Global bifurcations from the center of mass in the Sitnikov problem

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Dedicated to Professor Peter Kloeden

Abstract

The Sitnikov problem is a restricted three body problem where the eccentricity of the primaries acts as a parameter. We find families of symmetric periodic solutions bifurcating from the equilibrium at the center of mass.

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1 Introduction

The Sitnikov problem is a restricted three-body problem where two primaries of equal mass move in elliptic orbits lying on the plane x, y and the particle of zero mass moves on the z axis. In appropiate units the equation of motion of the small particle is

$$\ddot{z} + \frac{z}{(z^2 + r(t,e)^2)^{3/2}} = 0, \tag{1}$$

where $e \in [0, 1[$ is the eccentricity of the ellipses described by the primaries and $r(t, e)$ denotes the distance from the primaries to the origin (center of mass). The function $r(\cdot, e)$ has minimal period 2π and is implicity defined in terms of Kepler's equation, namely

$$r = \frac{1}{2}(1 - e\cos u), \quad u - e\sin u = t.$$

The dynamics of the periodic equation (1) has been studied by many authors since the sixties and we refer to [1, 15] for the most classical results. Probably one of the first questions that can be posed about (1) is the study of families of periodic solutions which depend continuously on the eccentricity. This question becomes simpler if one is restricted to the symmetric case: families of even or odd periodic solutions. Notice that the function $r(\cdot, e)$ is even an so (1) is invariant under the symmetries

$$(t, z, \dot{z}) \to (-t, z, -\dot{z}), \qquad (t, z, \dot{z}) \to (-t, -z, \dot{z}).$$

In this paper we shall concentrate on the search of even harmonic and subharmonic solutions satisfying

$$z(-t) = z(t), \quad z(t + 2N\pi) = z(t), \quad t \in \mathbb{R},$$

for some $N = 1, 2, \ldots$ Then we are lead to the boundary value problem associated to (1) and the Neumann boundary conditions

$$\dot{z}(0) = \dot{z}(N\pi) = 0. \tag{2}$$

This problem has been considered in several papers and the basic approach has been the continuation from the circular problem (see [16], [2], [3],[4], [19], [18]). For $e = 0$ the primaries move on a common circumference of radius $r(t,0) = 1/2$. Now the equation is autonomous and the boundary value problem for $e = 0$ can be studied via a phase portrait analysis. The solutions with $z(0) > 0$ are labelled according to its number of zeros, going from $p = 1$ to $\nu_N := [2\sqrt{2}N]$. The continuation of these solutions to families defined for small $e > 0$ can be achieved with the techniques of the theory of local bifurcation. The method of global continuation of Leray and Schauder has been applied more recently in [10] and its has been shown that some of these families can be continued to all the values of the eccentricity. These families are labelled according to the number of zeros in the same fashion as it occurs in the well known work by Rabinowitz [17] for other non-linearities. We also refer to [5] for recent applications of the method of global continuation in classical mechanics. From a numerical point of view, a global study of the Neumann and Dirichlet problem for (1) can be found in [8]. This paper contains suggestive diagrams where the families of periodic solutions are sketched. Besides the families described above a new type was found in [8]. These new families arise as bifurcations from the equilibrium $z = 0$ at certain positive values of the eccentricity, say $e = e_*$, and they are continued for all larger values $e_* < e < 1$. The main purpose of our paper is to provide a rigorous analysis of properties of these families. Our study is organized in three steps that we now describe.

i) ***Linearization at z=0.*** The Sturm Liouville problem

$$\ddot{y} + \frac{1}{r(t,e)^3} y = 0, \quad \dot{y}(0) = \dot{y}(N\pi) = 0, \tag{3}$$

is non-standard since it has a nonlinear dependence with respect to the parameter e. It follows from [12] and [10] that there exists a sequence of critical values $e = E_{n,N}$ resembling the eigenvalues in classical problems. We will refine this sequence and select those values of $E_{n,N}$ with some extra properties required if one expecting a bifurcation for $e > E_{n,N}$.

ii) ***The picthfork bifurcation.*** This is the type of bifurcation one expects at $z = 0$. The standard transversality conditions are not easy to check and we present a slight variant of the theorem on pitchfork bifurcation adapted to our problem.

iii) ***Global continuation.*** The initial step in the application of the Leray-Schauder method is the following observation on the oscillatory properties: the periodic solutions of the nonlinear problem have less zeros than those of the linearized problem (3). As a consequence the critical values $e = E_{n,N}$ act as barriers for the families and this allows us to control their behaviour. This technique is also applicable to the families emerging from the circular problem and we also derive a sharpened version of the main result in [10].

There are other global techniques that can be applied to the periodic problem for (1) and we refer to [21] and [13]. The first paper applies the Poincaré-Birkhoff theorem and the second employs variational techniques. Some of the results obtained in these papers can be deduced as consequences of our study. In particular our result implies that the Sitnikov problem has at least $2N + 1$ even solutions of period $2N\pi$ for every eccentricity $e \in [0, 1[$. Also we can prove a result observed in [8], page 114. The Sitnikov problem has an arbitrarily large number of 2π periodic solutions if e is close enough to 1. Finally we mention a question on the circular families that we cannot answer. The families with number of zeros $p = 1, \dots N$ are defined for all values of the eccentricity but we do not know if this is the case for those with number of zeros between $N + 1$ and ν_N. At the end of the paper we show how to reduce this problem to a question concerning a linear equation of Hill's type.

2 Global families

For each integer $N \geq 1$ consider the boundary value problem

$$\ddot{z} + \frac{z}{(z^2 + r(t,e)^2)^{3/2}} = 0, \quad \dot{z}(0) = \dot{z}(N\pi) = 0. \tag{4}$$

The solutions of this problem can be extended as even periodic solutions of period $2N\pi$.
Given $\xi, \eta \in \mathbb{R}$ and $e \in [0, 1[$ the solution of the Sitnikov equation with initial conditions

$$z(0) = \xi, \quad \dot{z}(0) = \eta,$$

is globally defined and will be denoted by $z(t; \xi, \eta, e)$. Solving (4) is equivalent to finding the zeros of the function

$$F_N : \mathbb{R} \times [0, 1[\to \mathbb{R}, \quad F_N(\xi, e) = \dot{z}(N\pi; \xi, 0, e).$$

This function is real analytic and so the set of zeros

$$\Sigma = \{(\xi, e) : F_N(\xi, e) = 0\} .$$

cannot be too pathological excepting around the boundary $e = 1$. Actually the theorems on the local structure of analytic sets and Puiseux Series are applicable in each point of Σ (see [9] for more details). This observation has many consequences and we now present the simplest.

Proposition 1. ***(i)*** *For each* $e \in [0, 1[$ *the problem (4) has a finite number of solutions.*

(ii) *Every connected subset of* Σ *is arcwise connected.*

Proof. (i) The function $F_N(\cdot, e)$ is real analytic an does not vanish outside a compact set (see Proposition 5.1 in [10]). This implies that $F_N(\cdot, e)$ has a finite number of zeros.
(ii) It follows from theorems 6.1.3 and 4.2.8 in [9] that Σ is locally arcwise connected. The conclusion follows from elementary results in Topology. □

We have included statement (ii) because it shows that the intuitive notion of "branch of solutions" can be formalized as a connected subset of Σ.
From now on we will identify solutions $z(t)$ of the problem (4) with points in the plane (ξ, e), $\xi = z(0)$. A set S of solutions of (4) will be identified to a subset S^* of Σ in the obvious way. For instance, if S_0 is the set of trivial solutions of (4), then

$$S_0^* = \{(0, e) : e \in [0, 1[\,\} .$$

The function F_N is odd in ξ and so Σ is symmetric with respect to the ξ-axis. This leads us to restrict to the region

$$\xi > 0, \quad 0 \le e < 1.$$

The set of zeros of F_N lying in this region will be denoted by Σ^+.

Definition 1. *A global family of non-trivial solutions is a set* $\mathcal{F}$ *of solutions of (4) such that* $\mathcal{F}^*$ *is a connected component of* Σ^+.

In view of Proposition 1 we observe that every global family is arcwise connected. Another important property is the invariance of the number of zeros along the family. By uniqueness of the initial value problem, the zeros of the non-trivial solutions must be non-degenerate. This explains why the number of zeros remains constant along a connected family. This is a very classical observation already employed by Rabinowitz [17].

We are ready to state the main result.

Theorem 1. *For each* $N \geq 1$ *there exists a number* $p_0 = p_0(N) > 1$ *and global families of non-trivial solutions* $\mathcal{F}_{p,N}$*,* $p = p_0, p_0 + 1, \dots$ *satisfying*

(i) $\mathcal{F}_{p,N}$ *has* p *zeros in* $[0, N\pi]$*.*

(ii) $cl(\mathcal{F}^*_{p,N}) \bigcap S^*_0 = \left\{(0, E^*_{p,N})\right\}$*.*

(iii) $Proy_2(\mathcal{F}^*_{p,N}) =]E^*_{p,N}, 1[$

where $\left\{E^*_{p,N}\right\}_{p \geq p_0}$ *is a strictly increasing sequence converging to* 1 *as* $p \to \infty$*.*

The notation cl employed in (ii) refers to the closure of a set relative to $[0, \infty[\times[0, 1[$. In (iii) we have used the notation $Proy_2(\xi, e) = e$.
By (i) the families $\mathcal{F}_{p,N}$ have different number of zeros and so they are pairwise disjoint. The condition (ii) shows that the family bifurcates from the center of mass $\xi = 0$ at the some value of the eccentricity $e = E^*_{p,N}$. The condition (iii) says that the horizontal line $e = E^*_{p,N}$ acts as a barrier for the family. The next figure is in agreement with the previous result and also with the qualitative features observed in the numerical study developed in [8]. However we have no reason to excluded more complicated situations: Secondary bifurcations, loss of monotonicity away from $\xi = 0$, unbounded branches as $e \uparrow 1$,...

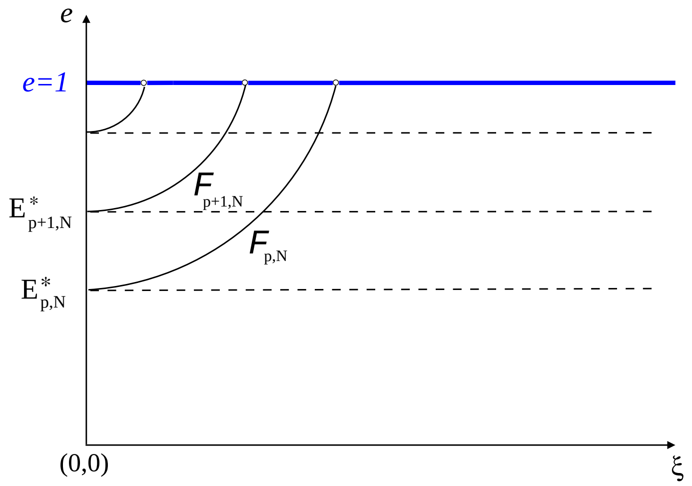

Figure 1: Families of periodic orbits emanating from the equilibrium.

The numbers $E^*_{p,N}$ will be obtained via a theoretical characterization and can be computed numerically. The first bifurcation for $N = 1$ occurrs at

$$E^*_{3,1} = 0,5444689...$$

We finish this section with a consequence of the above theorem.

Corollary 1. *Given $\alpha \geq 1$ there exists $e_* = e_*(\alpha) \in [0, 1[$ such that (1) has at least α even 2π-periodic solutions if $e \geq e_*$.*

3 The linear problem

Consider the problem of Sturm-Liouville type

$$\ddot{y} + \frac{1}{r(t,e)^3} y = 0, \quad \dot{y}(0) = \dot{y}(N\pi) = 0. \tag{5}$$

The difference with respect to classical problems is that the dependence with respect to the parameter e is nonlinear. Indeed problems with nonlinear dependence have been considered in [7] and [21] but in those works the equation depends monotonically upon the parameter, a condition that is not satisfied by $r = r(t, e)$.

Following [12] and [10] we know that there exists a sequence $\{E_{n,N}\}_{n\geq 1}$ with

$$0 < E_{1,N} < E_{2,N} \cdots < E_{n,N} < \cdots < 1, \quad \lim_{n\to\infty} \{E_{n,N}\} = 1,$$

such that the problem (5) has a non-trivial solution whenever $e = E_{n,N}$ for some n. It is also known that the number of zeros of this solution tends to infinity as $n \to \infty$.

The ideas in [14] and [12] can be applied to compute numerically the numbers $E_{n,N}$. The main observation in those works is that the Fourier expansion of solutions of (5) can be obtained via a linear recurrence. An alternative method can be applied after a change of independent variable. When the time is replaced by the eccentric anomaly,

$$u - e \sin u = t, \quad y(u) = y, \quad y' = \frac{dy}{du},$$

the problem (5) becomes

$$(1 - e \cos u)y'' - (e \sin u)y' + 8y = 0, \quad y'(0) = y'(N\pi) = 0.$$

This is an equation of Ince type (see [11]) and the advantage is that now the coefficients are elementary functions. It is now easy to solve it numerically via the shooting method and some standard software. As an example the reader can check that $E_{1,1} = 0.5444689...$, (see [12]) and the corresponding solution has three zeros in $[0, \pi]$.

From now on we employ the notation $y_1(t, e)$ and $y_2(t, e)$ to refer to the solutions of

$$\ddot{y} + \frac{1}{r(t, e)^3} y = 0, \tag{6}$$

satisfying the initial conditions

$$y_1(0) = 1, \quad \dot{y}_1(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad \dot{y}_2(0) = 1,$$

respectively. Notice that $\dot{y}_1(N\pi, e) = 0$ is equivalent to $e = E_{n,N}$ for some n.

The numbers $E_{n,N}$ resemble the classical eigenvalues of the Sturm-Liouville problems. We will extract a subsequence of $\{E_{n,N}\}_{n\geq 1}$ leading to a more refined analogy with the eigenvalues.

Lemma 1. *Assume that $N \geq 1$ and let $p_0 = p_0(N)$ be the minimum number of zeros that a solution of (5) can have. Then*

(i) $p_0(N) > N$.

(ii) *For each $p \geq p_0$ there exists some $e \in [0, 1[$ such that the problem (5) has a solution with p zeros in $[0, N\pi]$. Define*

$$E^*_{p,N} = \sup \Big\{ e \in [0,1[: \ (5) \textit{ has a solution with } p \textit{ zeros in} \, [0, N\pi] \Big\}.$$

Then $\{E^*_{p,N}\}_{p \geq p_0}$ is a subsequence of $\{E_{n,N}\}_{n \geq 1}$ that is strictly increasing and such that $\lim_{p\to\infty} E^*_{p,N} = 1$.

(iii) *For each $p \geq p_0$ there exists $\epsilon > 0$ such that*

$$(-1)^p \, \dot{y}_1(N\pi, e) < 0 \quad \textit{if} \quad E^*_{p,N} < e \leq E^*_{p,N} + \epsilon.$$

Proof. We introduce polar coordinates in (5)

$$y = \rho \cos\theta, \quad \dot{y} = \rho \sin\theta,$$

leading to the equation for the argument

$$\dot{\theta} = -\frac{1}{r(t,e)^3} \cos^2\theta - \sin^2\theta.$$

Let $\theta(t, e)$ be solution with initial condition $\theta(0) = 0$. The function $e \in [0, 1[\mapsto \theta(N\pi, e)$ is real analytic and the numbers $E_{n,N}$ are solutions of the inclusion $\theta(N\pi, e) \in \pi\mathbb{Z}$. The angle $\theta(t, e)$ rotates in the clockwise sense ($\dot{\theta} < 0$) and so the solutions of (5) corresponding to

$$\theta(N\pi, e) = -k\pi, k = 1, 2 \ldots$$

have precisely k zeros. We are ready to prove the lemma with the help of the argument. To prove (i) we observe that $r(t, e) < 1$ everywhere and so $\theta(t, e)$ satisfies the differential inequality

$$\dot{\theta} < -\cos^2\theta - \sin^2\theta = -1.$$

Thus, $\theta(N\pi, e) < -N\pi$ and so the number of crossings with the vertical axis $\big(\theta \in \pi/2 + \pi\mathbb{Z}\big)$ is greater than N. This implies that every solution of (5) has more than N zeros.

To prove (ii) we first observe that the number of zeros in $[0, N\pi]$ of $y_1(t, e)$ tends to infinity as $e \uparrow 1$. This was proven in Proposition 6.4 of [10]. Each zero produces a crossing of the orbit $(y, \dot{y})$ with the vertical axis and since the rotation is always clockwise we conclude that

$$\lim_{e \to 1^-} \theta(N\pi, e) = -\infty.$$

Now we can deduce that $\theta(N\pi, e)$ takes each value $-p\pi$, $p \geq p_0$, for some value of the eccentricity but not for e close to 1. Then $E^*_{p,N}$ can be characterized as the largest root of the equation

$$\theta(N\pi, e) = -p\pi.$$

Finally, to prove (iii) we observe that

$$\theta(N\pi, e) < -p\pi, \quad \text{if} \quad e > E^*_{p,N}.$$

Taking into account the definition of θ we observe that for $e > E^*_{p,N}$ close enough to $E^*_{p,N}$, the point $\big(y_1(N\pi, e), \dot{y}_1(N\pi, e)\big)$ lies in the fourth quadrant if p is even and in the second if p is odd. This implies (iii). $\square$

In the figure 2 we present a hypothetical situation for $p = 3$. Notice that the orbit $(y, \dot{y})$ could have self-intersections.

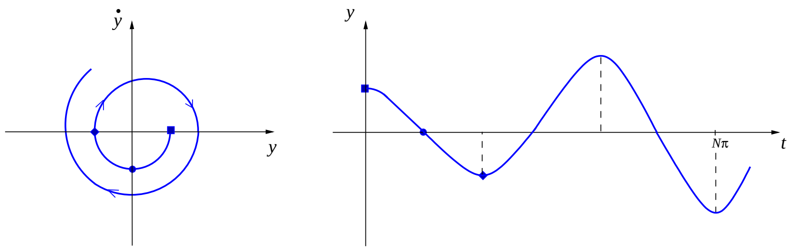

Figure 2: The orbit $(y, \dot{y})$ for the case $p = 3$.

Remark. The proof of (i) can be obtained via Sturm Comparison Theory. The theoretical estimate $p_0(1) \geq 2$ probably is not sharp. Numerical computations suggest that $E^*_{3,1} = E_{1,1}$ and $p_0(1) = 3$, but we do not have a proof of this fact.

4 Looking for a pitchfork bifurcation

The equation

$$F_N(\xi, e) = \dot{z}(N\pi; \xi, 0, e) = 0,$$

has the trivial branch $\xi = 0, e \in [0, 1[$ and any possible bifurcation must occur at a value of the eccentricity $e = e_*$ where the derivative

$$\frac{\partial F_N}{\partial \xi}(0, e_*) = \dot{y}_1(N\pi, e_*)$$

vanishes. This implies that $e_* = E_{n,N}$ for some $n \geq 1$. The function F_N is odd in ξ and it seems plausible to expect a pitchfork bifurcation at $(0, e_*)$. However this require to verify the transversality conditions

$$\frac{\partial^2 F_N}{\partial \xi \partial e}(0, e_*) \neq 0 \quad \text{and} \quad \frac{\partial^3 F_N}{\partial \xi^3}(0, e_*) \neq 0.$$

We refer to [20] for more details on this type of bifurcation. The computation of these derivatives requires to differentiate the Sitnikov equation with respect to initial conditions and parameters. Up to the third order this equation reads as

$$\ddot{z} + \frac{1}{r(t, e_*)^3} z - \frac{3}{2r(t, e_*)^5} z^3 + \cdots = 0.$$

The functions

$$w_1(t) = \frac{\partial^2 z}{\partial \xi \partial e}(t; 0, 0, e_*), \quad w_2(t) = \frac{\partial^2 z}{\partial \xi^3}(t; 0, 0, e_*),$$

are the solutions of

$$\ddot{w} + \frac{1}{r(t, e_*)^3} w + p_i(t) = 0, \quad w(0) = \dot{w}(0) = 0$$

with

$$p_1(t) = -\frac{3}{r(t, e_*)^4} \frac{\partial r}{\partial e}(t, e_*) y_1(t, e_*),$$
$$p_2(t) = -\frac{9}{r(t, e_*)^5} y_1(t, e_*)^3.$$

The derivative $\dfrac{\partial r}{\partial e}(t,e)$ can be expressed in terms of the eccentric anomaly. After differentiating in

$$u - e\sin u = t, \quad \text{and} \quad r = \frac{1}{2}(1 - e\cos u)$$

one arrives at

$$\frac{\partial r}{\partial e} = -\frac{1}{2}\cos u + \frac{e}{4r}\sin^2 u.$$

We notice that this function changes sign.
Going back to the functions w_1 and w_2 we apply the formula of variations of constants and obtain

$$w_i(t) = \int_0^t \left[y_1(t,e_*)y_2(s,e_*) - y_1(s,e_*)y_2(t,e_*)\right] p_i(s)\,ds, \quad i = 1,2.$$

From this expression and the definition of F_N we deduce that

$$\frac{\partial^2 F_N}{\partial\xi\partial e}(0,e_*) = \dot{w}_1(N\pi) = -\dot{y}_2(N\pi,e_*)\int_0^{N\pi} y_1(s,e_*)p_1(s)\,ds$$
$$\frac{\partial^3 F_N}{\partial\xi^3}(0,e_*) = \dot{w}_2(N\pi) = -\dot{y}_2(N\pi,e_*)\int_0^{N\pi} y_1(s,e_*)p_2(s)\,ds.$$

The product $y_1(s,e_*)p_2(s)$ is negative almost everywhere and so the sign of $\dfrac{\partial^3 F_N}{\partial\xi^3}(0,e_*)$ coincides with the sign of $\dot{y}_2(N\pi,e_*)$. Indeed we can control this sign with the number of zeros of $y_1(t,e_*)$. Since $\dot{y}_1(N\pi,e_*) = 0$ we can apply the Liouville formula to conclude that $y_1(N\pi,e_*)\dot{y}_2(N\pi,e_*) = 1$. The sign of $y_1(N\pi,e_*)$ is $(-1)^p$ where p is the number of zeros of $y_1(t,e_*)$ and so

$$\operatorname{sign}\left\{\frac{\partial^3 F_N}{\partial\xi^3}(0,e_*)\right\} = (-1)^p. \tag{7}$$

As mentioned earlier the function $\partial r/\partial e$ changes sign and this fact refrain us from employing a similar analysis for the study of the sign of $\dfrac{\partial^2 F_N}{\partial\xi\partial e}(0,e_*)$. Indeed we do not know if this quantity can vanish in some exceptional cases. This situation has lead us to develop a variant of the pitchfork bifurcation that is applicable at each $e = E^*_{p,N}$.

For the sake of clarity we now work with an abstract bifurcation problem of the type

$$F(x, \lambda) = 0, \quad x \in \mathbb{R}, \quad \lambda \in [a, b]$$

where $F : \mathbb{R} \times [a, b] \to \mathbb{R}$ is real analytic and $F(-x, \lambda) = -F(x, \lambda)$. In particular $x = 0$ is a solution for any λ.

Proposition 2. *In addition assume that, for some $\lambda_* \in]a, b[$ and $\Delta > 0$,*

$$\frac{\partial F}{\partial x}(0, \lambda_*) = 0, \quad \frac{\partial^3 F}{\partial x^3}(0, \lambda_*) < 0, \quad \frac{\partial F}{\partial x}(0, \lambda) > 0 \quad \textit{if} \quad \lambda \in]\lambda_*, \lambda_* + \Delta]$$

Then there exist numbers $r > 0, \delta > 0$ and a real analytic function $\psi : [\lambda_*, \lambda_* + \delta] \to \mathbb{R}$ with

$$0 < \delta \leq \Delta, \quad \psi(\lambda_*) = 0, \quad \psi(\lambda) > 0 \quad \textit{if} \quad \lambda > \lambda_*$$

such that the set of zeros of F is described locally by the equations $x = 0$ and $x^2 = \psi(\lambda)$; that is,

$$F^{-1}(0) \bigcap \big([-r, r] \times [\lambda_*, \lambda_* + \delta]\big) = \Big\{(x, \lambda) : \; x = 0 \qquad \textit{or} \quad x^2 = \psi(\lambda), \lambda \in [\lambda_*, \lambda_* + \delta]\Big\}$$

Proof. The function F is odd in x and admits an expansion of the type

$$F(x, \lambda) = \sum_{n=0}^{\infty} a_n(\lambda) x^{2n+1}.$$

This expansion is uniformly convergent in a neighborhood of $(0, \lambda_*)$ with coefficients

$$a_n(\lambda) = \frac{1}{(2n+1)!} \partial_x^{2n+1} F(0, \lambda).$$

Define the function

$$\Phi(y, \lambda) = \sum_{n=0}^{\infty} a_n(\lambda) y^n.$$

This is also a real analytic function defined in a neighborhood of $(0, \lambda_*)$ and satisfies

$$F(x, \lambda) = x \, \Phi(x^2, \lambda).$$

Consider now the problem of implicit functions

$$\Phi(y, \lambda) = 0$$

at the point $(0, \lambda_*)$. We notice that

$$\Phi(0, \lambda_*) = \frac{\partial F}{\partial x}(0, \lambda_*) = 0, \quad \frac{\partial \Phi}{\partial y}(0, \lambda_*) = \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(0, \lambda_*) \neq 0.$$

Therefore the set $\Phi = 0$ is locally described by an analytic curve $y = \Psi(\lambda)$ with $\Psi(\lambda_*) = 0$. The proof will be complete if we show that $\Psi(\lambda) > 0$ if $\lambda > \lambda_*$. To this end we decompose Φ as

$$\Phi(y, \lambda) = F_x(0, \lambda) + \varphi(y, \lambda)y.$$

The number $\varphi(0, \lambda_*) = \dfrac{1}{6}\partial_x^3 F(0, \lambda_*)$ is negative and so the same will happen to the function $\varphi(y, \lambda)$ when $|y| + |\lambda - \lambda_*|$ is small. The implicit function satisfies

$$\Psi(\lambda) = -\frac{F_x(0, \lambda)}{\varphi(\Psi(\lambda), \lambda)},$$

and this formula shows that Ψ is positive when $\lambda - \lambda_*$ is small and positive. $\square$

Remark. The previous proof also works when the signs of $\partial_x^3 F(0, \lambda_*)$ and $\partial_x F(0, \lambda)$ are reversed simultaneously. The bifurcation described above is of pitchfork type only in a generalized sense. With respect to the classical situation we have lost information: no description of the set of zeros in $\lambda < \lambda_*$ is provided and the bifurcating function $x = \pm\sqrt{\Psi(\lambda)}$ is not necessarily of order $o(\sqrt{\lambda})$ at $\lambda = \lambda_*$.

After this digression we go back to our concrete problem with $F = F_N$. We notice that for $e = E^*_{p,N}$ we can combine (7) with (iii) in lemma 1 to deduce that the conditions of the above Proposition are satisfied. In this way we have proved that there is generalized pitchfork bifurcation at each $E^*_{p,N}$.

5 Oscillatory properties of solutions

In this section we compare the number of zeros of solutions of the linear problem (5) with those corresponding to the nonlinear problem (4).

Proposition 3. *Assume that* $e = E_{n,N}$ *for some* $n \geq 1$ *and let* m *be the number of zeros of the non-trivial solutions of (5). Then any non-trivial solution of (4) with* $e = E_{n,N}$ *has less than* m *zeros.*

Proof. It is based on Sturm Comparison Theory and we present it in terms of the argument function. Let $y(t)$ be a solution of (5) with $y(0) > 0$. As in the proof of the Lemma 1 we introduce polar coordinates $y + i\dot{y} = \rho e^{i\theta}$ and find that the argument satisfies

$$\dot{\theta} = -\frac{1}{r(t, E_{n,N})^3} \cos^2\theta - \sin^2\theta, \quad \theta(0) = 0. \tag{8}$$

The derivative $\dot{\theta}$ is always negative and so the rotation is clockwise. Every zero of $y(t)$ corresponds to the solution of $\theta(t) = \frac{\pi}{2} - j\pi$ with $j = 1, 2, \ldots m$. This implies that $\theta(N\pi) = -m\pi$. Assume now that $z_*(t)$ is solution of (4) with $z_*(0) > 0$. This function is also a solution of the linear equation

$$\ddot{y} + a(t)y = 0, \quad a(t) := (z_*^2(t) + r(t, E_{n,N})^2)^{-3/2}.$$

The corresponding argument $\Theta(t)$ with $z_* + i\dot{z}_* = R\,e^{i\Theta}$ satisfies

$$\dot{\Theta} = -a(t)\cos^2\Theta - \sin^2\Theta, \quad \Theta(0) = 0.$$

The definition of $a(t)$ implies that

$$a(t) \leq \frac{1}{r(t, E_{n,N})^3}, \quad t \in [0, N\pi]$$

and this inequality is strict excepting at a finite number of instants (the zeros of z_*). In consequence $\Theta(t)$ is an upper solution of (8) and therefore

$$\theta(t) < \Theta(t) \quad \text{for each} \quad t \in]0, N\pi].$$

The Neumann boundary conditions are satisfied by $z_*(t)$ and this says that $\Theta(N\pi) = -\mu\pi$ where $\mu \geq 1$ is the number of zeros of z_*. The above inequality implies that $\mu < m$. □

The previous result has many useful consequences. Next we derive from it a uniqueness criterion.

Lemma 2. *Given* $p \geq p_0$ *there exists* $\delta > 0$ *such that for* $e \in]E^*_{p,N}, E^*_{p,N} + \delta]$ *the problem (4) has a unique solution with* $z(0) > 0$ *and having exactly* p *zeros.*

Proof. We know from the previous section that there is a generalized pitchfork bifurcation for $e = E^*_{p,N}$. This implies that if $e \in]E^*_{p,N}, E^*_{p,N} + \delta]$ then (4) has a unique solution satisfying

$$0 < z(0) \leq r.$$

Here δ and r are given by Proposition 2. Actually $z(0) = \sqrt{\Psi(e)}$ and so this solution has exactly p zeros. Here we are applying lemma 7.1 in [10]. This argument proves the existence and local uniqueness of the solution postulated by the lemma. To complete the proof we show that any solution of (4) with p zeros and $z(0)$ positive must also satisfy $z(0) \leq r$. Possibly in this process we must decrease the size of δ. Assume by a contradiction argument the existence of a sequence $e_n \searrow E^*_{p,N}$ such that for $e = e_n$ the problem (4) has a solution $z_n(t)$ with p zeros and $z_n(0) > r$. From proposition 5.1 in [10] we obtain a bound $M > 0$, independent of n, such that

$$z_n(0) \leq M.$$

By continuous dependence with respect to initial conditions and parameters we find that $z_n(t)$, $\dot{z}_n(t)$ and $\ddot{z}_n(t)$ are uniformly bounded in $[0, N\pi]$. By compactness we can extract a subsequence $z_k(t)$ such that

$$z_k(t) \to Z(t), \quad \dot{z}_k(t) \to \dot{Z}(t) \quad \text{uniformly in} \quad t \in [0, N\pi].$$

Since $r \leq Z(0) \leq M$, it is easy to prove that $Z(t)$ is a non-trivial solution of (4) for $e = E^*_{p,N}$. The zeros of $Z(t)$ are simple and so their number is just the number of zeros of $z_k(t)$ for a k large enough. In this way we have produced a solution of (4) for $e = E^*_{p,N}$ having exactly p zeros. This is not compatible with Proposition 3. □

6 Global continuation and proof of the main result

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an isolated zero $x_0 \in \mathbb{R}$, the index of f at x_0 is denoted by $\text{index}(f, x_0)$. Let us recall that $\text{index}(f, x_0) = 1$ whenever $(x - x_0) \cdot f(x) > 0$ for $x \neq x_0$ close to x_0. When the inequality is reversed the value of the index is -1. In any other case $\text{index}(f, x_0) = 0$.

Typically bifurcation phenomena are associated to changes of index and this is the case for the pitchfork bifurcation. In the abstract setting of the previous section we can consider the problem

$$F(x, \lambda) = 0,$$

where F is real analytic and odd in x. The conditions imposed in Proposition 2 imply that

$$\text{index}(F(\cdot, \lambda_*), 0) = -1 \quad \text{and} \quad \text{index}(F(\cdot, \lambda), 0) = 1 \quad \text{if} \quad \lambda \in]\lambda_*, \lambda_* + \Delta].$$

To check this it is sufficient to observe that in a small neighborhood of the origin $F(\cdot, \lambda_*)$ is strictly decreasing while $F(\cdot, \lambda)$ is strictly increasing. Next we want to compute the index at the non-trivial zero $x = +\sqrt{\Psi(\lambda)}$ if $\lambda \in]\lambda_*, \lambda_* + \delta]$. We select $r_+ > 0$ small enough so that $F(r_+, \lambda_*) < 0$ and $r_+ \leq r$ where r is given by Proposition 2. After a restriction in the size of $\delta > 0$ we can assume by continuity that $F(r_+, \lambda) < 0$ if $\lambda \in]\lambda_*, \lambda_* + \delta]$. Since $F(\cdot, \lambda)$ is increasing around $x = 0$ and $x = \sqrt{\Psi(\lambda)}$ is the unique zero in $]0, r_+]$ we conclude that

$$\text{index}(F(\cdot, \lambda), \sqrt{\Psi(\lambda)}) = -1 \quad \text{if} \quad \lambda \in]\lambda_*, \lambda_* + \delta].$$

As we already mentioned the conclusion of Proposition 2 still holds when the inequalities in the assumption are reversed. In such a case the index at $x = \sqrt{\Psi(\lambda)}$ becomes $+1$.
The above discussion is applicable to the function F_N coming from the Sitnikov problem. The variables (x, λ) are now replaced by (ξ, e) and we can concluded that

$$\text{index}\big(F_N(\cdot, e), \sqrt{\Psi(e)}\big) = \pm 1, \quad \text{if} \quad e \in]E^*_{p,N}, E^*_{p,N} + \delta], \tag{9}$$

where $\xi = \sqrt{\Psi(e)}$ is the branch appearing in the pitchfork bifurcation. From now on we assume that δ is fixed but small enough so that also the conclusion of Lemma 2 holds.

Once we have a non-vanishing index, as in condition (9), the method of global continuation developed by Leray and Schauder can be applied. We analyze the equation $F_N(\xi, e) = 0$ for eccentricities e lying in the interval $[a, b]$ with $a = E^*_{p,N} + \delta$ and $b = 1 - \epsilon$. Here $\epsilon > 0$ is an arbitrarily small number. We intend to apply Theorem 4.4 in [10] and so we check that the set of zeros of $F_N(\xi, e) = 0$, $e \in [a, b]$, is bounded and finite for fixed e. This is a consequence of Proposition 5.1 in [10] and Proposition 1 in the present paper. The global continuation from $\xi = \sqrt{\Psi(a)}$, $e = a$, produces a continuous path $\alpha : [0, 1] \to \mathbb{R} \times [a, b]$, $\alpha(s) = (\xi(s), e(s))$ satisfying

$$F_N(\xi(s), e(s)) = 0 \quad \text{if} \quad s \in [0, 1], \quad e(0) = a, \quad \xi(0) = \sqrt{\Psi(a)},$$

and either $e(1) = b$ or $e(1) = a$ and $\xi(1) \neq \xi(0)$.

We claim that $\xi(s) \neq 0$ for each $s \in [0, 1]$. Otherwise there should exist $\hat{s} \in]0, 1]$ such that $\xi(s) > 0$ if $s \in [0, \hat{s}[$ and $\xi(\hat{s}) = 0$. Denote by $z_s(t) = z(t; \xi(s), 0, e(s))$ the branch of solutions of (4) produced by the path $\alpha(s)$. Since the number of zeros is invariant along non-trivial families we deduce that $z_s(t)$ has exactly p zeros for $s \in [0, \hat{s}[$. Notice that Lemma 7.1 in [10] implies that the solutions produced by the pitchfork bifurcation have exactly p zeros. The same lemma would imply that $e(\hat{s}) = E_{n,N}$ for some n. Moreover, the corresponding solutions of (5) should also have p zeros. This is not possible because the definition of $E^*_{p,N}$ would imply that $e(\hat{s}) = E_{n,N} \leq E^*_{p,N} < a$. Once we know that $\xi(s) \neq 0$ for each $s \in [0, 1]$ we can say that the solution $z_s(t)$ has exactly p zeros for every s, in particular for $s = 1$. Now we are ready to prove that $e(1) = b$. Otherwise $z_0(t)$ and $z_1(t)$ would be different solutions of (4) for $e = a = E^*_{p,N} + \delta$ having exactly p zeros. We know this is not possible thanks to Lemma 2. The previous reasoning can be repeated for each $\epsilon \in]0, \epsilon_*[$ and so we obtain a family of paths $\alpha_\epsilon(s) = (\xi_\epsilon(s), e_\epsilon(s))$ satisfying

$$e_\epsilon(0) = E^*_{p,N} + \delta, \quad e_\epsilon(1) = 1 - \epsilon, \quad \xi_\epsilon(s) > 0, \quad \forall s \in [0, 1].$$

The set

$$\mathcal{G} = \left\{ (\sqrt{\Psi(e)}, e) : e \in]E^*_{p,N}, E^*_{p,N} + \delta] \right\} \cup \bigcup_{0<\epsilon<\epsilon_*} \alpha_\epsilon([0, 1]),$$

is a connected subset of Σ^+. Let $\mathcal{F}^*_{p,N}$ be the connected component of Σ^+ containing $\mathcal{G}$. By construction every solution in $\mathcal{F}_{p,N}$ has p zeros in $[0, N\pi]$. Moreover $(0, E^*_{p,N})$ is in the closure of $\mathcal{F}^*_{p,N}$ and $\text{proy}_2(\mathcal{F}^*_{p,N}) \supset]E^*_{p,N}, 1[$. The proof of Theorem 1 will be complete once we show that there are no other points in $\text{cl}(\mathcal{F}^*_{p,N}) \cap S^*_0$ and $\text{proy}_2(\mathcal{F}^*_{p,N})$ coincides with $]E^*_{p,N}, 1[$. From Proposition 3 we deduce that $E^*_{p,N}$ cannot belong to $\text{proy}_2(\mathcal{F}^*_{p,N})$. Since this set is an interval we conclude that $\text{proy}_2(\mathcal{F}^*_{p,N}) =]E^*_{p,N}, 1[$. Assume now that

$(0, e)$ is a point in the closure of $\mathcal{F}_{p,N}^*$. From the previous discussions we deduce that $e \geq E_{p,N}^*$ and from lemma 7.1 in [10] and the definition of $E_{p,N}^*$ we conclude that $e \leq E_{p,N}^*$. In this way we have checked that the conditions $(i), (ii)$ and (iii) of the Theorem 1 hold and the proof is complete.

7 Continuation from the circular problem

For $e = 0$ the orbits of the primaries are circular and the Sitnikov problem is reduced to the autonomous equation

$$\ddot{z} + \frac{z}{(z^2 + 1/4)^{3/2}} = 0.$$

For this equation the origin is a center and all solutions of type $z(t; \xi, 0, 0)$ are periodic. The minimal period, denoted by $T(\xi)$, is an increasing function satisfying $T(0+) = \frac{\pi}{\sqrt{2}}$, $T(+\infty) = +\infty$ see [2] and also [6]. The non-trivial solutions of the boundary value problem (4) for $e = 0$ are in correspondence with the solutions of

$$T(\xi) = \frac{2N\pi}{p}, \tag{10}$$

where $p \geq 1$ is the number of zeros of the solution in $[0, N\pi]$. Hence we can only expect a solution if $\frac{2N\pi}{p} > \frac{\pi}{\sqrt{2}}$ or, equivalently,

$$1 \leq p \leq \nu_N := [2\sqrt{2}N],$$

where [.] denotes the integer part of a number. Let $\xi_{p,N}$ denotes the solution of (10) for $p = 1, \ldots, \nu_N$. In the next result we give a description of the global families emanating from $z(t; \xi_{p,N}, 0, 0)$. It is an improvement of the main result in [10].

First we need some notation concerning the linear problem. We recall that $p_0(N)$ is the smallest number of zeros of solutions of (5). In analogy with $E_{p,N}^*$ we define

$$e_{p,N}^* = \min \{ e \in [0, 1[: (5) \text{ has a solution with } p \text{ zeros in } [0, N\pi] \} .$$

In general $e_{p,N}^* \leq E_{p,N}^*$ but we do not know if the inequality can be strict in some case.

Theorem 2. *For each $N \geq 1$ and $p = 1, 2, \ldots \nu_N$ there exists a global family $\mathcal{C}_{p,N}$ of non-trivial solutions satisfying*

1. *$\mathcal{C}_{p,N}$ has p zeros in $[0, N\pi]$*

2. *$\mathcal{C}^*_{p,N} \bigcap \{e = 0\} = \{(\xi_{p,N}, 0)\}$.*

Moreover if $p < p_0(N)$ then $\mathcal{C}^*_{p,N}$ is closed (relative to $[0, \infty[\times[0, 1[$) and $proy_2(\mathcal{C}^*_{p,N}) = [0, 1[$. If $p \geq p_0(N)$ then $cl(\mathcal{C}^*_{p,N}) \bigcap S^*_0 = \{(0, e^*_{p,N})\}$ and $proy_2(\mathcal{C}^*_{p,N}) = [0, e^*_{p,N}[$.

We do not present a detailed proof of this results. It follows along the lines of the proof of Theorem 1 in combination with the ideas developed in [10]. The new ingredient with respect to [10] is Proposition 3. After the statement of the main result in [10] there was a figure containing several hypothetical families of periodic orbits. Now we know that the situation described by the family A_2 cannot occur. Nevertheless the situation is far from being satisfactory since we do not know if the second alternative described for $p \geq p_0(N)$ can ever occurr. For instance, for $N = 1$ there are two families $\mathcal{C}_{1,1}$ and $\mathcal{C}_{2,1}$. According to Lemma 1 we know that $p_0(1) > 1$. This proves that the family $\mathcal{C}_{1,1}$ is defined for all eccentricities $e \in [0, 1[$. Numerical experiments seem to indicate that $p_0(1) = 3$ and so most probably the family $\mathcal{C}_{2,1}$ can also be continued up to $e = 1$. However we only know that $p_0(1) \geq 2$ and cannot discard at the moment the second alternative.

In view of the above discussions and the conclusion of the Theorem 1, the following question concerning Ince's equation seems to be relevant.

Is it true that for any non-trivial solution of

$$(1 - e\cos u)y'' - e(\sin u)y' + 8y = 0, \quad y'(0) = y'(N\pi) = 0,$$

the number of zeros is greater than ν_N?

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