \mathbb{N}$ be fixed. Then for one of the bounded orbits from the family $\mathcal{B}$, denoted by $(t_n^*, v_n^*)_{n\in\mathbb{Z}}$, and some $N \in \mathbb{N}_0$ we have

$$|t_n - t_n^*| + |p(t_n)^2v_n^2 - p(t_n^*)^2(v_n^*)^2| \le \varepsilon \quad \textit{for} \quad N \le n \le N + T.$$

On the basis of Theorems 3.2 and 3.3, it is natural to raise the question of the existence of unbounded orbits for the Fermi-Ulam ping-pong model. Zharnitsky [10] found an interesting example of an unbounded orbit for some periodic forcing $p \in C(\mathbb{R})$. In the next result we shall construct a function $p \in C^\infty(\mathbb{R})$ such that the associated Fermi-Ulam model supports at least one complete and unbounded orbit. Due to the work [6] this p cannot be periodic.

Theorem 3.4 *Let $0 < a < b$, $M > 0$ and $m \in \mathbb{N}$. Then there exists a function $p \in C^\infty(\mathbb{R})$ satisfying*

$$a \le p(t) \le b \quad (t \in \mathbb{R}), \quad \|\dot{p}\|_\infty + \|\ddot{p}\|_\infty + \ldots + \|p^{(m)}\|_\infty \le M, \tag{3.6}$$

and such that there is at least one complete and unbounded orbit of the Fermi-Ulam ping-pong model.

The proofs of Theorems 3.2 and 3.3 will be obtained by an application of our previous results and will be given later. The proof of Theorem 3.4 is of more elementary nature so that we present it directly. To construct the function $p = p(t)$, we shall take the point of view of a player who moves the racket $x = p(t)$. The racket is made to oscillate between $x = a$ and $x = b$. When going from a to b, it will be blocked at the impact time, meaning that $\dot{p}(\tilde{t}_n) = 0$; recall that $\tilde{t}_n$ is the time of the n'th impact to the wall. Then, when going back from b to a, the ball will be hit with an impulse, so that $\dot{p}(t) < 0$ shortly after $t = \tilde{t}_n$. We shall make this procedure rigorous by use of the following two lemmas, whose proof is given in the appendix; see Sections 4.3 and 4.4 below.

Lemma 3.5 *Let $0 < a < b$, $M > 0$ and $m \in \mathbb{N}$. For every (t_0, v_0) with $v_0 > 0$ large enough there exists a function $p_+ \in C^\infty(\mathbb{R})$ satisfying (3.6) and such that for the complete orbit $(t_n, v_n)_{n\in\mathbb{Z}}$ for $p = p_+$ it holds that $t_n \to \pm\infty$ for $n \to \pm\infty$ and $v_n = v_0$ for all $n \in \mathbb{Z}$. Furthermore, for the derivatives*

$$p_+^{(k)}(t_n) = 0 = p_+^{(k)}(\tilde{t}_n) \quad \textit{for} \quad k \geq 1 \quad \textit{and} \quad n \in \mathbb{Z}.$$

Finally, for some integer $N_+ \geq 1$,

$$p_+(t) = a \ \ (t \leq t_0) \quad \textit{and} \quad p_+(t) = b \ \ (t \geq t_{N_+}).$$

Lemma 3.6 *Let $0 < a < b$, $M > 0$, and $m \in \mathbb{N}$. Given (t_0, v_0) with $v_0 > 0$ large enough, there exists a function $p_- \in C^\infty(\mathbb{R})$ satisfying (3.6) and*

$$\dot{p}_-(t) \leq 0 \quad \textit{for} \quad t \in \mathbb{R} \tag{3.7}$$

such that for the complete orbit $(t_n, v_n)_{n\in\mathbb{Z}}$ for $p = p_-$ we have $v_{n+1} \geq v_n$ and

$$\max_{n\in\{0,\dots,N_- -1\}} |\dot{p}_-(\tilde{t}_n)| \geq C$$

for some integer $N_- \geq 1$ as well as

$$p_-(t) = b \ \ (t \leq t_0) \quad \textit{and} \quad p_-(t) = a \ \ (t \geq t_{N_-}).$$

Here $C > 0$ is a suitable constant that only depends on a, b, M, and m.

Now we are ready for the

Proof of Theorem 3.4: First we construct the function p for $t \geq 0$. To this end we apply Lemma 3.5 for $t_0 = 0$ and $v_0 > 0$ large enough. Thereby we obtain a function $p_+ \in C^\infty(\mathbb{R})$ and $N_+ \geq 1$ with the properties as described in Lemma 3.5. Next select $N \in \mathbb{N}$ so that $N \geq N_+$ and $t_N \geq 1$ holds; this is possible in view of $t_n \to \infty$ as $n \to \infty$. Then we define $p(t) = p_+(t)$ for $t \in [0, t_N]$ and observe that $v_n = v_0$ for the associated velocities and moreover $\dot{p}(\tilde{t}_n) = 0$. The next step is to apply Lemma 3.6 to the initial condition (t_N, v_0) and to put $p(t) = p_-(t)$ for $t \in [t_N, t_{N_-}]$. For the associated velocities we find $v_{j+1} \geq v_j \geq v_0$, and furthermore we have $\dot{p}(\tilde{t}_j) \leq 0$. Also there is $j_1 \in \{N, \dots, N_- - 1\}$ so that $|\dot{p}(\tilde{t}_{j_1})| \geq C$ for $C = C(a, b, M, m)$. Thus, since the velocities are increasing and C does only depend on a, b, M, and m, we see that this procedure can be repeated infinitely many times to construct a function $p \in C^\infty([0, \infty[)$ satisfying (3.6) which oscillates between a and b and which has a forward orbit $(t_n, v_n)_{n\in\mathbb{N}_0}$ so that $\dot{p}(\tilde{t}_n) \leq 0$ for $n \in \mathbb{N}_0$ and furthermore $|\dot{p}(\tilde{t}_{n_k})| \geq C$ for infinitely many $n_k \in \mathbb{N}_0$. We claim that $(v_n)_{n\in\mathbb{N}_0}$ is unbounded. For, otherwise we would have $v_n \to v_\infty < \infty$ as $n \to \infty$, since the sequence is increasing. But

$$v_{n+1} = v_n - 2\dot{p}(\tilde{t}_n) = v_n + 2|\dot{p}(\tilde{t}_n)|$$

then leads to

$$v_\infty = v_0 + 2\sum_{j=0}^{\infty} |\dot{p}(\tilde{t}_j)|,$$

which however contradicts the fact that $|\dot{p}(\tilde{t}_{n_k})| \geq C$ for infinitely many $n_k \in \mathbb{N}_0$. Finally we extend p to the whole real line as an even function. Then the unbounded orbit becomes complete and symmetric, i.e., we have $t_{-n} = -t_n$ and $v_{-n} = v_n$. □

3.2 The generating function

The map $(t_0, v_0) \mapsto (t_1, v_1)$ from (3.3) is not symplectic, as a short computations shows that $v_1\, dt_1 \wedge dv_1 = v_0\, dt_0 \wedge dv_0$. Since we intend to apply the general results from Section 2, which concern symplectic maps, we need to reformulate the model in terms of time t and energy $E = \frac{1}{2}\, v^2$. Thus we rewrite (3.3) in terms of the energies E_0 and E_1 given by $v_0 = \sqrt{2E_0}$ and $v_1 = \sqrt{2E_1}$ as

$$\begin{aligned} &\Psi : (t_0, E_0) \mapsto (t_1, E_1), \\ &\qquad t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2E_1}}, \quad E_1 = E_0 - 2\sqrt{2E_0}\, \dot{p}(\tilde{t}) + 2\dot{p}(\tilde{t})^2 = (\sqrt{E_0} - \sqrt{2}\, \dot{p}(\tilde{t}))^2, \end{aligned} \tag{3.8}$$

where $\tilde{t} = \tilde{t}(t_0, E_0)$ is implicitly defined by the relation

$$\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}. \tag{3.9}$$

The map Ψ from (3.8) is exact symplectic and has a surprisingly simple (semi-)explicit generating function; also see [6, Section 4.1]. First it will be derived by a non-rigorous physics-style argument. Thereafter we are going to verify by direct computation that the function which we found has the desired properties.

Motivated by the general theory [7, p. 84] we observed in [5] that if L is a smooth Lagrangian of the type $L(t, x, \dot{x}) = \frac{1}{2}\, \dot{x}^2 - U(t, x)$ with $U(t, 0) = 0$, then the restricted action

$$H(t_0, t_1) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt$$

is a generating function for the successor map $\Psi : (t_0, E_0) \mapsto (t_1, E_1)$. Here we are assuming that $t_1 - t_0 > 0$ is small enough and $x = x(t; t_0, t_1)$ is the unique solution to the Dirichlet problem

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = 0, \quad x(t_0) = x(t_1) = 0, \quad x(t) > 0 \quad \text{on} \quad]t_0, t_1[,$$

where $E_0 = \dot{x}(t_0+)^2/2$ as well as $E_1 = \dot{x}(t_1-)^2/2$. In principle, this fact does not apply to the ping-pong problem, since the Lagrangian is not smooth. In this case it is given by

$$L(t, x, \dot{x}) = \frac{1}{2}\, \dot{x}^2 - U(t, x), \quad U(t, x) = \left\{ \begin{array}{ccc} +\infty & : & x > p(t) \\ 0 & : & 0 \le x \le p(t) \\ +\infty & : & x < 0 \end{array} \right. .$$

In what follows, for calculating the generating function in our singular case we simply pretend that this connection between the action and the successor map will continue to be valid. First we rewrite the Dirichlet problem as

$$\ddot{x} = 0, \quad x(t_0) = x(t_1) = 0, \quad x(\hat{t}) = p(\hat{t}), \quad \dot{x}(\hat{t}+) = -\dot{x}(\hat{t}-) + 2\dot{p}(\hat{t});$$

here it is understood that $\hat{t} \in]t_0, t_1[$ has to be found such that the two last conditions are satisfied. Then $x(t) = \alpha(t - t_0)$ for $t \in [t_0, \hat{t}]$ yields $\dot{x}(\hat{t}-) = \alpha$, and hence

$$x(t) = (-\alpha + 2\dot{p}(\hat{t}))(t - \hat{t}) + \alpha(\hat{t} - t_0), \quad t \in [\hat{t}, t_1],$$

by the continuity of x at $t = \hat{t}$. The condition $x(t_1) = 0$ becomes

$$\alpha = 2\Big(\frac{t_1 - \hat{t}}{t_0 + t_1 - 2\hat{t}}\Big)\dot{p}(\hat{t}),$$

and furthermore we have

$$\alpha(\hat{t} - t_0) = x(\hat{t}) = p(\hat{t}). \tag{3.10}$$

Eliminating α, it follows that $\hat{t} = \hat{t}(t_0, t_1) \in]t_0, t_1[$ has to solve

$$(t_0 + t_1 - 2\hat{t})\, p(\hat{t}) = 2(t_1 - \hat{t})(\hat{t} - t_0)\, \dot{p}(\hat{t}). \tag{3.11}$$

Then for the action

$$H(t_0, t_1) = \int_{t_0}^{t_1} \frac{1}{2}\, \dot{x}(t; t_0, t_1)^2\, dt$$

the relations (3.10) and (3.11) imply that

$$\begin{aligned} H(t_0, t_1) &= \frac{\alpha^2}{2}\, (\hat{t} - t_0) + \frac{1}{2}\, (-\alpha + 2\dot{p}(\hat{t}))^2\, (t_1 - \hat{t}) \\ &= \frac{1}{2}\, \frac{p(\hat{t})^2}{(\hat{t} - t_0)} + \frac{1}{2}\Big(- \frac{p(\hat{t})}{\hat{t} - t_0} + 2\dot{p}(\hat{t})\Big)^2\, (t_1 - \hat{t}) \\ &= \frac{1}{2}\, \frac{p(\hat{t})^2}{(\hat{t} - t_0)} + \frac{1}{2}\, \frac{p(\hat{t})^2}{(t_1 - \hat{t})}. \end{aligned} \tag{3.12}$$

It will be checked rigorously that this H is a generating function for Ψ from (3.8), in the sense of Definition 2.1. First it has to be shown that $\hat{t}$ is a well-defined function in the variables t_0 and t_1. Denote

$$\tilde{\Delta}_* = \min\Big\{1, \frac{a}{4(\|\dot{p}\|_\infty + \|\ddot{p}\|_\infty + 1)}\Big\} > 0.$$

Lemma 3.7 *Let (3.4) be satisfied. If* $0 < t_1 - t_0 < \tilde{\Delta}_*$*, then the relation*

$$(t_0 + t_1 - 2\hat{t})\, p(\hat{t}) = 2(t_1 - \hat{t})(\hat{t} - t_0)\, \dot{p}(\hat{t}) \tag{3.13}$$

has a unique solution $\hat{t} = \hat{t}(t_0, t_1) \in]t_0, t_1[$*. Furthermore,* $\hat{t}$ *is a* C^1*-function and*

$$|\hat{t} - t_m| \;\le\; \frac{1}{a}\, \|\dot{p}\|_\infty (t_1 - t_0)^2, \tag{3.14}$$

$$\Big|\partial_{t_0}\hat{t} - \frac{1}{2}\Big| + \Big|\partial_{t_1}\hat{t} - \frac{1}{2}\Big| \;\le\; \frac{4}{11a}\, (8\|\dot{p}\|_\infty + a + 1)\, (t_1 - t_0), \tag{3.15}$$

where $t_m = (t_0 + t_1)/2$ *is the midpoint.*

The proof of Lemma 3.7 is given in the appendix, Section 4.5. Note that by the lemma H from (3.12) is defined on

$$\tilde{\Omega} = \{(t_0, t_1) : 0 < t_1 - t_0 < \tilde{\Delta}_*\}.$$

Let $0 < \Delta_* \le \frac{a}{10b}\, \tilde{\Delta}_* < \tilde{\Delta}_*$ and put

$$\begin{aligned} \Omega &= \{(t_0, t_1) : 0 < t_1 - t_0 < \Delta_*\} \quad \text{and} \\ D &= \{(t_0, E_0) : E_0 = \partial_{t_0} H(t_0, t_1) \text{ for some } (t_0, t_1) \in \Omega\}. \end{aligned} \tag{3.16}$$

Lemma 3.8 *Let (3.4) be satisfied. Then the set D from (3.16) is a domain. A generating function H on Ω for the map Ψ from (3.8) on D is given by*

$$H(t_0,t_1)=\frac{1}{2}\,p(\hat{t})^2\Big(\frac{1}{\hat{t}-t_0}+\frac{1}{t_1-\hat{t}}\Big),$$

where $\hat{t}=\hat{t}(t_0,t_1)\in]t_0,t_1[$ is from Lemma 3.7. The generating function has the expansions

$$H(t_0,t_1)=p(t_m)^2\,\frac{2}{t_1-t_0}+\mathcal{O}(1) \tag{3.17}$$

and

$$\partial_{t_0}H(t_0,t_1)=\frac{2p(t_m)^2}{(t_1-t_0)^2}+\mathcal{O}((t_1-t_0)^{-1}),\quad \partial_{t_1}H(t_0,t_1)=-\frac{2p(t_m)^2}{(t_1-t_0)^2}+\mathcal{O}((t_1-t_0)^{-1}), \tag{3.18}$$

for $t_1-t_0>0$ small. Furthermore,

$$\partial^2_{t_0t_0}H,\ \partial^2_{t_0t_1}H,\ \partial^2_{t_1t_1}H=\mathcal{O}((t_1-t_0)^{-3})\quad\text{and}\quad -\partial^2_{t_0t_1}H(t_0,t_1)\ge\frac{c}{(t_1-t_0)^3} \tag{3.19}$$

for some $c>0$ and $t_1-t_0>0$ small.

The proof of this result is somewhat technical and given in a further appendix; see Section 4.6.

We can now try to apply the results of Section 2 to the generating function H. This seemingly natural approach has the drawback that the factor $p(\hat{t})^2$ appears in H. Then in order to satisfy (2.6) from Theorem 2.3 for $\kappa=1$, it would lead to an additional restriction on the size of b/a; more precisely, $b/a<\sqrt{19}/4$ is sufficient. With respect to Theorem 2.4, it cannot be applied in the variables $\theta=t$ and $r=E$, since (3.8) implies that

$$\Delta(E)=\mathcal{O}(E^{1/2}),$$

and so the action variable is not an adiabatic invariant in this case. To remedy these defects, we introduce a further change of variables that has the effect of stopping the moving plate.

3.3 The change of variables

Define

$$\tau(t)=\int_0^t\frac{ds}{p(s)^2},\qquad t\in\mathbb{R},$$

together with the inverse function $\tau\mapsto t(\tau)$ which satisfies $\tau=\int_0^{t(\tau)}\frac{ds}{p(s)^2}$. In particular, we then have $t'(\tau)=p(t(\tau))^2$. This transformation was already used in [9, 10]. Next consider the symplectic map

$$\Gamma:\quad (t,E)\mapsto(\tau,W),\quad\text{where}\quad \tau=\tau(t),\ W=p(t)^2E,$$

along with

$$\Phi=\Gamma\circ\Psi\circ\Gamma^{-1}:\quad(\tau_0,W_0)\mapsto(t_0,E_0)\mapsto(t_1,E_1)\mapsto(\tau_1,W_1), \tag{3.20}$$

which is symplectic also. Let $\delta_*=b^{-2}\Delta_*$,

$$\begin{aligned}\Omega'&=\{(\tau_0,\tau_1):0<\tau_1-\tau_0<\delta_*\},\quad\text{and}\\ D'&=\{(\tau_0,W_0):W_0=\partial_{\tau_0}h(\tau_0,\tau_1)\text{ for some }(\tau_0,\tau_1)\in\Omega'\},\end{aligned} \tag{3.21}$$

where h is the transformed generating function and given in (3.22) below.

Lemma 3.9 *The set D' from (3.21) is a domain such that $D' \subset \Gamma(D)$. A generating function h on Ω' for Φ from (3.20) on D' is given by*

$$h(\tau_0, \tau_1) = H(t(\tau_0), t(\tau_1)). \tag{3.22}$$

It has the expansions

$$h(\tau_0, \tau_1) = \frac{2}{\tau_1 - \tau_0} + \mathcal{O}(1) \tag{3.23}$$

and

$$\partial_{\tau_0} h(\tau_0, \tau_1) = \frac{2}{(\tau_1 - \tau_0)^2} + \mathcal{O}((\tau_1 - \tau_0)^{-1}), \quad \partial_{\tau_1} h(\tau_0, \tau_1) = -\frac{2}{(\tau_1 - \tau_0)^2} + \mathcal{O}((\tau_1 - \tau_0)^{-1}), \tag{3.24}$$

for $\tau_1 - \tau_0 > 0$ small. Furthermore,

$$\partial^2_{\tau_0\tau_0} h,\ \partial^2_{\tau_0\tau_1} h,\ \partial^2_{\tau_1\tau_1} h = \mathcal{O}((\tau_1 - \tau_0)^{-3}) \quad \text{and} \quad -\partial^2_{\tau_0\tau_1} h(\tau_0, \tau_1) \geq \frac{c}{(\tau_1 - \tau_0)^3} \tag{3.25}$$

for some $c > 0$ and $\tau_1 - \tau_0 > 0$ small. If $(\tau_n)_{n\in\mathbb{Z}}$ is a complete orbit for h, i.e., if it satisfies

$$\partial_2 h(\tau_{n-1}, \tau_n) + \partial_1 h(\tau_n, \tau_{n+1}) = 0, \quad n \in \mathbb{Z},$$

and moreover $\delta \leq \tau_{n+1} - \tau_n \leq \Delta$ holds for $n \in \mathbb{Z}$, then $(t_n)_{n\in\mathbb{Z}} = (t(\tau_n))_{n\in\mathbb{Z}}$ is a complete orbit for H, and it verifies $a^2\delta \leq t_{n+1} - t_n \leq b^2\Delta$ for $n \in \mathbb{Z}$.

The proof of this lemma is elaborated in Section 4.7 of the appendix. Now we are in the position to prove our first main result on the ping-pong model, which relies on the application of Theorem 2.3 to h.

Proof of Theorem 3.2: Put

$$\underline{\alpha} = \frac{23}{12} \quad \text{and} \quad \overline{\alpha} = \frac{25}{12}.$$

Then $\overline{\alpha}/\underline{\alpha} = 25/23 < 19/16$, and (2.5) for $\kappa = 1$ is verified on $\{(\tau_0, \tau_1) : 0 < \tau_1 - \tau_0 < \delta_{**}\}$, where

$$\delta_{**} = \min\left\{1, \delta_*, \frac{1}{12C_*}\right\} \tag{3.26}$$

with $C_* > 0$ such that $|h(\tau_0, \tau_1) - 2/(\tau_1 - \tau_0)| \leq C_*$; see Lemma 3.9. In fact, if $0 < \tau_1 - \tau_0 < \delta_{**}$, then

$$\left|h(\tau_0, \tau_1) - \frac{2}{\tau_1 - \tau_0}\right| \leq C_* \leq \frac{1}{12\delta_{**}} \leq \frac{1}{12(\tau_1 - \tau_0)}.$$

Thus (2.5) follows from $2 - \underline{\alpha} = 1/12 = \overline{\alpha} - 2$. Fix sequences (δ_j) and (Δ_j) such that $\Delta_{j+1} < \delta_j < \Delta_j < \delta_{**}$ and also $\delta_j < \sigma_{**}^{-2}\Delta_j$ and $b^2\Delta_{j+1} < a^2\delta_j < b^2\Delta_j$ are satisfied for all $j \in \mathbb{N}$. Then Theorem 2.3 can be used for h on every

$$\Omega_j = \{(\tau_0, \tau_1) : \delta_j \leq \tau_1 - \tau_0 \leq \Delta_j\}.$$

Hence by Theorem 2.3 for every $j \in \mathbb{N}$ there is a sequence $(\tau_n^{(j)})_{n\in\mathbb{Z}}$ such that

$$\delta_j \leq \tau_{n+1}^{(j)} - \tau_n^{(j)} \leq \Delta_j, \quad \text{and} \quad \partial_2 h(\tau_{n-1}^{(j)}, \tau_n^{(j)}) + \partial_1 h(\tau_n^{(j)}, \tau_{n+1}^{(j)}) = 0$$

holds for $n \in \mathbb{Z}$. Defining $t_n^{(j)} = t(\tau_n^{(j)})$, Lemma 3.9 implies that

$$a^2\delta_j \le t_{n+1}^{(j)} - t_n^{(j)} \le b^2\Delta_j \quad \text{and} \quad \partial_2 H(t_{n-1}^{(j)}, t_n^{(j)}) + \partial_1 H(t_n^{(j)}, t_{n+1}^{(j)}) = 0$$

for $n \in \mathbb{Z}$. Thus each $(t_n^{(j)})_{n\in\mathbb{Z}}$ is a complete orbit for H and the orbits j and j' are distinct for $j \neq j'$. It remains to outline how a complete orbit gives rise to a solution of bounded energy. For the energies we have

$$E_n^{(j)} = \partial_{t_0} H(t_n^{(j)}, t_{n+1}^{(j)}) = \frac{p(\hat{t}_n^{(j)})^2}{2(\hat{t}_n^{(j)} - t_n^{(j)})^2}.$$

It follows that

$$\frac{a^2}{2b^4(\Delta_j)^2} \le E_n^{(j)} \le 16\,\frac{b^2}{a^4(\delta_j)^2}, \tag{3.27}$$

which proves the boundedness of the energies $(E_n^{(j)})_{n\in\mathbb{Z}}$ for every $j \in \mathbb{N}$. Regarding the estimate (3.27), it was used that for $(t_0, t_1) \in \tilde{\Omega}$ by (3.14) and by definition of $\tilde{\Delta}_*$,

$$|\hat{t} - t_0| \ge |t_m - t_0| - |t_m - \hat{t}| \ge \frac{1}{2}(t_1 - t_0) - \frac{1}{2a}\|\dot{p}\|_\infty (t_1 - t_0)^2 \ge \frac{1}{4}(t_1 - t_0), \tag{3.28}$$

and similarly $|t_1 - \hat{t}| \ge (t_1 - t_0)/4$. Also from (3.27) it can be read off that $\lim_{j\to\infty} \inf_{n\in\mathbb{Z}} E_n^{(j)} = \infty$, i.e., the energies of the constructed bounded orbits tend to infinity. □

Remark 3.10 We return to the point made in Remark 3.1. In the above application we always have $v_0 > 3\|\dot{p}\|_\infty$ and also $v_1 > 3\|\dot{p}\|_\infty$, since for instance

$$E_0 = \frac{1}{2}\,\dot{x}(t_0+)^2 = \frac{1}{2}\,\alpha^2 = \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2} \ge \frac{a^2}{2(t_1 - t_0)^2} \ge \frac{a^2}{2\Delta_*^2} > 8\|\dot{p}\|_\infty^2$$

by definition of Δ_*. Thus $v_0 = \sqrt{2E_0}$ yields $v_0 > 3\|\dot{p}\|_\infty$.

3.4 The adiabatic invariant

It remains to prove Theorem 3.3, where we will make use of Theorem 2.4. From the expansions (3.23) and (3.24) of h and its first derivatives the hypotheses (2.7), (2.8), and (2.9) of Theorem 2.4 are straightforward to check; the more crucial points are (2.10) and the Lipschitz continuity of Φ. Now the proof of the estimate (2.10), i.e.,

$$\partial_{\tau_0} h(\tau_0, \tau_1) + \partial_{\tau_1} h(\tau_0, \tau_1) = \mathcal{O}(\tau_1 - \tau_0), \tag{3.29}$$

will be a consequence of the following result, whose proof is given in an appendix; see Section 4.8 below.

Lemma 3.11 *If $p \in C^2(\mathbb{R})$ and $\varphi = p^2$, then*

$$\partial_{\tau_0} h(\tau_0, \tau_1) + \partial_{\tau_1} h(\tau_0, \tau_1) = \frac{1}{2}\,p(\hat{t})^2 \int_0^1 (1 - \lambda)\Big[\ddot{\varphi}((1 - \lambda)\hat{t} + \lambda t_0) - \ddot{\varphi}((1 - \lambda)\hat{t} + \lambda t_1)\Big]\,d\lambda.$$

Since in fact p is C^3-bounded by assumption (3.5), also $\dddot{\varphi} = 2p\dddot{p} + 6\dot{p}\ddot{p}$ is bounded. Now note that $a \le p(t) \le b$ implies that

$$b^{-2}(t_1 - t_0) \le \int_{t_0}^{t_1} \frac{ds}{p(s)^2} \le a^{-2}(t_1 - t_0),$$

and thus

$$b^{-2}(t_1 - t_0) \le \tau_1 - \tau_0 \le a^{-2}(t_1 - t_0) \tag{3.30}$$

for $t_0 = t(\tau_0)$ and $t_1 = t(\tau_1)$. Then Lemma 3.11 leads to

$$\begin{aligned} |\partial_{\tau_0} h(\tau_0, \tau_1) + \partial_{\tau_1} h(\tau_0, \tau_1)| &\le \frac{b^2}{2} \|\dddot{\varphi}\|_\infty \Big(\int_0^1 (1-\lambda)\lambda \, d\lambda \Big) (t_1 - t_0) \\ &\le \frac{b^4}{12} \|\dddot{\varphi}\|_\infty (\tau_1 - \tau_0), \end{aligned}$$

which is (3.29). With (3.29) being proved, we finally need to check that Φ is Lipschitz continuous. Suppose for a moment that we could just take the derivatives ∂_{τ_0} and ∂_{τ_1} in (3.29), resulting in

$$\partial^2_{\tau_0\tau_0} h(\tau_0, \tau_1) + \partial^2_{\tau_0\tau_1} h(\tau_0, \tau_1) = \mathcal{O}(1), \tag{3.31}$$

$$\partial^2_{\tau_1\tau_0} h(\tau_0, \tau_1) + \partial^2_{\tau_1\tau_1} h(\tau_0, \tau_1) = \mathcal{O}(1). \tag{3.32}$$

Then for the Monge-Ampère operator

$$\begin{aligned} Mh &= (\partial^2_{\tau_0\tau_0} h)(\partial^2_{\tau_1\tau_1} h) - (\partial^2_{\tau_0\tau_1} h)^2 \\ &= \Big(-\partial^2_{\tau_0\tau_1} h + \alpha \Big)\Big(-\partial^2_{\tau_0\tau_1} h + \beta \Big) - (\partial^2_{\tau_0\tau_1} h)^2 \\ &= -(\alpha + \beta)\, \partial^2_{\tau_0\tau_1} h + \alpha\beta, \end{aligned} \tag{3.33}$$

where α and β are bounded functions of τ_0 and τ_1. Taking into account (3.33) and (3.25), it thus follows from (2.11) that $D\Phi$ is bounded. In other words, the point is to verify (3.31) and (3.32) rigorously. However, this is easily achieved by differentiating the relation from Lemma 3.11 and combining the result with the estimate (3.15). Notice that again this requires the C^3-boundedness of p.

Finally let us give a complete

Proof of Theorem 3.3: Due to Lemma 3.9 we may suppose that $\delta_* > 0$ is so small that

$$\Big| h(\tau_0, \tau_1) - \frac{2}{\tau_1 - \tau_0} \Big| \le C_*,$$

$$\Big| \partial_{\tau_0} h(\tau_0, \tau_1) - \frac{2}{(\tau_1 - \tau_0)^2} \Big| \le C_*(\tau_1 - \tau_0)^{-1}, \tag{3.34}$$

$$\Big| \partial_{\tau_1} h(\tau_0, \tau_1) + \frac{2}{(\tau_1 - \tau_0)^2} \Big| \le C_*(\tau_1 - \tau_0)^{-1}, \tag{3.35}$$

holds for $0 < \tau_1 - \tau_0 < \delta_*$, where $C_* > 0$ is a suitable constant. Let

$$\delta_{**} = \min \Big\{ 1, \delta_*, \frac{1}{12 C_*} \Big\}$$

as in (3.26). From the proof of Theorem 3.2 we already know that (2.7) is satisfied for $\underline{\alpha} = 23/12$ and $\overline{\alpha} = 25/12$. Since $C_*(\tau_1 - \tau_0) \le C_*\delta_{**} < 1$ we moreover obtain from (3.34), (3.35) that $|(\tau_1 - \tau_0)^2 \partial_{\tau_0} h(\tau_0, \tau_1) - 2| < 1$ and $|(\tau_1 - \tau_0)^2 \partial_{\tau_1} h(\tau_0, \tau_1) + 2| < 1$, so that (2.8), (2.9) follow with $a = 1$ and $A = B = 3$. Above it has been verified that (3.29) holds and that Φ is Lipschitz continuous. Therefore we are in the position to use Theorem 2.4. $\square$

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4 Appendix: Some technicalities

In this section we shall collect the proofs of several technical results that were needed previously.

4.1 Proof of Theorem 2.3

As was mentioned earlier, we can follow the method developed in [4, Thm. 2.1] and [5, Thm. 3.1]. After somewhat increasing δ and decreasing Δ we may assume that h is defined on $\{(\theta, \theta') \in \mathbb{R}^2 : \delta \le \theta' - \theta \le \Delta\}$. To construct $(\theta_n^*)_{n\in\mathbb{Z}}$ on finite segments $-N \le n \le N$, fix $A > 0$, $N \in \mathbb{N}$, and $\Delta > \delta > 0$. Denote

$$\Sigma^{(N)} = \Big\{\Theta = (\theta_n)_{-N\le n\le N} : \theta_{\pm N} = \pm A,\ \delta \le \theta_{n+1} - \theta_n \le \Delta \text{ for } n = -N, \dots, N-1\Big\}.$$

From [4, Lemma 2.3] we recall the following result.

Lemma 4.1 *If $\delta \le \frac{A}{N} \le \Delta$, then $\Sigma^{(N)} \ne \emptyset$, and $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is compact.*

Next put

$$S(\Theta) = \sum_{n=-N}^{N-1} h(\theta_n, \theta_{n+1}), \quad \Theta = (\theta_n)_{-N\le n\le N} \in \Sigma^{(N)}.$$

Since $S : \Sigma^{(N)} \to \mathbb{R}$ is continuous, there exists a minimizer, i.e.,

$$S(\Theta^{(N)}) = \min_{\Theta\in\Sigma^{(N)}} S(\Theta)$$

for a suitable $\Theta^{(N)} = (\theta_n^{(N)})_{-N\le n\le N} \in \Sigma^{(N)}$, which henceforth we consider to be fixed.

The essence of the following proof is to show that this minimum is attained in the interior of $\Sigma^{(N)}$. Then $\Theta^{(N)}$ is a critical point of S, and so (2.1) is satisfied on the segment $-N \le n \le N$. All this is explained in detail in our previous papers. In particular, the passage to the limit $N \to \infty$ is based on a standard use of the product topology in the space of sequences. Proving that the minimum is interior is more delicate. For completeness we include those lemmas where there are modifications with respect to [4, 5].

Lemma 4.2 *Define*

$$\sigma_* = 2^{1+1/\kappa}(\overline{\alpha}/\underline{\alpha})^{1/\kappa}.$$

Then for all $N \in \mathbb{N}$,

$$\sigma_*^{-1}(\theta_n^{(N)} - \theta_{n-1}^{(N)}) \le \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}), \quad -N+1 \le n \le N-1.$$

Proof: To derive the upper bound as in [4, Lemma 2.4], denote $\theta_n^{(N)} - \theta_{n-1}^{(N)} = L$ and $\theta_{n+1}^{(N)} - \theta_n^{(N)} = \sigma L$ for $L, \sigma > 0$. Then it follows that

$$h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \leq h(\theta_{n-1}^{(N)}, s) + h(s, \theta_{n+1}^{(N)}),$$

where $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_{n-1}^{(N)})$. Thus by (2.5),

$$\begin{aligned} \underline{\alpha} L^{-\kappa}(1+\sigma^{-\kappa}) &= \underline{\alpha}(\theta_n^{(N)} - \theta_{n-1}^{(N)})^{-\kappa} + \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^{-\kappa} \\ &\leq 2^{\kappa+1}\,\overline{\alpha} L^{-\kappa}(1+\sigma)^{-\kappa}. \end{aligned}$$

Hence we obtain

$$\sigma^\kappa \leq (1+\sigma^{-\kappa})(1+\sigma)^\kappa \leq 2^{\kappa+1}\,(\overline{\alpha}/\underline{\alpha}),$$

and consequently $\sigma \leq \sigma_*$. Concerning the lower bound, it is sufficient to use the upper bound for the function $h_0(\theta, \theta') = h(-\theta', -\theta)$ for $(\theta, \theta') \in \Omega$ as in [4, Lemma 2.4]. □

Let

$$\delta^{(N)} = \min_{-N \leq n \leq N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \quad \text{and} \quad \Delta^{(N)} = \max_{-N \leq n \leq N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)})$$

for $N \in \mathbb{N}$. Then $\Theta^{(N)} = (\theta_n^{(N)})_{-N \leq n \leq N} \in \Sigma^{(N)}$ implies that $\delta \leq \delta^{(N)} \leq \Delta^{(N)} \leq \Delta$. For the next result cf. [4, Lemma 2.5].

Lemma 4.3 *Suppose that* $\overline{\alpha} < (1 + 3 \cdot 2^{-(\kappa+3)})\,\underline{\alpha}$ *and define*

$$\sigma_{**} = 2^{3(1+2/\kappa)}(\overline{\alpha}/\underline{\alpha})^{2/\kappa}.$$

Then for all $N \in \mathbb{N}$*,*

$$\Delta^{(N)} \leq \sigma_{**}\delta^{(N)}.$$

Proof: Let $-N \leq m, n \leq N-1$ be such that

$$\delta^{(N)} = \theta_{m+1}^{(N)} - \theta_m^{(N)} \quad \text{and} \quad \Delta^{(N)} = \theta_{n+1}^{(N)} - \theta_n^{(N)}.$$

Since in particular $\sigma_{**} \geq 2\sigma_* \geq 2$ holds, we may, as in [4, Lemma 2.5], suppose that $\delta^{(N)} \leq \frac{1}{2\sigma_*}\,\Delta$, $\Delta^{(N)} \geq 2\delta$, and $m+2 \leq n$ are satisfied. In complete analogy to the earlier proof it is then derived that

$$\begin{aligned} &h(\theta_m^{(N)}, \theta_{m+1}^{(N)}) + h(\theta_{m+1}^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\quad \leq h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}) \end{aligned}$$

for $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_n^{(N)})$. Since $h \geq 0$ we obtain from (2.5) and Lemma 4.2 that

$$\begin{aligned} \underline{\alpha}(\delta^{(N)})^{-\kappa} &= \underline{\alpha}(\theta_{m+1}^{(N)} - \theta_m^{(N)})^{-\kappa} \leq \overline{\alpha}(\theta_{m+2}^{(N)} - \theta_m^{(N)})^{-\kappa} + \overline{\alpha}(s - \theta_n^{(N)})^{-\kappa} + \overline{\alpha}(\theta_{n+1}^{(N)} - s)^{-\kappa} \\ &\leq \overline{\alpha}((\sigma_*^{-1} + 1)\delta^{(N)})^{-\kappa} + 2^{\kappa+1}\,\overline{\alpha}(\Delta^{(N)})^{-\kappa}. \end{aligned} \tag{4.1}$$

Denoting $q = \overline{\alpha}/\underline{\alpha} > 1$, first note that

$$\begin{aligned} \underline{\alpha} - \overline{\alpha}(\sigma_*^{-1}+1)^{-\kappa} &= (\sigma_*^{-1}+1)^{-\kappa}\underline{\alpha}\Big((2^{-(1+1/\kappa)}q^{-1/\kappa} + 1)^\kappa - q\Big) \\ &\geq (\sigma_*^{-1}+1)^{-\kappa}\underline{\alpha}\Big(2^{-(\kappa+1)}q^{-1} + 1 - q\Big) \\ &= (\sigma_*^{-1}+1)^{-\kappa}\underline{\alpha}\,q^{-1}(-q^2 + q + 2^{-(\kappa+1)}) \\ &\geq (\sigma_*^{-1}+1)^{-\kappa}\underline{\alpha}\,q^{-1}\Big(\frac{\varepsilon}{10}\Big), \end{aligned}$$

the latter estimate being a consequence of $q \in]1, 1 + 3\varepsilon/4]$ for $\varepsilon = 2^{-(\kappa+1)} \leq 1/4$ and the fact that the function $\psi(q) = -q^2 + q + \varepsilon$ is decreasing for such q, so that $\psi(q) \geq \psi(1 + 3\varepsilon/4) = (\varepsilon/4)(1 - 9\varepsilon/4) \geq \varepsilon/10$. Therefore (4.1) shows that

$$(\sigma_*^{-1} + 1)^{-\kappa} \underline{\alpha}\, q^{-1}\Big(\frac{\varepsilon}{10}\Big)(\delta^{(N)})^{-\kappa} \leq \Big[\underline{\alpha} - \overline{\alpha}(\sigma_*^{-1} + 1)^{-\kappa}\Big](\delta^{(N)})^{-\kappa} \leq 2^{\kappa+1}\, \overline{\alpha}(\Delta^{(N)})^{-\kappa},$$

which is equivalent to

$$(\Delta^{(N)})^{\kappa} \leq 10 \cdot 2^{2(\kappa+1)}(\sigma_*^{-1} + 1)^{\kappa} q^2\, (\delta^{(N)})^{\kappa}.$$

Since $\sigma_*^{-1} \leq 1$, we furthermore have $10 \cdot 2^{2(\kappa+1)}(\sigma_*^{-1} + 1)^{\kappa} \leq 10 \cdot 2^{3\kappa+2} \leq 2^{3(\kappa+2)}$, which gives the claim. □

Using Lemma 4.3 we obtain as in [4, Cor. 2.6] the following

Corollary 4.4 *Suppose that the assumptions of Lemma 4.3 are satisfied. If*

$$\sigma_{**}\delta < \frac{A}{N} < \sigma_{**}^{-1}\Delta,$$

then for all $N \in \mathbb{N}$ and $-N \leq n \leq N - 1$,

$$\delta < \delta^{(N)} \leq \theta_{n+1}^{(N)} - \theta_n^{(N)} \leq \Delta^{(N)} < \Delta.$$

From the preceding results the proof of Theorem 2.3, and in particular the passage to the limit $N \to \infty$, can be carried out by a verbatim adaption of [4, Thm. 2.1], choosing again $A = A_N = \frac{1}{2}(\sigma_{**}^{-1}\Delta + \sigma_{**}\delta)N$ in the definition of $\Sigma^{(N)}$. □

4.2 Proof of Lemmas 2.5 and 2.6

Proof of Lemma 2.5: It suffices to note that

$$r' = -\partial_2 h(\theta, \theta') \leq B(\theta' - \theta)^{-(\kappa+1)} \leq \frac{B}{a}\, \partial_1 h(\theta, \theta') = \frac{B}{a}\, r$$

by (2.9) and (2.8). □

Proof of Lemma 2.6: Denote by $\sigma_{**} \geq 1$ the constant from Theorem 2.3 and let

$$\gamma_1 = \frac{1}{2}\,\sigma_{**}^{-2} < 1. \tag{4.2}$$

Next fix $\Delta \in]0, \Delta_*[$ and take $\delta = \sigma_{**}^{-2}\Delta/2 = \gamma_1\Delta$. Then Theorem 2.3 applies to yield $(\theta_n^\Delta)_{n\in\mathbb{Z}}$ such that $\delta \leq \theta_{n+1}^\Delta - \theta_n^\Delta \leq \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}^\Delta, \theta_n^\Delta) + \partial_1 h(\theta_n^\Delta, \theta_{n+1}^\Delta) = 0, \quad n \in \mathbb{Z}. \tag{4.3}$$

Thus if we let $r_n^\Delta = \partial_1 h(\theta_n^\Delta, \theta_{n+1}^\Delta)$, then $(\theta_n^\Delta, r_n^\Delta)_{n\in\mathbb{Z}}$ is an orbit of Φ, since

$$r_{n+1}^\Delta = \partial_1 h(\theta_{n+1}^\Delta, \theta_{n+2}^\Delta) = -\partial_2 h(\theta_n^\Delta, \theta_{n+1}^\Delta)$$

in view of (4.3), and hence in particular $(\theta_n^\Delta, r_n^\Delta) \in D$. Therefore $(\theta_{n+1}^\Delta, r_{n+1}^\Delta) = \Phi(\theta_n^\Delta, r_n^\Delta)$ by the generating function property. Next,

$$r_n^\Delta = \partial_1 h(\theta_n^\Delta, \theta_{n+1}^\Delta) \geq a(\theta_{n+1}^\Delta - \theta_n^\Delta)^{-(\kappa+1)} \geq a\Delta^{-(\kappa+1)}$$

by (2.8), and the upper bound on r_n^Δ is obtained similarly. □

4.3 Proof of Lemma 3.5

Fix a function $w \in C^\infty([0,1])$ such that $w(0)=0$, $w(1)=1$, $\dot{w}(s)>0$ for $s \in]0,1[$, and $w^{(k)}(0) = w^{(k)}(1) = 0$ for $k \in \mathbb{N}$. Next let

$$\mu = \frac{M}{\sum_{k=1}^{m} \|w^{(k)}\|_\infty (\frac{v_0}{a})^k} > 0,$$

where we take $v_0 > 0$ so large that $\mu < b-a$. Now select an integer $N_+ \geq 2$ so that

$$a + (N_+ - 1)\mu < b \leq a + N_+\mu.$$

Thereafter we introduce the sequence $(\tau_n)_{n\in\mathbb{Z}}$ by $\tau_0 = t_0$ and

$$\tau_{n+1} - \tau_n = \begin{cases} \frac{2a}{v_0} & : \quad n \leq -1 \\ 2(\frac{a+(n+1)\mu}{v_0}) & : \quad 0 \leq n \leq N_+ - 1 \\ \frac{2b}{v_0} & : \quad n \geq N_+ \end{cases} .$$

Using this sequence and the number $\hat{\mu} \in]0,\mu]$ given by $\hat{\mu} = b - a - (N_+ - 1)\mu$, we define the function $p_+ : \mathbb{R} \to [a,b]$ as

$$p_+(t) = \begin{cases} a & : \quad t \leq \tau_0 \\ a + n\mu + \mu w(\frac{v_0}{a}(t-\tau_n)) & : \quad n = 0,\dots,N_+ - 2 \text{ and } t \in [\tau_n, \tau_n + \frac{a}{v_0}] \\ a + (n+1)\mu & : \quad n = 0,\dots,N_+ - 2 \text{ and } t \in [\tau_n + \frac{a}{v_0}, \tau_{n+1}] \\ a + (N_+ - 1)\mu + \hat{\mu} w(\frac{v_0}{a}(t - \tau_{N_+-1})) & : \quad t \in [\tau_{N_+-1}, \tau_{N_+-1} + \frac{a}{v_0}] \\ b & : \quad t \geq \tau_{N_+-1} + \frac{a}{v_0} \end{cases} .$$

Let $(t_n, v_n)_{n\in\mathbb{Z}}$ denote the complete orbit of the Fermi-Ulam model for $p = p_+$ that agrees at $n = 0$ with (t_0, v_0). We claim that

$$t_n = \tau_n \quad \text{and} \quad v_n = v_0 \quad \text{for all} \quad n \in \mathbb{Z}. \tag{4.4}$$

For, we already know that $t_0 = \tau_0$. Then the unique solution $\tilde{t}_0$ of

$$(\tilde{t}_0 - t_0)v_0 = p_+(\tilde{t}_0)$$

is

$$\tilde{t}_0 = t_0 + \frac{a+\mu}{v_0},$$

since $t_0 + \frac{a}{v_0} < t_0 + \frac{a+\mu}{v_0} < t_0 + \frac{2(a+\mu)}{v_0} = \tau_1$ yields $p_+(t_0 + \frac{a+\mu}{v_0}) = a + \mu$. Furthermore, $\dot{p}_+(\tilde{t}_0) = 0$ implies that $v_1 = v_0$ and

$$t_1 = \tilde{t}_0 + \frac{p_+(\tilde{t}_0)}{v_1} = t_0 + \frac{a+\mu}{v_0} + \tilde{t}_0 - t_0 = t_0 + \frac{2(a+\mu)}{v_0} = \tau_1$$

by the definition of the map (3.3). This argument can be applied repeatedly to deduce that $\tilde{t}_n = t_n + \frac{a+(n+1)\mu}{v_0}$ for $n \in \{0,\dots,N_+ - 1\}$ and $v_n = \ldots = v_1 = v_0$ as well as $t_n = \tau_n$ for $n \in \{0,\dots,N_+\}$. Due to $p_+(t) = b$ for $t \geq \tau_{N_+} = t_{N_+}$, it then follows that $\tilde{t}_n = t_n + \frac{b}{v_0}$ for $n \geq N_+$, which yields the desired relations $v_n = v_0$ and $t_n = \tau_n$ for $n \geq N_+ + 1$. Then a similar reasoning proves that $v_n = v_0$ and $t_n = \tau_n$ for $n \leq -1$. From (4.4) the remaining assertions of the lemma including (3.6) are straightforward to check. $\square$

4.4 Proof of Lemma 3.6

Once again we fix a function $w \in C^\infty([0,1])$ such that $w(0)=0$, $w(1)=1$, $\dot{w}(s)>0$ for $s\in]0,1[$, and $w^{(k)}(0)=w^{(k)}(1)=0$ for $k\in\mathbb{N}$. For $\sigma>0$ so large that

$$\sum_{k=1}^{m} \sigma^{-k}\|w^{(k)}\|_\infty \le \frac{M}{b-a}$$

we put

$$p_-(t) = \begin{cases} b & : \quad t \le t_0 \\ b-(b-a)w(\sigma^{-1}(t-t_0)) & : \quad t\in[t_0, t_0+\sigma] \\ a & : \quad t \ge t_0+\sigma \end{cases} .$$

Then clearly (3.6) and (3.7) are verified. Next we check that $|\dot{p}_-(\tilde{t}_n)| \ge C = C(a,b,M,m)$ for some $n\in\mathbb{N}_0$. For, from (3.7) and the definition of the map (3.3) it follows that $v_{n+1}\ge v_n$ for $n\in\mathbb{Z}$, so that in particular $v_n\ge v_0$ for $n\in\mathbb{N}_0$. Thus we can enforce v_n to be as large as needed, if we choose $v_0>0$ large enough initially. Observing

$$t_{n+1}-t_n = p_-(\tilde{t}_n)\Big(\frac{1}{v_n}+\frac{1}{v_{n+1}}\Big) = \mathcal{O}\Big(\frac{1}{v_n}\Big)$$

for v_n large we deduce that

$$0<\tilde{t}_{n+1}-\tilde{t}_n = t_{n+1}-t_n+\frac{p_-(\tilde{t}_{n+1})}{v_{n+1}}-\frac{p_-(\tilde{t}_n)}{v_n} \le \frac{C_1}{v_0} \quad \text{for} \quad n\in\mathbb{N}_0. \tag{4.5}$$

From the mean value theorem there is $\tau\in]t_0,t_0+\sigma[$ such that

$$b-a = p_-(t_0)-p_-(t_0+\sigma) = -\dot{p}_-(\tau)\sigma = |\dot{p}_-(\tau)|\,\sigma.$$

By (4.5) and noting that also $\tilde{t}_0-t_0=\frac{p_-(\tilde{t}_0)}{v_0}$, we find $n\in\mathbb{N}_0$ so that $|\tilde{t}_n-\tau|\le\frac{C_1}{2v_0}$. Thus by (3.6) we arrive at

$$|\dot{p}_-(\tilde{t}_n)| \ge |\dot{p}_-(\tau)| - M|\tilde{t}_n-\tau| \ge \frac{b-a}{\sigma} - M\frac{C_1}{2v_0} \ge \frac{b-a}{2\sigma} =: C,$$

provided that v_0 is chosen sufficiently large. Therefore the claim is obtained by taking N_- so large that $t_{N_-}\ge t_0+\sigma$; note that this can be done, since $t_{n+1}\ge\tilde{t}_n = t_n+\frac{p_-(\tilde{t}_n)}{v_n}\ge t_n+\frac{a}{v_n}$ implies that $t_{n+1}\ge t_0+\sum_{j=0}^{n}\frac{a}{v_j}$ for $n\in\mathbb{N}_0$. On the other hand, $|v_{n+1}-v_n| = 2|\dot{p}_-(\tilde{t}_n)|\le C_2$ yields $v_n\le nC_2+v_0$ for $n\in\mathbb{N}_0$, so that $t_{n+1}\ge t_0+\sum_{j=0}^{n}\frac{a}{jC_2+v_0}$ for $n\in\mathbb{N}_0$. □

4.5 Proof of Lemma 3.7

Write $\hat{t}=t_m+\delta$ for $|\delta|<\eta=(t_1-t_0)/2$. Then (3.13) is equivalent to

$$\phi(\delta) = \delta p(t_m+\delta)+(\eta^2-\delta^2)\,\dot{p}(t_m+\delta) = 0.$$

Note that

$$\begin{aligned} \phi'(\delta) &= p(t_m+\delta)+(\eta^2-\delta^2)\,\ddot{p}(t_m+\delta)-\delta\dot{p}(t_m+\delta) \\ &\ge a-\eta^2\|\ddot{p}\|_\infty-|\delta|\,\|\dot{p}\|_\infty \ge \frac{a}{2}, \quad |\delta|<\eta, \end{aligned} \tag{4.6}$$

by the assumptions on $t_1 - t_0$. Defining $M = (2/a)(\|\dot{p}\|_\infty + 1)$, it follows that

$$\phi(M\eta^2) = \phi(0) + \int_0^{M\eta^2} \phi'(\delta)\, d\delta \geq \eta^2\Big(\dot{p}(t_m) + \frac{Ma}{2}\Big) > 0,$$

and in the same way $\phi(-M\eta^2) < 0$ is obtained. Since $|\pm M\eta^2| < \eta$, (4.6) implies that ϕ has a unique zero δ_0 in $]-\eta,\eta[$, and furthermore $|\delta_0| < M\eta^2$. Therefore the unique solution to (3.13) is $\hat{t} = t_m + \delta_0$ and the smoothness of $\hat{t}$ follows from (4.6) and the Implicit Function Theorem. Then (3.13) yields

$$|\hat{t} - t_m| = |(t_1 - \hat{t})(\hat{t} - t_0)|\, \frac{|\dot{p}(\hat{t})|}{|p(\hat{t})|} \leq \frac{1}{a}\, \|\dot{p}\|_\infty (t_1 - t_0)^2,$$

i.e., (3.14). For (3.15), differentiating (3.13) w.r.t. t_0 gives

$$\partial_{t_0}\hat{t} = \frac{p(\hat{t}) + 2(t_1 - \hat{t})\dot{p}(\hat{t})}{2(t_1 - \hat{t})(\hat{t} - t_0)\ddot{p}(\hat{t}) + (t_0 + t_1 - 2\hat{t})\dot{p}(\hat{t}) + 2p(\hat{t})}.$$

Thus

$$\begin{aligned}
\Big|\partial_{t_0}\hat{t} - \frac{1}{2}\Big| &= \Big|\frac{4(t_1 - \hat{t})\dot{p}(\hat{t}) - 2(t_1 - \hat{t})(\hat{t} - t_0)\ddot{p}(\hat{t}) - (t_0 + t_1 - 2\hat{t})\dot{p}(\hat{t})}{4(t_1 - \hat{t})(\hat{t} - t_0)\ddot{p}(\hat{t}) + 2(t_0 + t_1 - 2\hat{t})\dot{p}(\hat{t}) + 4p(\hat{t})}\Big| \\
&\leq \frac{4\|\dot{p}\|_\infty (t_1 - t_0) + 2\|\ddot{p}\|_\infty (t_1 - t_0)^2 + \frac{2}{a}\,\|\dot{p}\|_\infty (t_1 - t_0)^2}{4a - 4\|\ddot{p}\|_\infty (t_1 - t_0)^2 - \frac{4}{a}\|\dot{p}\|_\infty^2 (t_1 - t_0)^2} \\
&\leq \frac{4\|\dot{p}\|_\infty + \frac{a}{2} + \frac{1}{2}}{4a - \frac{5}{4}\,a}\,(t_1 - t_0) \\
&= \frac{2}{11a}\,(8\|\dot{p}\|_\infty + a + 1)\,(t_1 - t_0)
\end{aligned}$$

by (3.14) and the choice of $\tilde{\Delta}_*$, using once more that $0 < t_1 - \hat{t} < t_1 - t_0$ and $0 < \hat{t} - t_0 < t_1 - t_0$. Next,

$$\partial_{t_1}\hat{t} = \frac{p(\hat{t}) - 2(\hat{t} - t_0)\dot{p}(\hat{t})}{2(t_1 - \hat{t})(\hat{t} - t_0)\ddot{p}(\hat{t}) + (t_0 + t_1 - 2\hat{t})\dot{p}(\hat{t}) + 2p(\hat{t})}. \tag{4.7}$$

From this the bound on $|\partial_{t_1}\hat{t} - 1/2|$ can be obtained in an analogous way, proving (3.15). □

4.6 Proof of Lemma 3.8

First note that

$$\partial_{t_0} H(t_0, t_1) = \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2} \quad \text{and} \quad \partial_{t_1} H(t_0, t_1) = -\frac{p(\hat{t})^2}{2(t_1 - \hat{t})^2} \tag{4.8}$$

for $(t_0, t_1) \in \tilde{\Omega}$. For, differentiating (3.12) it follows that

$$\begin{aligned}
\partial_{t_0} H &= p(\hat{t})\dot{p}(\hat{t})\,(\partial_{t_0}\hat{t})\Big(\frac{1}{\hat{t} - t_0} + \frac{1}{t_1 - \hat{t}}\Big) + \frac{1}{2}\,p(\hat{t})^2\Big(-\frac{1}{(\hat{t} - t_0)^2}\,(\partial_{t_0}\hat{t} - 1) + \frac{1}{(t_1 - \hat{t})^2}\,(\partial_{t_0}\hat{t})\Big) \\
&= (\partial_{t_0}\hat{t})p(\hat{t})\Big[\dot{p}(\hat{t})\Big(\frac{1}{\hat{t} - t_0} + \frac{1}{t_1 - \hat{t}}\Big) - \frac{1}{2}\,p(\hat{t})\Big(\frac{1}{(\hat{t} - t_0)^2} - \frac{1}{(t_1 - \hat{t})^2}\Big)\Big] + \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2}.
\end{aligned}$$

Then

$$[\dots] = \frac{(t_1 - t_0)}{2(\hat{t} - t_0)^2 (t_1 - \hat{t})^2} \Big[2(\hat{t} - t_0)(t_1 - \hat{t}) \dot{p}(\hat{t}) - (t_0 + t_1 - 2\hat{t}) p(\hat{t}) \Big] = 0$$

by (3.13). Since an analogous calculation is possible for $\partial_{t_1} H$, we obtain (4.8). Next we differentiate (4.8) one more time to see that

$$\begin{aligned} \partial^2_{t_0 t_1} H &= \frac{p(\hat{t})}{(\hat{t} - t_0)^3} \Big[(\hat{t} - t_0) \dot{p}(\hat{t}) - p(\hat{t}) \Big] (\partial_{t_1} \hat{t}) \\ &= -\frac{p(\hat{t})}{(\hat{t} - t_0)^3} \frac{[p(\hat{t}) - (\hat{t} - t_0)\dot{p}(\hat{t})] \, [p(\hat{t}) - 2(\hat{t} - t_0)\dot{p}(\hat{t})]}{2(t_1 - \hat{t})(\hat{t} - t_0)\ddot{p}(\hat{t}) + (t_0 + t_1 - 2\hat{t})\dot{p}(\hat{t}) + 2p(\hat{t})}, \end{aligned} \tag{4.9}$$

where (4.7) was used. Therefore by (3.14),

$$\begin{aligned} -\partial^2_{t_0 t_1} H &\geq \frac{a}{(t_1 - t_0)^3} \frac{(a - \tilde{\Delta}_* \|\dot{p}\|_\infty)\,(a - 2\tilde{\Delta}_* \|\dot{p}\|_\infty)}{2\tilde{\Delta}_*^2 \|\ddot{p}\|_\infty + (2/a)\tilde{\Delta}_*^2 \|\dot{p}\|_\infty^2 + 2b} \\ &\geq \frac{3a^3}{8(t_1 - t_0)^3} \frac{1}{a + 2b}. \end{aligned} \tag{4.10}$$

We infer that for fixed $t_0 \in \mathbb{R}$ the function $t_1 \mapsto \partial_{t_0} H(t_0, t_1)$ is strictly decreasing on $]t_0, t_0 + \tilde{\Delta}_*[$ and tends to infinity for $t_1 \searrow t_0$. Thus if we define

$$\tilde{D} = \{(t_0, E_0) : E_0 = \partial_{t_0} H(t_0, t_1) \text{ for some } (t_0, t_1) \in \tilde{\Omega}\},$$

then we obtain the characterizations

$$\begin{aligned} \tilde{D} &= \Big\{ (t_0, E_0) : t_0 \in \mathbb{R}, E_0 \in]\partial_{t_0} H(t_0, t_0 + \tilde{\Delta}_*), \infty[\Big\}, \\ D &= \Big\{ (t_0, E_0) : t_0 \in \mathbb{R}, E_0 \in]\partial_{t_0} H(t_0, t_0 + \Delta_*), \infty[\Big\}. \end{aligned} \tag{4.11}$$

In particular, this shows that D is open and connected. Now observe that in general $0 < \hat{t} - t_0 < t_1 - t_0$. Therefore (4.8) and the definition of Δ_* yields

$$\partial_{t_0} H(t_0, t_0 + \Delta_*) = \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2} \geq \frac{a^2}{2\Delta_*^2} \geq \frac{a^2}{2\tilde{\Delta}_*^2} > 8\|\dot{p}\|_\infty^2,$$

for $\hat{t} = \hat{t}(t_0, t_0 + \Delta_*)$. Recalling (4.11), we thus have shown that

$$(t_0, E_0) \in D \quad \Longrightarrow \quad E_0 > \frac{a^2}{2\Delta_*^2} > 8\|\dot{p}\|_\infty^2. \tag{4.12}$$

The next step is to check Definition 2.1 for the current setup. For, suppose first that $(t_0, E_0) \in D$ and $(t_1, E_1) = \Psi(t_0, E_0)$. Then $E_0 = \partial_{t_0} H(t_0, t_1')$ for some $0 < t_1' - t_0 < \Delta_*$. From $(t_1, E_1) = \Psi(t_0, E_0)$ we deduce that

$$\begin{aligned} 2(t_1 - \tilde{t})(\tilde{t} - t_0)\,\dot{p}(\tilde{t}) &= (t_1 - \tilde{t})(\tilde{t} - t_0)\,(\sqrt{2E_0} - \sqrt{2E_1}) = (t_1 - \tilde{t})\,p(\tilde{t}) - (\tilde{t} - t_0)\,p(\tilde{t}) \\ &= (t_0 + t_1 - 2\tilde{t})\,p(\tilde{t}). \end{aligned} \tag{4.13}$$

Using (3.8) and (3.9) we furthermore get $t_0 < \tilde{t} < t_1$. Now we claim that

$$t_1 - t_0 < \tilde{\Delta}_*. \tag{4.14}$$

For, by (3.8) and (3.9), and due to $(t_1, E_1) = \Psi(t_0, E_0)$,

$$t_1 - t_0 = (t_1 - \tilde{t}) + (\tilde{t} - t_0) = p(\tilde{t})\Big(\frac{1}{\sqrt{2E_1}} + \frac{1}{\sqrt{2E_0}}\Big). \tag{4.15}$$

Since $E_1 = (\sqrt{E_0} - \sqrt{2}\,\dot{p}(\tilde{t}))^2$ and $E_0 > 8\|\dot{p}\|_\infty^2$ by (4.12), it follows that

$$\sqrt{2E_1} = |\sqrt{2E_0} - 2\dot{p}(\tilde{t})| \geq \sqrt{2E_0} - 2\|\dot{p}\|_\infty \geq \Big(\sqrt{2} - \frac{1}{\sqrt{2}}\Big)\sqrt{E_0} \geq \frac{1}{2}\sqrt{E_0}.$$

Thus (4.15) and (4.12) yields

$$t_1 - t_0 \leq b\Big(\frac{2}{\sqrt{E_0}} + \frac{1}{\sqrt{2E_0}}\Big) \leq \frac{3b}{\sqrt{E_0}} \leq \frac{3\sqrt{2}b}{a}\,\Delta_* < \tilde{\Delta}_*,$$

which completes the proof of (4.14). If we summarize (4.13), $t_0 < \tilde{t} < t_1$, and (4.14), then the uniqueness assertion from Lemma 3.7 implies that $\tilde{t}(t_0, E_0) = \hat{t}(t_0, t_1)$. Therefore we conclude from (3.9) and (4.8) that

$$\partial_{t_0} H(t_0, t_1') = E_0 = \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)^2} = \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2} = \partial_{t_0} H(t_0, t_1).$$

Since $t_1', t_1 \in]t_0, t_0 + \tilde{\Delta}_*[$, the foregoing observations imply that $t_1' = t_1$ must be satisfied. In particular, $0 < t_1 - t_0 < \Delta_*$ and hence $(t_0, t_1) \in \Omega$. Finally, (3.8), $\tilde{t} = \hat{t}$, and (4.8) leads to

$$E_1 = \frac{p(\tilde{t})^2}{2(t_1 - \tilde{t})^2} = \frac{p(\hat{t})^2}{2(t_1 - \hat{t})^2} = -\partial_{t_1} H(t_0, t_1).$$

Conversely, take $(t_0, t_1) \in \Omega$ and $E_0, E_1 \in \mathbb{R}$ such that

$$E_0 = \partial_{t_0} H(t_0, t_1) \quad \text{and} \quad E_1 = -\partial_{t_1} H(t_0, t_1)$$

holds. Then $(t_0, E_0) \in D$ by the definition of D. From (4.8) we see that

$$E_0 = \partial_{t_0} H(t_0, t_1) = \frac{p(\hat{t})^2}{2(\hat{t} - t_0)^2} \quad \text{and} \quad E_1 = -\partial_{t_1} H(t_0, t_1) = \frac{p(\hat{t})^2}{2(t_1 - \hat{t})^2} \tag{4.16}$$

for $\hat{t} = \hat{t}(t_0, t_1)$, which in turn implies that

$$t_1 = \hat{t} + \frac{p(\hat{t})}{\sqrt{2E_1}} \quad \text{and} \quad \hat{t} = t_0 + \frac{p(\hat{t})}{\sqrt{2E_0}}.$$

Denoting $\tilde{t} = \tilde{t}(t_0, E_0)$, (3.9) and (4.12) yields

$$|\hat{t} - \tilde{t}| = \Big|\frac{p(\hat{t})}{\sqrt{2E_0}} - \frac{p(\tilde{t})}{\sqrt{2E_0}}\Big| \leq \frac{\|\dot{p}\|_\infty}{\sqrt{2E_0}}\,|\hat{t} - \tilde{t}| \leq \frac{1}{4}\,|\hat{t} - \tilde{t}|,$$

so that $\hat{t} = \tilde{t}$. In particular, we have

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2E_1}}.$$

Furthermore, from (4.16) and (3.13) we obtain

$$\sqrt{2E_0} - \sqrt{2E_1} = \frac{p(\hat{t})}{\hat{t} - t_0} - \frac{p(\hat{t})}{t_1 - \hat{t}} = \frac{(t_0 + t_1 - 2\hat{t})}{(\hat{t} - t_0)(t_1 - \hat{t})}\, p(\hat{t}) = 2\dot{p}(\hat{t}) = 2\dot{p}(\tilde{t}).$$

This finally gives

$$E_1 = (\sqrt{E_0} - \sqrt{2}\,\dot{p}(\tilde{t}))^2$$

and completes the proof that $\Psi(t_0, E_0) = (t_1, E_1)$. Therefore H on Ω is a generating function for Ψ on D, in the sense of Definition 2.1.

Concerning the expansion (3.17), we insert the definition of H and apply (3.14) to bound

$$\begin{aligned}
\Big|H(t_0, t_1) - p(\hat{t})^2 \,\frac{2}{t_1 - t_0}\Big| &= \frac{1}{2}\Big|\Big(\frac{1}{\hat{t} - t_0} - \frac{2}{t_1 - t_0}\Big) + \Big(\frac{1}{t_1 - \hat{t}} - \frac{2}{t_1 - t_0}\Big)\Big|\, p(\hat{t})^2 \\
&\le C\Big|\frac{2\hat{t} - (t_0 + t_1)}{(\hat{t} - t_0)(t_1 - t_0)}\Big| + C\Big|\frac{2\hat{t} - (t_0 + t_1)}{(t_1 - \hat{t})(t_1 - t_0)}\Big| \\
&\le C\Big|\frac{t_1 - t_0}{\hat{t} - t_0}\Big| + C\Big|\frac{t_1 - t_0}{t_1 - \hat{t}}\Big| \le C,
\end{aligned}$$

where $C > 0$ depends on a, $\|p\|_\infty$, and $\|\dot{p}\|_\infty$. For the last estimate we made use of (3.28). Since

$$\Big|\frac{p(t_m)^2 - p(\hat{t})^2}{t_1 - t_0}\Big| \le C\Big|\frac{t_m - \hat{t}}{t_1 - t_0}\Big| \le C(t_1 - t_0)$$

by (3.14), we get (3.17). Now we turn to (3.18). In view of (4.8) we can use similar arguments as before to derive the desired expansions. For (3.19) concerning the second order derivatives, we obtain from (4.9), (3.28), and (4.10) that

$$\partial^2_{t_0 t_1} H = \mathcal{O}((t_1 - t_0)^{-3}) \quad \text{and} \quad -\,\partial^2_{t_0 t_1} H(t_0, t_1) \ge \frac{c}{(t_1 - t_0)^3}.$$

Also from (4.8),

$$\begin{aligned}
\partial^2_{t_0 t_0} H &= \frac{p(\hat{t})}{(\hat{t} - t_0)^3}\Big[\Big((\hat{t} - t_0)\dot{p}(\hat{t}) - p(\hat{t})\Big)(\partial_{t_0}\hat{t}) + p(\hat{t})\Big], \\
\partial^2_{t_1 t_1} H &= -\frac{p(\hat{t})}{(t_1 - \hat{t})^3}\Big[\Big((t_1 - \hat{t})\dot{p}(\hat{t}) + p(\hat{t})\Big)(\partial_{t_1}\hat{t}) - p(\hat{t})\Big],
\end{aligned}$$

and this yields

$$\partial^2_{t_0 t_0} H,\ \partial^2_{t_1 t_1} H = \mathcal{O}((t_1 - t_0)^{-3})$$

by (3.15) and (3.28). □

4.7 Proof of Lemma 3.9

First note that $0 < \tau_1 - \tau_0 < \delta_*$ and (3.30) imply that $b^{-2}(t_1 - t_0) \le \tau_1 - \tau_0 < \delta_*$ for $t_0 = t(\tau_0)$ and $t_1 = t(\tau_1)$, so that $0 < t_1 - t_0 < \Delta_*$ and $H(t_0, t_1)$ is well-defined. If we consider τ_0 and τ_1 to be the independent variables, then (4.16) leads to

$$\partial_{\tau_0} h(\tau_0, \tau_1) = \partial_{t_0} H(t_0, t_1)\, t'(\tau_0) = E_0\, p(t_0)^2 = W_0, \tag{4.17}$$

and similarly $\partial_{\tau_1} h(\tau_0,\tau_1) = -W_1$.

In order to check Definition 2.1, we introduce

$$\begin{aligned} \Omega_1 &= \{(t_0,t_1) : 0 < \tau(t_1) - \tau(t_0) < \delta_*\} \quad \text{and} \\ D_1 &= \{(t_0,E_0) : E_0 = \partial_{t_0} H(t_0,t_1) \text{ for some } (t_0,t_1) \in \Omega_1\}. \end{aligned}$$

The map $(t_0,t_1) \mapsto (\tau(t_0),\tau(t_1))$ is a diffeomorphism between Ω_1 and Ω'. As in the proof of Lemma 3.8 it can be checked that D_1 is a domain, and it is easy to verify that the restriction of H to Ω_1 is a generating function for the restriction of Ψ to D_1. Moreover $\Gamma(D_1) = D'$, and the conclusion follows by transporting the domains Ω_1 and D_1 to Ω' and D', respectively.

Turning to the expansion (3.23), we recall from (3.30) that

$$b^{-2}(t_1 - t_0) \le \tau_1 - \tau_0 \le a^{-2}(t_1 - t_0).$$

Hence $\tau_1 - \tau_0 > 0$ small corresponds to $t_1 - t_0 > 0$ small, and accordingly (3.17) gives

$$h(\tau_0,\tau_1) = H(t_0,t_1) = p(t_m)^2 \, \frac{2}{t_1 - t_0} + \mathcal{O}(1). \tag{4.18}$$

Then

$$\begin{aligned} \Big|(\tau_1 - \tau_0) - (t_1 - t_0)\,\frac{1}{p(t_m)^2}\Big| &= \Big|\int_{t_0}^{t_1} \frac{p(t_m)^2 - p(s)^2}{p(s)^2 p(t_m)^2}\,ds\Big| \\ &\le C\int_{t_0}^{t_1} |t_m - s|\,ds \\ &\le C(t_1 - t_0)^2 \le C(\tau_1 - \tau_0)^2. \end{aligned}$$

It follows that

$$\begin{aligned} \Big|\frac{p(t_m)^2}{t_1 - t_0} - \frac{1}{\tau_1 - \tau_0}\Big| &= \frac{p(t_m)^2}{(t_1 - t_0)(\tau_1 - \tau_0)}\,\Big|(\tau_1 - \tau_0) - (t_1 - t_0)\,\frac{1}{p(t_m)^2}\Big| \\ &\le C\frac{(\tau_1 - \tau_0)^2}{(t_1 - t_0)(\tau_1 - \tau_0)} \le C, \end{aligned}$$

and hence (3.23) is obtained from (4.18). Concerning (3.24), we get by (4.17) and (3.18),

$$\begin{aligned} &\Big|\partial_{\tau_0} h(\tau_0,\tau_1) - \frac{2}{(\tau_1 - \tau_0)^2}\Big| \\ &= \Big|\partial_{t_0} H(t_0,t_1)\, p(t_0)^2 - \frac{2}{(\tau_1 - \tau_0)^2}\Big| \\ &\le \Big|\partial_{t_0} H(t_0,t_1) - \frac{2p(t_m)^2}{(t_1 - t_0)^2}\Big|\, p(t_0)^2 + 2\,\Big|\frac{p(t_m)^2}{(t_1 - t_0)^2}\, p(t_0)^2 - \frac{1}{(\tau_1 - \tau_0)^2}\Big| \\ &\le C(t_1 - t_0)^{-1} + 2\,\Big|\frac{p(t_m)^2}{(t_1 - t_0)^2}\, p(t_0)^2 - \frac{1}{(\tau_1 - \tau_0)^2}\Big|. \end{aligned}$$

For the second term note that

$$\tau_1 - \tau_0 = \frac{1}{p(\zeta)^2}\,(t_1 - t_0)$$

for some $\zeta \in]t_0, t_1[$ by the mean value theorem. Hence

$$\Big|\frac{p(t_m)^2}{(t_1-t_0)^2}\,p(t_0)^2 - \frac{1}{(\tau_1-\tau_0)^2}\Big| = \frac{1}{(t_1-t_0)^2}\,|p(t_m)^2 p(t_0)^2 - p(\zeta)^4| \le C(t_1-t_0)^{-1},$$

since $|t_m - \zeta| \le t_1 - t_0$ and $|t_0 - \zeta| \le t_1 - t_0$. This completes the proof of (3.24), and for $\partial_{\tau_1} h$ the argument is similar. Next we show (3.25). Differentiating (4.17) and the analogous relation for $\partial_{\tau_1} h$, it is found that

$$\begin{aligned}
\partial^2_{\tau_0\tau_0} h &= p(t_0)^3\Big[p(t_0)(\partial^2_{t_0t_0} H) + 2\dot p(t_0)(\partial_{t_0} H)\Big],\\
\partial^2_{\tau_1\tau_1} h &= p(t_1)^3\Big[p(t_1)(\partial^2_{t_1t_1} H) + 2\dot p(t_1)(\partial_{t_1} H)\Big],\\
\partial^2_{\tau_0\tau_1} h &= p(t_0)^2 p(t_1)^2\,(\partial^2_{t_0t_1} H).
\end{aligned}$$

Hence (3.25) is a consequence of (3.5), (3.30), and (3.18), (3.19) from Lemma 3.8.

To verify the last assertion of the lemma, we can employ (4.17) and the relation for $\partial_{\tau_1} h$ to see that

$$\partial_1 h(\tau_n, \tau_{n+1}) = \partial_1 H(t_n, t_{n+1})\,p(t_n)^2 \quad\text{and}\quad \partial_2 h(\tau_{n-1}, \tau_n) = \partial_2 H(t_{n-1}, t_n)\,p(t_n)^2.$$

As a consequence,

$$\partial_2 H(t_{n-1}, t_n) + \partial_1 H(t_n, t_{n+1}) = \frac{1}{p(t_n)^2}\Big[\partial_2 h(\tau_{n-1}, \tau_n) + \partial_1 h(\tau_n, \tau_{n+1})\Big] = 0$$

for $n \in \mathbb{Z}$. □

4.8 Proof of Lemma 3.11

First we need

Lemma 4.5 *Let $\varphi \in C^2([a,b])$ and suppose that $c \in]a,b[$. Then*

$$\begin{aligned}
&(b-c)^2\varphi(a) - (c-a)^2\varphi(b)\\
&= (b-a)\Big[\varphi(c)(b+a-2c) - \dot\varphi(c)(b-c)(c-a)\Big] + R(a,b,c)
\end{aligned}$$

for

$$R(a,b,c) = (b-c)^2(c-a)^2 \int_0^1 (1-\lambda)\Big[\ddot\varphi((1-\lambda)c + \lambda a) - \ddot\varphi((1-\lambda)c + \lambda b)\Big]\,d\lambda.$$

Proof: This follows from a Taylor expansion of φ about c to express $\varphi(a)$ and $\varphi(b)$. □

Proof of Lemma 3.11: By (3.22) and (4.16),

$$\begin{aligned}
\partial_{\tau_0} h(\tau_0, \tau_1) + \partial_{\tau_1} h(\tau_0, \tau_1) &= \partial_{t_0} H(t_0, t_1)\,p(t_0)^2 + \partial_{t_1} H(t_0, t_1)\,p(t_1)^2\\
&= \frac{p(\hat t)^2 p(t_0)^2}{2(\hat t - t_0)^2} - \frac{p(\hat t)^2 p(t_1)^2}{2(t_1 - \hat t)^2}\\
&= \frac{p(\hat t)^2}{2(\hat t - t_0)^2 (t_1 - \hat t)^2}\,\Big[(t_1 - \hat t)^2 p(t_0)^2 - (\hat t - t_0)^2 p(t_1)^2\Big]. \qquad (4.19)
\end{aligned}$$

Now it suffices to apply Lemma 4.5 for $a = t_0$, $c = \hat t$, $b = t_1$, and $\varphi = p^2$, and to make use of (3.13) to see that only the remainder R survives. □

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