EXISTENCE OF PERIODIC ORBITS FOR HIGH-DIMENSIONAL AUTONOMOUS SYSTEMS

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ABSTRACT. We give a result on existence of periodic orbits for autonomous differential systems with arbitrary finite dimension. It is based on a Poincaré-Bendixson property enjoyed by a new class of monotone systems introduced in L. A. Sanchez, *Cones of rank* 2 *and the Poincaré-Bendixson property for a new class of monotone systems*, Journal of Differential Equations 216 (2009), 1170-1190. A concrete application is done to a scalar differential equation of order 4.

1. INTRODUCTION

The study of periodic orbits in autonomous systems is a quite nontrivial issue in the theory of differential equations. The main reason seems to be the decisive role played by the dimension of the phase space. In the two-dimensional case the classical Poincaré-Bendixson theorem (see [4]) provides an utmost powerful tool which actually yields to a thorough understanding of the global behavior of planar systems. In higher dimensions no such a general result can exist as long as chaotic behavior comes into the scene. This fact reduced for a long time the search of periodic orbits, at least for dissipative systems, to local approaches in the setting of bifurcation theory.

Over the last thirty years partial extensions of the Poincaré-Bendixson property for new classes of autonomous systems have been achieved. All of them share the same underlying idea: to prove that compact limit sets of the flows are topologically conjugate to invariant sets of planar flows. A first example is the work by H.L. Smith in [10] on three-dimensional competitive systems, which is based on the theory of monotone systems developed mainly by M. W. Hirsch and H. L. Smith himself (see the monographs [5] and [11]). A second theory to be mentioned is that by R. A. Smith in [12, 13], where quadratic Lyapunov-like functions are employed to construct globally attracting two-dimensional Lipschitz manifolds. A third example was provided by J. Mallet-Paret and H. L. Smith in [8] for monotone cyclic feedback systems thanks to the existence of a discrete-valued Lyapunov functional. The ideas contained in this work have even been applied to delay differential systems and one-dimensional scalar parabolic equations.

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Very recently we have obtained in [9] a new result of this nature. It is based on an extended notion of monotone flow with respect to some generalized cones called cones of rank 2. These cones were already considered by M. A. Krasnoselskii *et al.* in [7] to study spectral properties of certain operators (see also [3]). We have taken advantage of such spectral properties, in combination with the theory of invariant manifolds and the Closing Lemma, to establish that some limit sets of these flows are essentially two-dimensional. A first consequence is that we provide a new theoretical framework which is intended to encompass the aforementioned results by H. L. Smith and R. A. Smith.

In the present paper we initiate some further developments of the theory described in [9] in order to get deeper insights into the behavior of these new monotone flows. Concretely we just deal here with the problem of the existence of periodic orbits. To do that we require the flow to be dissipative and to have a unique equilibrium with certain instability properties. The existence of at least one periodic orbit will follow from the Poincaré-Bendixson property proved in [9] and the existence of a convenient locally invariant manifold at the equilibrium point. We shall also show how our results can be employed in practice by studying the monotonicity induced by indefinite quadratic forms and applying it to a particular four-order autonomous scalar equation. This application is meant to be merely illustrative and we leave more detailed analysis of more relevant models for future works.

Let us outline how this paper is organized. Next section is just devoted to summarize the main results of [9]. This includes the introduction of the cones of rank 2 and the corresponding notion of monotone flow. We also present the key properties enjoyed by these systems proved in that paper.

In the third section we give our result on existence of periodic orbits. To do this we have to show some implications of the monotonicity over the local structure around an equilibrium point.

Section 4 is devoted to introduce what we call P-cooperative systems. These are systems that are monotone with respect to generalized orders induced by a indefinite matrix P. We put the emphasis on the computational aspects which this monotonicity notion entails.

In the final section we check the criterion developed in sections 3 and 4 for a four-order autonomous equation.

2. C-cooperative systems

We consider a general autonomous system

$$\dot{X} = F(X), \quad X \in \mathbb{R}^N \tag{1}$$

where F is a smooth vector field defined in $\mathbb{R}^N$. The semiflow induced by (1) is denoted by $\Phi(t, p)$, and we assume for simplicity that it is defined for all $t \geq 0$ and $p \in \mathbb{R}^N$.

Given a solution $X(t) = \Phi(t, p)$ of (1), its positive semiorbit is the set

$$\mathrm{O}^+(p) = \{X(t) : t \geq 0\}.$$

If $X(t)$ is defined for all $t \in \mathbb{R}$ then

$$\mathrm{O}(p) = \{X(t) : t \in \mathbb{R}\}$$

is called the orbit of $X(t)$.

Constant solutions of (1) are of the form $X(t) \equiv p$ where $F(p) = 0$. The point p is then said to be an equilibrium point of the system.

The orbit of nonconstant periodic solutions is an oriented simple closed curve. We call it a periodic orbit of (1).

Our aim in this section is to recall the results of [9] that will be employed later on. We begin with a basic definition.

Definition 1. *A subset $C \subset \mathbb{R}^N$ is said a cone of rank k if*

1. *It is closed.*
2. *It is homogeneous, i. e. $x \in C, \lambda \in \mathbb{R} \Rightarrow \lambda x \in C$.*
3. *It contains a subspace of dimension k but no subspace of dimension greater than k.*

C is said k-solid if there is a subspace H of dimension k with $H - \{0\} \subset \overset{\circ}{C}$. It is said complemented if there is a subspace H^c of dimension $N-k$ verifying $H^c \cap C = \{0\}$.

Next definition looks like somewhat technical, but it has an amenable expression in concrete cases as we shall see in section 4. We denote by $F'(X)$ the derivative of the vector field F.

Definition 2. *System (1) is C-cooperative if the following condition is fulfilled:*

Let $p, q \in \mathbb{R}^n$ and define the matrices

$$A^{pq}(t) = \int_0^1 F'(s\Phi(t,p) + (1-s)\Phi(t,q))ds$$

and $U^{pq}(t)$ the solution of

$$\dot{U} = A^{pq}(t)U, \ U(0) = I.$$

Then

$$U^{pq}(t)C - \{0\} \subset \overset{\circ}{C} \quad \text{for all } t > 0. \tag{2}$$

Remark 1. *Inclusion (2) is referred as that matrices $U^{pq}(t)$ are strongly positive with respect to C.*

In next theorem we recall the main properties of solutions of C-cooperative systems when $k = 2$. We suppose then that system (1) is C-cooperative where C is a cone of rank 2 that is 2-solid and complemented.

Theorem 1. *Let $X(t)$ be a nonconstant solution of (1) such that $\dot{X}(t_0) \in C$ for some $t_0 \geq 0$.*

I) Invariance Property*: $\dot{X}(t) \in \overset{\circ}{C}$ for all $t > t_0$.*

II) Poincaré-Bendixson Property*: If in addition $X(t)$ is bounded in $[0, +\infty[$ and its omega-limit set Ω has no equilibrium points, then Ω is a periodic orbit.*

Property I follows from (2) and the identity

$$\dot{X}(t) = U^{X(t_0)X(t_0)}(t)(\dot{X}(t_0)). \tag{3}$$

Property II is theorem 1 in [9].

Remark 2. *Property I) is inherited by any nonconstant solution $Y(t)$ belonging to the omega-limit set of $X(t)$. To see that let us fix $s \in \mathbb{R}$ arbitrary. There exists a sequence $\{t_n\} \to +\infty$ such that $\{X(t_n)\} \to Y(s)$. On the other hand*

$$\{\dot{X}(t_n)\} = \{F(X(t_n))\} \to F(Y(s)) = \dot{Y}(s).$$

Since $\dot{X}(t_n) \in \overset{\circ}{C}$ for large n we have that $\dot{Y}(s) \in C$. Furthermore since s is arbitrary and applying again (3) and Property I we actually have that $\dot{Y}(s) \in \overset{\circ}{C}$.

In consequence for a nonconstant solution $X(t)$ only two possibilities occur: either $\dot{X}(t) \in \overset{\circ}{C}$ for all t large enough or $X(t) \in \mathbb{R}^N - C$ for all $t \in \mathbb{R}$. We will say that $X(t)$ is eventually infinitesimally ordered in the first case and that it is infinitesimally unordered in the second case.

Finally we stress another result that was extremely important for proving the Poincaré-Bendixson property in [9] and which will play a big role hereinafter. Its proof can be found in [3] and [7].

Theorem 2 (**Perron-Frobenius Property**)**.** *Let $U^{pq}(t)$ be the operators introduced in definition 2 and consider $L = U^{pq}(t_0)$ for certain fixed $t_0 > 0$. Let the spectrum of L be*

$$\mathrm{Sp} = \{\mu_1, \ldots, \mu_N\}$$

where $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_N|$. Then

$$|\mu_2| > |\mu_3|. \tag{4}$$

In addition if H and H^c are the (generalized) eigenspaces of L associated to $\{\mu_1, \mu_2\}$ and $\{\mu_3, \ldots, \mu_N\}$ respectively, then it holds that

$$H - \{0\} \subset \overset{\circ}{C} \quad \text{and} \quad H \cap C = \{0\}. \tag{5}$$

Inequality (4) is usually known as a spectral gap property.

3. Existence of periodic orbits

The Poncaré-Bendixson property described in theorem 1 yields to the existence of a nontrivial periodic orbit as soon as there is one infinitesimally ordered bounded solution verifying that its omega-limit set has no equilibrium points. We prove that this happens in a standard situation.

We first need to introduce some well-known definitions.

Definition 3. *System (1) is said dissipative if there exists a bounded set $D \subset \mathbb{R}^N$ such that for each $p \in \mathbb{R}^N$ there is $t_0 > 0$ such that $\Phi(t, p) \in D$ for all $t > t_0$.*

In particular all solutions of a dissipative system are bounded.

Definition 4. *An equilibrium point $p_0 \in \mathbb{R}^N$ of (1) is stable if for any open neighborhood U of p_0 there exists another open neighborhood V of p_0 such that for each $p \in V$ there exists $t_0 > 0$ verifying that $\Phi(t, p) \in U$ for all $t > t_0$. We say that p_0 is unstable if it is not stable.*

Recall that we denote the derivative of F at a point $X \in \mathbb{R}^N$ by $F'(X)$.

Definition 5. *An equilibrium point $p_0 \in \mathbb{R}^N$ of (1) is said hyperbolic if no eigenvalue of $F'(p_0)$ has zero real part.*

Remark 3. *For an hyperbolic equilibrium p_0 to be unstable it is necessary and sufficient that $F'(p_0)$ has at least one eigenvalue with positive real part.*

We state now the main result of this paper.

Theorem 3. *Let us suppose that system (1) is C-cooperative, dissipative and has a unique equilibrium point p_0 that in addition is hyperbolic and unstable. Then it has at least one nontrivial periodic orbit.*

In the rest of the section we assume the hypotheses of this theorem in order to prove it. The key point is the local structure of the flow around the equilibrium point p_0.

We first study how many eigenvalues with positive real parts can exist.

Proposition 1. *The number of eigenvalues of $F'(p_0)$ having positive real part is even.*

Proof: This proposition follows by the same argument employed in Theorem 52.1 in [6]. We outline it here for the reader's convenience. Since system (1) is supposed dissipative the topological degree over large balls of $I - \Phi(t, \cdot)$ equals 1 for all t large enough. The maps $I - \Phi(t, \cdot)$ and $-F$ are homotopic over these balls and so $-F$ has degree 1 as well. Since this vector field has an unique zero at p_0 that in addition is not degenerate, we can assert that $\text{sgn}(\text{Det } (-F'(p_0))) = 1$. Obviously this sign must be $(-1)^m$ where m is the number of real positive eigenvalues of $F'(p_0)$. Thus m must be even. Since non-real complex eigenvalues appear in pairs the proposition is proved.$\square$

From this proposition and the instability hypotheses over p_0 directly we can state:

Corollary 1. *If the spectrum of $F'(p_0)$ is*

$$\mathrm{Sp}(F'(p_0)) = \{\lambda_1, \dots, \lambda_N\} \quad \textit{with} \quad \mathrm{Re}(\lambda_i) \geq \mathrm{Re}(\lambda_j) \textit{ for } i < j,$$

then

$$\mathrm{Re}(\lambda_1),\ \mathrm{Re}(\lambda_2) > 0. \tag{6}$$

In consequence the local unstable manifold W^u has dimension at least 2. Let us call W^s, in case that it exists, to the local stable manifold at p_0.

On the other hand we can take $p = q = p_0$ in definition 2 and so we know that the matrix solution $U^{p_0}(t) = U^{p_0 p_0}(t)$ of the initial value problem

$$\dot{U} = F'(p_0)U, \quad U(0) = I$$

satisfies in particular $U^{p_0}(1)C - \{0\} \subset \overset{\circ}{C}$.

By the Perron-Frobenius property we have that $|\mu_2| > |\mu_3|$ where the spectrum of $U^{p_0}(1)$ is

$$\mathrm{Sp}(U^{p_0}(1)) = \{\mu_1, \mu_2, \dots, \mu_N\} \quad \text{with} \quad |\mu_i| \geq |\mu_j| \text{ for } i < j.$$

Since $U^{p_0}(1) = \mathrm{Exp}(F'(p_0))$ we can assert that

$$\mathrm{Re}(\lambda_2) > \mathrm{Re}(\lambda_3) \tag{7}$$

(recall that λ_i's are the eigenvalues of $F'(p_0)$ defined in corollary 1).

In addition let us call Π_1 and Π_2 to the generalized eigenspaces associated to $\{\lambda_1, \lambda_2\}$ and $\{\lambda_3, \dots, \lambda_N\}$ respectively. These eigenspaces are the same that those appearing in the splitting of the matrix $U^{p_0}(1)$ in the Perron-Frobenius Property. Therefore we deduce that

$$\Pi_1 - \{0\} \subset \overset{\circ}{C} \quad \text{and} \quad \Pi_2 \cap C = \{0\}. \tag{8}$$

The spectral gap (7) is rather important since it leads to distinctions among different invariant manifolds around the equilibrium point according to the rate of convergence to it. Concretely in our situation we can establish:

Theorem 4. *There exists a smooth locally invariant manifold W_1 containing p_0 satisfying:*

i) $\dim(W_1) = 2$.
ii) *The tangent space to W_1 at p_0 is Π_1.*
iii) *For each $r \in]\lambda_3, \lambda_2[$ and any p lying on W_1 there exists $M > 0$ such that*

$$|\Phi(t, p) - p_0| \leq M e^{rt} \quad \textit{for all } t < 0.$$

In particular $W_1 \subset W^u$.

Proof: This theorem can be deduced either from Lemma 5.1 of [4] or Theorem 4.1 of [2]. Notice that this last result is stated under the hypothesis $\mathrm{Re}(\lambda_2) < \mathrm{Re}(\lambda_3)$ instead of (7). The fact that we can apply it just follows through a time-reversal argument.□

Corollary 2. *If $X(t)$ is a solution of (1) whose orbit lies in $W_1 - \{p_0\}$ then*

$$\dot{X}(t) \in \overset{\circ}{C} \quad \textit{for all } t \in \mathbb{R}.$$

Proof: Due to the Perron-Frobenius property we can assert that the tangent plane Π_1 to W_1 at p_0 satisfies that $\Pi_1 - \{0\} \subset \overset{\circ}{C}$. Obviously the same inclusion is fulfilled for any subspace that is close enough to Π_1, in particular for the tangent spaces to W_1 at points near p_0. Since $X(t)$ tends to p_0 as $t \to -\infty$ we deduce that $\dot{X}(t_0) \in \overset{\circ}{C}$ for all t near $-\infty$. The Invariance Property directly gives the corollary.□

A similar property is enjoyed by the stable manifold.

Corollary 3. *If W^s is nontrivial, then any solution $X(t)$ lying in $W^s - \{p_0\}$ satisfies that $\dot{X}(t) \in \mathbb{R}^N - C$ for all t.*

Proof: Recall that Π_2 the eigenspace associated to $\{\lambda_3, \dots, \lambda_N\}$ verifies that $\Pi_2 \cap C = \{0\}$. On the other hand the tangent space of W^s at p_0 is contained in Π_2. From this we can reason as in the preceding proof to obtain this corollary.□

Proof of theorem 3: Let us take $X(t)$ any solution starting at W^1 and prove that its omega-limit set does not contain p_0. Otherwise the Butler-McGhee lemma (see [1]) would imply that the omega-limit set of $X(t)$ contains an orbit $Y(t)$ lying in W^s. Corollary 2 and remark 2 then contradict corollary 3.□

Once we have proved our main theorem it is time to prove its usefulness in concrete applications. To do that we introduce in next section a cone defined by means of indefinite forms which provides a quite flexible notion of monotone flow.

4. P-cooperative systems

We work in $\mathbb{R}^N$ with $N \geq 3$. The usual scalar product of vectors $x, y \in \mathbb{R}^N$ will be denoted by $\langle x, y \rangle$.

Let us consider P a symmetric matrix having 2 negative eigenvalues and $N-2$ positive eigenvalues. The associated indefinite bilinear form is given by $Q(X) = \langle X, PX \rangle$.

Let us define the set

$$C = C(P) = \{X \in \mathbb{R}^N : Q(X) \leq 0\}.$$

Proposition 2. *C is a cone of rank 2. In addition it is complemented and 2-solid.*

Proof: The continuity of Q firstly implies that C is closed by definition. The equality $Q(\alpha x) = \alpha^2 Q(X)$ for every $\alpha \in \mathbb{R}^N$ gives the homogeneity. Let us call H (resp. H^c) to the 2-dimensional (resp. $(N-2)$-dimensional) linear subspace associated to the negative (resp. positive) eigenvalues of P. It holds that

(9) $Q(X) < 0$ for all $X \in H - \{0\}$, $\qquad Q(X) > 0$ for all $X \in H^c - \{0\}$.

Again the continuity of Q says that

$$H - \{0\} \subset \overset{\circ}{C}, \qquad H^c \cap C = \{0\}.$$

From these inclusions the proposition easily follows. □

Now let us consider again the autonomous system (1). The transpose of a matrix A is denoted by A^*.

Proposition 3. *System (1) is C-cooperative provided that for every $X \in \mathbb{R}^N$ there exists $\lambda(X) \in \mathbb{R}$ such that the matrices*

$$F'(X)^*P + PF'(X) + \lambda(X)P$$

are negative definite for all $X \in \mathbb{R}^N$.

Proof: See Proposition 7 and the afterwards discussion in [9]. □

Remark 4. *When the preceding proposition holds true we will say that system (1) is P-cooperative in order to explicit the role of matrix P.*

Consequently we get:

Corollary 4. *If system (1) is P-cooperative then Theorem 3 can be applied.*

Let us study some basic facts on the concept of P-cooperativeness. Given the matrix P as above, we define

$$\mathbb{M}_P = \{A : A^*P + PA + \lambda P < 0 \text{ for certain } \lambda \in \mathbb{R}\}.$$

Every matrix A belonging to $\mathbb{M}_P$ will be said P-cooperative as well.

For a fixed P-cooperative matrix A let us define

$$\Lambda = \{\lambda \in \mathbb{R} : A^*P + PA + \lambda P < 0\}.$$

Proposition 4. Λ *is an open interval.*

Proof: It is obvious that Λ is open. On the other hand take $\lambda_1 < \lambda_2$ belonging to Λ and let us prove that any $\lambda \in]\lambda_1, \lambda_2[$ also belongs to Λ. If $X \in \mathbb{R}^N$ and $Q(X) < 0$ then

$$\langle X, (A^*P + PA)X\rangle + \lambda\langle X, PX\rangle = \langle X, (A^*P + PA)X\rangle + \lambda_1\langle X, PX\rangle + (\lambda - \lambda_1)Q(x) < 0. \tag{10}$$

A similar argument in the case that $Q(X) \geq 0$ employing now λ_2 in the middle term of (10) proves the proposition.□

Next lemma follows directly from the definitions.

Lemma 1. *Next properties hold true:*

i) $\mathbb{M}_P$ *is convex.*
ii) *If U is a $N \times N$ invertible matrix, then $U^{-1}\mathbb{M}_P U = \mathbb{M}_{U^*PU}$.*

Remark 5. *Item i) of the las lemma implies for instance that if the set $\{F'(X) : X \in \mathbb{R}^N\}$ is a segment of matrices then the property of being C-cooperative will be fulfilled simply provided that the extreme matrices of the segment are C-cooperative. On the other hand item ii) says how the P-cooperativeness property transforms through linear changes of coordinates.*

We finish the section with a discussion on some computational aspects of the P-cooperativeness in the particular case

$$P = P_\alpha = \begin{pmatrix} -\alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\alpha & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}, \qquad \alpha > 0.$$

The parameter α measures the aperture of the corresponding cone C: the larger α is the bigger the cone becomes. Let us take $A = (a_{ij})$ of order N. Then the matrix $Q(\lambda) = A^* P_\alpha + P_\alpha A + \lambda P_\alpha$ is equal to

$$\begin{pmatrix} -\alpha(2a_{11}+\lambda) & -\alpha(a_{12}+a_{21}) & -\alpha a_{13}+a_{31} & \cdots & -\alpha a_{1N}+a_{N1} \\ -\alpha(a_{12}+a_{21}) & -\alpha(2a_{22}+\lambda) & -\alpha a_{23}+a_{32} & \cdots & -\alpha a_{2N}+a_{N2} \\ -\alpha a_{13}+a_{31} & -\alpha a_{23}+a_{32} & 2a_{33}+\lambda & \cdots & a_{3N}+a_{N3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha a_{1N}+a_{N1} & -\alpha a_{2N}+a_{N2} & a_{3N}+a_{N3} & \cdots & 2a_{NN}+\lambda \end{pmatrix}.$$

Let us call $Q_j(\lambda)$ the submatrix of $Q(\lambda)$ formed by the first j's rows and columns for $j = 1, \ldots, N$. Let $p_j(\lambda)$ be the determinant of $Q_j(\lambda)$. Clearly $p_j(\lambda)$ are polynomials of degree j that in addition have positive leading coefficients for $j > 1$.

Next lemma expresses the condition for $Q(\lambda)$ to be negative definite through a very well-known criterion involving polynomials $p_j(\lambda)$.

Lemma 2. *Matrix $Q(\lambda)$ is negative definite for certain $\lambda = \lambda_0$ if and only if*

$$\operatorname{sign}(p_j(\lambda_0)) = (-1)^j.$$

Actually we give a more precise description of the situation in next statement.

Proposition 5. *Matrix A of order N is P_α-cooperative if and only if $p_j(\lambda)$ has two roots $\mu_1^j, \mu_2^j \in \mathbb{R}$ for $j = 2, \ldots, N$ satisfying*

(1) $\mu_1^2 \le \mu_2^2 \le \mu_1^3$
(2) $\mu_1^j \le \mu_1^{j+1} < \mu_2^{j+1} \le \mu_2^j$ *for* $j = 3, \ldots, N$.
(3) $\operatorname{sign}(p_j(\lambda)) = (-1)^j$ *on* $]\mu_1^j, \mu_2^j[$ *for* $j = 3, \ldots, N$.

In addition the permitted values of λ which appear in the definition of P-cooperativeness are those in the interval $]\mu_1^N, \mu_2^N[$.

Proof of proposition 5: It is easy to see that $p_2(\lambda)$ has two roots $\mu_1^2 \le \mu_2^2$ and that

$$p_1(\lambda) < 0, \quad p_2(\lambda) > 0 \Leftrightarrow \lambda > \mu_2^2.$$

Let us see that $p_3(\mu_2^2) \ge 0$. Otherwise for $\lambda(\epsilon) = \mu_2^2 + \epsilon$ with $\epsilon > 0$ small enough we would have that $p_1(\lambda(\epsilon)) < 0, p_2(\lambda(\epsilon)) > 0, p_3(\lambda(\epsilon)) < 0$ and hence $Q_3(\lambda(\epsilon))$ is negative definite. Letting ϵ tend to 0 we have that $Q_3(\mu_2^2)$

is at least negative semidefinite. Since $p_3(\mu_2^2)$ is supposed to be nonzero $Q_3(\mu_2^2)$ is negative definite indeed. But the equality $p_2(\mu_2^2) = 0$ contradicts lemma 2.

From this we deduce that $Q_3(\lambda)$ is negative definite if and only if $p_3(\lambda)$ has two zeroes $\mu_1^3 < \mu_2^3$ in $[\mu_2^2, +\infty[$ such that $p_3(\lambda) < 0$ in $]\mu_1^3, \mu_2^3[$.

Likewise $p_4(\lambda)$ takes nonpositive values at μ_1^3 and μ_2^3. Therefore $Q_4(\lambda)$ is negative definite for certain λ if and only if $p_4(\lambda)$ has two roots $\mu_1^4 < \mu_2^4$ in $[\mu_1^3, \mu_2^3]$ such that $p_4(\lambda) > 0$ in $]\mu_1^4, \mu_2^4[$. Simply reiterating this argument the proposition follows.□.

Remark 6. *Notice that due to proposition 4 the roots μ_i^j in the preceding proposition are unique.*

5. Application to a scalar four-order equation

Just to exemplify our results we consider the equation

$$x^{iv)} + 2x''' + 2x'' + 2x' + x = f(x) \tag{11}$$

The characteristic values of the equation

$$x^{iv)} + 2x''' + 2x'' + 2x' + x = 0$$

are $\pm i$ and -1 with multiplicity 2. So we expect that the certain nonlinearities $f(x)$ provoke the appearance of periodic orbits. To be precise we assume that $f(x)$ satisfies:

i) f is continuously derivable.
ii) $f(x) = x \Leftrightarrow x = 0$.
iii) $f'(0) < 0$.
iv) $\lim_{|x|\to\infty} \frac{f(x)}{x} = L \in]0, 1[$.

Hypothesis ii) ensures that $x = 0$ is the only equilibrium point of (11). Hypothesis iii) implies that this equilibrium point is unstable. Finally hypothesis iv) means that (11) can be rewritten as

$$x^{iv)} + 2x''' + 2x'' + 2x' + (1 - L)x = g(x) \tag{12}$$

where $g(x)$ is continuous and bounded. Since now the right hand of (12) is stable a straightforward argument proves that (11) is dissipative.

We are going now to impose (11) to be P-cooperative with respect to certain matrix P. This will always be possible as far as $|f'(x)|$ is small enough. To get concrete estimates let us rewrite equation (11) as an equivalent four-dimensional system as follows:

$$\dot{X} = AX + G(X), \tag{13}$$

where $X = (x, x', x'', x''')$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{pmatrix}$$

and $G(X) = (0, 0, 0, f(x))^*$.

Define

$$w_+ = \sup\{f'(x) : x \in \mathbb{R}\}, \quad w_- = \inf\{f'(x) : x \in \mathbb{R}\}.$$

According to remark 5 system (13) is P-cooperative provided that matrices

$$A_\pm = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 + w_\pm & -2 & -2 & -2 \end{pmatrix}$$

are so.

The matrix

$$V = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -2 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

induces a change of variables that transform A into its Jordan canonical form, that is

$$U^{-1}AU = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Furthermore we have that

$$B_\pm = U^{-1}A_\pm U = \begin{pmatrix} -w_\pm/2 & 1 & -w_\pm/2 & 0 \\ -1 & 0 & 0 & 0 \\ w_\pm/2 & 0 & -1 + w_\pm/2 & 1 \\ w_\pm/2 & 0 & w_\pm/2 & -1 \end{pmatrix}.$$

We consider matrix P_α defined in the preceding section. By lemma 1 it is enough to prove that $B_\pm$ are both P_α-cooperative to show that $A_\pm$ are P-cooperative with respect another matrix P. So let us implement the computational tool described in that section.

First we compute

$$B_\pm^* P_\alpha + P_\alpha B_\pm + \lambda P_\alpha = \begin{pmatrix} \alpha(w_\pm - \lambda) & 0 & (1+\alpha)w_\pm/2 & w_\pm/2 \\ 0 & -\alpha\lambda & 0 & 0 \\ (1+\alpha)w_\pm/2 & 0 & -2 + w_\pm + \lambda & 1 + w_\pm/2 \\ w_\pm/2 & 0 & 1 + w_\pm/2 & -2 + \lambda \end{pmatrix}$$

We then have the polynomials

$$p_2(\lambda) = \alpha^2\lambda(\lambda - w_\pm),$$

$$p_3(\lambda) = \alpha\lambda[\alpha(\lambda - w_\pm)(-2 + w_\pm + \lambda) - (1+\alpha)^2 w_\pm^2/4]$$

and

$$p_4(\lambda) = -\alpha\lambda \mathrm{Det}\begin{pmatrix} \alpha(w_\pm - \lambda) & (1+\alpha)w_\pm/2 & w_\pm/2 \\ (1+\alpha)w_\pm/2 & -2 + w_\pm + \lambda & 1 + w_\pm/2 \\ w_\pm/2 & 1 + w_\pm/2 & -2 + \lambda \end{pmatrix}$$

Let us take $\alpha = 1$. We give values to $w_\pm$ up to a decimal figure for which proposition 5 applies.

Firstly for $w_+ = 0.2$ we have that

$$p_2(\lambda) = \lambda(\lambda - 0.2), \qquad \mu_2^2 = 0.2,$$

$$p_3(\lambda) = \lambda(\lambda^2 - 2\lambda + 0.4), \qquad \mu_3^1 = 0.2254,\ \mu_3^2 = 1.7746$$

and

$$p_4(\lambda) = \lambda(\lambda^3 - 4\lambda^2 + 3.2\lambda - 0.62), \qquad \mu_4^1 = 0.2936,\ \mu_4^2 = 0.7032.$$

Thus conditions of proposition 5 are fulfilled. Let us see that this does not occur for $w_+ = 0.3$. In fact it is straightforward that $p_4(\lambda)$ would have to have four real roots. But in this case

$$p_4(\lambda) = \lambda(\lambda^3 - 4\lambda^2 + 3.3\lambda - 0.945),$$

and its roots are

$$\lambda_1 = 0,\ \lambda_2 = 0.496 - 0.260i,\ \lambda_3 = 0.496 + 0.260i,\ \lambda_4 = 3.007.$$

Concerning w_- we check the value $w_- = -5.9$ which provides

$$p_2(\lambda) = \lambda(\lambda + 5.9), \qquad \mu_2^2 = 0,$$

$$p_3(\lambda) = \lambda(\lambda^2 - 2\lambda - 11.8), \qquad \mu_3^1 = 0,\ \mu_3^2 = 4.5777$$

and

$$p_4(\lambda) = \lambda(\lambda^3 - 4\lambda^2 - 2.9\lambda + 0.295), \qquad \mu_4^1 = 0,\ \mu_4^2 = 0.0906.$$

Again for $w_- = 0.6$ polynomial $p_4(\lambda)$ does not satisfy proposition 5. Actually

$$p_4(\lambda) = \lambda^2(\lambda^2 - 4\lambda - 3)$$

whose roots are

$$\lambda_1 = -0.646,\ \lambda_2, \lambda_3 = 0,\ \lambda_4 = 4.646.$$

This corresponds just to the case in which $\mu_1^4 = \lambda_2$ and $\mu_2^4 = \lambda_3$ coalesce into one double root.

So corollary 4 implies the existence of a periodic orbit provided that $-5.9 \leq f'(x) \leq 0.2$ for all $x \in \mathbb{R}$. Notice that in particular in the definition of P-cooperative matrix we must take λ in the interval $]0.2936, 0.703[$ for A_+ and in the interval $]0, 0.0906[$ for A_-. This means that we cannot take the same λ for both matrices, which makes a strong difference with the theory developed in [12, 13].

A natural question is if we can improve the bounds above by choosing conveniently the parameter α. We have explored this possibility and as far as w_+ is concerned no really good improvement can be achieved. The reason for this is the closeness between the characteristic values -1 and $\pm i$ for the unperturbed system and the fact that in the region where $f'(x)$ becomes positive these characteristic values get closer and closer until the spectral gap property (4) is violated.

On the other hand we do have succeeded in improving the lower bound for $f'(x)$. For instance we have obtained the condition $f'(x) \in]-11.9, .2[$ by taking $\alpha = 2$ whereas $f'(x) \in]-8.5, 0.2[$ if $\alpha = 2.5$. It seems that the best choice must be a value of α around 2. To accomplish such sharper estimates requires a deeper study of the P-cooperativeness condition on the line explained in proposition 5. We pretend to work out these kind of improvements in more specific papers to be done in the future.

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