Degree theory and almost periodic problems

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Dedicated to A. Cellina and J. Yorke

1 Introduction

This paper deals with differential equations which are almost periodic in time. Examples of these equations are

- the pendulum equation with quasi-periodic forcing

$$\ddot{u} + c\dot{u} + \sin u = \sin t + \sin \sqrt{2}t$$

- the limit-periodic Riccati equation

$$\dot{u} = u^2 - 1 + \sum_{n=1}^{\infty} 3^{-n} \sin(2^{-n}t).$$

Since the beginning of the last century, the development of a theory for these equations has been inspired by the known results in the simpler class of periodic equations. This is shown in the following quotation, taken from the introduction of the paper by Bohl [1],

Les méthodes de détermination des solutions périodiques ont été notablement perfectionées, ces derniers temps, grâce surtout aux travaux bien connus de M. Poincaré. On ne peut pas en dire autant à ce qu'il semble, des solutions trigonométriques plus générales.

By the time the paper [1] appeared, the notion of almost periodic function had not been introduced and Bohl just referred to *trigonometric solutions*. After the definition of almost periodicity by Bohr, many results for linear and nonlinear equations were obtained. Information on many of these results can be found in [11, 8, 5].

Going back to the periodic problem, we notice that nowadays there are several methods for proving the existence of periodic solutions. One of the most popular consists in a combination of Functional Analysis and Degree Theory. This method reduces the periodic problem to a fixed point equation in a space of periodic functions and then applies the theory of Leray and Schauder. We refer to the work by Krasnoselskii and his school [9] and by Mawhin [10] for more information. The purpose of this paper is to discuss what should be the analogous approach in the almost periodic case and to present some examples which seem to indicate that the degree theory is not applicable in this setting. In this context it is interesting to mention the related discussion by Fink in [5] chapter 8, section 3.

The paper is organized in two parts. First we will consider a second order equation of Newtonian type and transform the almost periodic problem in a fixed point equation of the type

$$u = \mathcal{K}u, \quad u \in AP,$$

where AP is the Banach space of almost periodic functions. In general the operator $\mathcal{K}$ is not compact on bounded sets and so the Leray-Schauder theory is not applicable. We will present an example where $\mathcal{K}$ has no fixed points and maps the unit ball into its interior; that is

$$\mathcal{K}(\overline{B}) \subset B, \quad B = \{u \in AP : \; ||u||_\infty < 1\}.$$

This shows that $\mathcal{K}$ cannot belong to any class of maps for which it is possible to define a degree with the standard properties. As a corollary it will be proved that the well known homotopy method for periodic problems does not extend to the almost periodic case.

The second part of the paper deals with a first order almost periodic equation having a prescribed module of frequencies. Given an additive subgroup of $\mathbb{R}$, which will be denoted by Ω, the space $AP(\Omega)$ is composed by those almost periodic functions having all their frequencies in Ω. The search of solutions in $AP(\Omega)$ leads to a fixed point equation in this space. It will be shown that, unless Ω is cyclic, the Schauder principle does not hold and so the degree is not applicable. The cases of cyclic groups correspond to periodic functions of a fixed period and so this result shows that periodic problems are rather special. At this point I would like to express my gratitude to Professor Corduneanu, for these results on prescribed frequencies were motivated by a question that he posed to me.

All the proofs in this paper are based on previous results in [13] and [14]. These papers, which were joint work with M. Tarallo, were inspired by some of the constructions by Opial [12], Fink and Frederickson [6], Zhikov and Levitan [15] and Johnson [7].

Before ending this introduction it is worth mentioning that there are several theories of weak almost periodic functions, leading to reinterpretations of the notion of *trigonometric solution.* It is conceivable that the change of AP and $AP(\Omega)$ by larger functional spaces could allow the use of degree theory.

2 Equations of the second order

For a fixed number $c > 0$ we consider the Newtonian equation with friction

$$\ddot{u} + c\dot{u} = f(t, u). \tag{1}$$

We shall restrict to the scalar case and assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. When f has the appropriate time dependence, the periodic and almost periodic problems are well defined and can be transformed in fixed point equations.

2.1 The periodic problem

Fix the period $T > 0$ and assume that f is periodic in time, that is

$$f(t + T, u) = f(t, u) \quad \text{for each } (t, u).$$

We work with the Banach space

$$C_T = \{u : \mathbb{R} \to \mathbb{R} / \ u \text{ is continuous and } T - \text{periodic}\},$$

endowed with the L^∞-norm

$$||u||_\infty = \sup_{t \in \mathbb{R}} |u(t)|.$$

Given $k > 0$, the linear equation

$$\ddot{u} + c\dot{u} = ku + p(t), \quad p \in C_T$$

has a unique solution in C_T. This solution can be expressed in terms of the Green function as

$$u(t) = -\int_0^T G(t, s)p(s)ds.$$

We do not need the explicit form of G and just recall that $G = G(t, s, k)$ is continuous and positive. After fixing k the equation (1) can be rewritten as

$$\ddot{u} + c\dot{u} = ku + g(t, u) \tag{2}$$

with $g(t, u) = f(t, u) - ku$. The search of T-periodic solutions of (1) or (2) becomes equivalent to

$$u = Ku, \quad u \in C_T$$

where

$$K : C_T \to C_T, \quad Ku(t) = -\int_0^T G(t, s)g(s, u(s))ds.$$

It is well known that the operator K is compact on bounded sets and the Leray-Schauder degree is applicable to $id - K$. Since G and g depend upon

k, we have a family of operators $K = K(u, k)$. This is irrelevant from the point of view of degree theory since the degree is independent of k. Indeed, all operators $K(\cdot, k)$ have the same fixed points and the formula $K = K(u, k)$ defines a homotopy on any bounded and open set $\mathcal{U} \subset C_T$ without fixed points on the boundary. This implies that

$$deg(id - K(\cdot, k_1), \mathcal{U}) = deg(id - K(\cdot, k_2), \mathcal{U})$$

for all $k_1, k_2 > 0$.

2.2 The almost periodic problem

We start with the Banach spaces

$$BC = \{u : \mathbb{R} \to \mathbb{R} / \ u \text{ is continuous and bounded}\}$$

$$AP = \{u : \mathbb{R} \to \mathbb{R} / \ u \text{ is almost periodic}\}$$

endowed with the L^∞-norm. For each $T > 0$, C_T is contained in AP and this space can be characterized as the smallest closed linear subspace of BC which contains $\bigcup_{T>0} C_T$. Alternatively AP can be characterized as the closure in BC of the space of trigonometric polynomials

$$u(t) = a_0 + \sum_{n=1}^{N} \{a_n \cos \omega_n t + \sin \omega_n t\}$$

where the frequencies ω_n are arbitrary. We refer to [4] for more information on the definition of almost periodic function.

The dependence of f with respect to t will be almost periodic. This means that

$$(i) \qquad f(\cdot, u) \in AP \ \text{ for each } u \in \mathbb{R}.$$

In order to apply the methodology of Functional Analysis we need the composition property

$$u \in AP \quad \Rightarrow \quad f(\cdot, u(\cdot)) \in AP.$$

However it is well known that the almost periodicity in t is not sufficient to guarantee this property (see [5], page 16). We say that f is in the class UAP if it satisfies (i) and the additional condition

(ii) For each $r > 0$ the family of functions $\{f(t, \cdot)\}_{t \in \mathbb{R}}$ is equicontinuous on $[-r, r]$.

The composition property holds if $f \in UAP$ (see [5], Chapter 2).

Our task will be to adapt the discussion on the periodic problem to this new setting. Again we fix $k > 0$ and observe that the linear equation

$$\ddot{u} + c\dot{u} = ku + p(t), \quad p \in AP$$

has a unique solution in AP. It can be expressed in terms of a Green function, but this time the integral is extended over the whole real line. Namely,

$$u(t) = -\int_{-\infty}^{\infty} \tilde{G}(t,s)p(s)ds$$

with

$$\tilde{G}(t,s) = \begin{cases} \frac{1}{r_+ - r_-} e^{r_-(t-s)} & \text{if } t \geq s \\ \frac{1}{r_+ - r_-} e^{r_+(t-s)} & \text{if } t \leq s, \end{cases}$$

and $r_\pm = \frac{-c \pm \sqrt{c^2+4k}}{2}$.

We observe that $\tilde{G}$ is continuous and positive. It is interesting to observe that the periodic Green function G can be obtained from $\tilde{G}$. Indeed,

$$G(t,x) = \sum_{n=-\infty}^{\infty} \tilde{G}(t, s+nT).$$

More information about linear almost periodic equations can be found in [3].

With the help of the Green function the search of almost periodic solutions of (1) or (2) is equivalent to

$$u = \mathcal{K}u, \quad u \in AP$$

with

$$\mathcal{K} : AP \to AP, \;\; \mathcal{K}u(t) = -\int_{-\infty}^{\infty} \tilde{G}(t,s)g(s,u(s))ds.$$

2.3 Non-applicability of Schauder's Principle

Typically the operator $\mathcal{K}$ is not compact on bounded sets. We show this in the particular case $g(t,u) = u$. Now $\mathcal{K}$ is linear and an easy computation leads to

$$\mathcal{K}(e^{i\omega t}) = \frac{1}{-(\omega^2+k) + ic\omega} e^{i\omega t}$$

for each $\omega \in \mathbb{R}$. In this way we have obtained an uncountable set of eigenvalues and so $\mathcal{K}$ cannot be compact. This observation explains why the theory of Leray and Schauder is not applicable to $id - \mathcal{K}$. Next we shall prove that a tentative version of Schauder Fixed Point Theorem cannot hold for $\mathcal{K}$. This excludes the possibility of defining a degree of $id - \mathcal{K}$.

Theorem 1. *For each $c > 0$ there exists $f \in UAP$ and $k > 0$ such that the associated operator $\mathcal{K}$ has no fixed points and*

$$\mathcal{K}(\overline{B}) \subset B$$

where

$$B = \{u \in AP / \; ||u||_\infty < 1\}.$$

The proof will be inspired by well known results for the periodic problem. In the periodic case, the region of C_T lying between a lower and an upper solution is invariant under K and contains a fixed point. Here we are assuming that k is large enough. We refer to [2] for more details. However it was proved in [13] that the method of upper and lower solutions fails in the almost periodic case. This will be the starting point for the proof.

Proof of Theorem 1. Given $u, v \in BC$ we introduce the notations

$$u < v \quad \text{ if } v(t) - u(t) > 0 \text{ for each } t \in \mathbb{R},$$

$$u \ll v \quad \text{if } \inf_{t\in\mathbb{R}} (v(t) - u(t)) > 0.$$

In the spaces C_T both notions are equivalent but not in AP. Notice that the strong inequality does not hold for the functions $u(t) = \sin t + \sin \sqrt{2}t$ and $v(t) = 2$. We observe that the unit ball in AP can be expressed as

$$B = \{u \in AP/ \ -1 \ll u \ll 1\}.$$

According to Theorem 10 in [13] it is possible to find $c > 0$, $f \in UAP$ and $\alpha < \beta$ such that the equation (1) has no almost periodic solutions and

$$f(\cdot, \alpha) \ll 0 \ll f(\cdot, \beta). \tag{3}$$

This inequality means that the numbers α and β are strict lower and upper solutions.

We must prove the Theorem for arbitrary c positive but it is enough to do it for a concrete value c. This is sufficient because we can change the friction coefficient by a re-scaling of time $t \mapsto \lambda t$ with $\lambda > 0$. Also, after translation and dilation of u we can assume that $\alpha = -1$ and $\beta = 1$. We do this but for convenience we keep the notation α and β, which now represent the numbers -1 and 1.

According to [13] the function f is smooth and the derivative $\frac{\partial f}{\partial u}$ is also in UAP. This implies that $|\frac{\partial f}{\partial u}(t, u)|$ is uniformly bounded in regions of the type $t \in \mathbb{R}, |u| \leq M$. We select $k > 0$ large enough so that the function $u \in [\alpha, \beta] \mapsto g(t, u) = f(t, u) - ku$ is decreasing for each $t \in \mathbb{R}$. The definition of $\mathcal{K}$ and the positivity of $\tilde{G}$ imply that $\mathcal{K}$ is monotone on $\overline{B}$; that is

$$\alpha \leq u \leq v \leq \beta \Rightarrow \mathcal{K}u \leq \mathcal{K}v.$$

Now we are going to use the condition (3). Given $\delta > 0$ with $f(t, \alpha) \leq -\delta$ everywhere,

$$\mathcal{K}\alpha(t) = -\int_{-\infty}^{\infty} \tilde{G}(t, s)[f(s, \alpha) - k\alpha]ds \geq (\delta + k\alpha) \int_{-\infty}^{\infty} \tilde{G}(t, s)ds = \frac{\delta}{k} + \alpha.$$

This implies that $\mathcal{K}\alpha \gg \alpha$. In the same way one obtains $\mathcal{K}\beta \ll \beta$. We are ready to prove that $\mathcal{K}(\overline{B}) \subset B$. Indeed, given $u \in \overline{B}$,

$$-1 = \alpha \ll \mathcal{K}\alpha \leq \mathcal{K}u \leq \mathcal{K}\beta \ll \mathcal{K}\beta = 1.$$

We know that the equation (1) has no almost periodic solutions and so $\mathcal{K}$ has no fixed points. This remark completes the proof.

2.4 The continuation method

A crucial property of the degree is the invariance under homotopies. The application of this property to the periodic problem leads to general continuation principles. As an example we consider the unit ball in C_T,

$$B_T = \{u \in C_T / \ ||u||_\infty < 1\},$$

and the family of equations

$$\ddot{u} + c\dot{u} = \lambda f(t,u) + (1-\lambda)u, \quad \lambda \in [0,1]. \tag{4}$$

Our initial equation (1) appears for $\lambda = 1$ while the equation is linear and has the unique T-periodic solution $u = 0$ for $\lambda = 0$. Assume that f is continuous and T-periodic in t and there are no T-periodic solutions of (4) in ∂B_T. The invariance under homotopies implis that the equation (1) has a T-periodic solution lying in B_T.

As the reader probably expects, the analogous principle does not hold for almost periodic problems. Going back to the proof of Theorem 1 we take the same function $f \in UAP$ with $\alpha = -1$, $\beta = 1$ and observe that the functions

$$f_\lambda(t,u) = \lambda f(t,u) + (1-\lambda)u$$

are also in UAP and satisfy

$$f_\lambda(\cdot,\alpha) \ll 0 \ll f_\lambda(\cdot,\beta).$$

Moreover these strong inequalities are uniform in λ. This allows us to repeat the strategy of the proof of Theorem 1 for each λ and conclude that

$$\mathcal{K}_\lambda(\overline{B}) \subset B \quad \text{for each } \lambda \in [0,1].$$

The operator $\mathcal{K}_\lambda$, defined on AP, refers to the equation (4) for fixed λ. The choice of the constant k is made independently of λ, this means that $u \in [\alpha,\beta] \mapsto f_\lambda(t,u) - ku$ is decreasing for all t and λ. We sum up the above discussions.

Corollary 1. *There exist $f \in UAP$ and $k > 0$ such that the operator $\mathcal{K}_\lambda$ associated to (4) satisfies*

- *$\mathcal{K}_1$ has no fixed points*
- *$\mathcal{K}_0$ is linear and has a unique fixed point at $u = 0$*
- *$\mathcal{K}_\lambda(\overline{B}) \subset B$ for each $\lambda \in [0,1]$.*

In the theory of almost periodic equations it is customary to impose the assumptions not only on the original equation but on all the equations lying in the hull. We will prove that this makes no difference, but first we recall the notion of hull. Given $f \in UAP$ we say that $f^\star : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is in the hull of f, denoted by $f^\star \in \mathcal{H}(f)$, if there exists a sequence of real numbers $\{h_n\}$ such that

$$f(t+h_n,u) \to f^\star(t,u) \quad \text{as } n \to \infty,$$

and the convergence is uniform in $(t,u) \in \mathbb{R} \times [-M,M]$ for each $M > 0$. It is easy to prove that $\mathcal{H}(f)$ is contained in UAP but the hull of a smooth function f can contain non-smooth functions.[1] By a passage to the limit we observe that the following two properties are inherited by each $f^\star \in \mathcal{H}(f)$,

$$f^\star(\cdot,\alpha) \ll 0 \ll f^\star(\cdot,\beta)$$

$$u \in [\alpha,\beta] \mapsto f^\star(t,u) - ku \quad \text{is monotone non} - \text{increasing}.$$

This allows to improve the conclusion of Corollary 1 with

$$\mathcal{K}_{\lambda,f^\star}(\overline{B}) \subset B$$

for each $\lambda \in [0,1]$ and $f^\star \in \mathcal{H}(f)$. At this point one must observe that $\mathcal{K}_{\lambda,f^\star}$ has fixed points in and only if $\mathcal{K}_{\lambda,f}$ does.

3 First order equations: prescribing the module of frequencies

3.1 The space $AP(\Omega)$

Given Ω, an additive subgroup of $\mathbb{R}$, the space $AP(\Omega)$ is defined as the closure in AP of the class of trigonometric polynomials having frequencies in Ω. These are polynomials of the type

$$a_0 + \sum_{n=1}^{N} \{a_n \cos \omega_n t + b_n \sin \omega_n t\}$$

with $\omega_1, \ldots, \omega_N \in \Omega$.

The space AP becomes a commutative Banach algebra with the standard product of functions and each space $AP(\Omega)$ is a subalgebra. This can be deduced from the additive formulas for trigonometric functions.

Given an almost periodic function $u(t)$, the module $mod(u)$ is the smallest additive subgroup of $\mathbb{R}$ containing all the non vanishing Fourier coefficients of u. With the help of Fourier analysis one can prove that

$$AP(\Omega) = \{u \in AP : \; mod(u) \subset \Omega\}.$$

Next we identify the space $AP(\Omega)$ for some groups.

[1] The function $f(t) = \sqrt{2 - \sin t - \sin \sqrt{2} t}$ is smooth but $f^\star(t) = \sqrt{2 - \cos t - \cos \sqrt{2} t}$ is not.

- The trivial group $\Omega_0 = \{0\}$, $AP(\Omega_0) = \{\text{constant functions}\}$
- Cyclic groups $\Omega_1 = T\mathbb{Z},\ T > 0$, $AP(\Omega_1) = C_T$
- A free group with two generators $\Omega_2 = \{n + m\sqrt{2}/\ n, m \in \mathbb{Z}\}$, $AP(\Omega_2) = \{\text{quasi} - \text{periodic functions with frequencies 1 and } \sqrt{2}\}$
- The rational numbers $\Omega_3 = \mathbb{Q}$, $AP(\Omega_3) = \{$uniform limits of periodic functions having periods commensurable with $2\pi\}$.

3.2 The Ω-almost periodic problem

We will study the equation

$$\dot{u} = f(t, u) \tag{5}$$

where f is in UAP and satisfies the additional condition

$$f(\cdot, u) \in AP(\Omega) \ \text{ for each } u \in \mathbb{R}. \tag{6}$$

In this setting we discuss the existence of solutions in $AP(\Omega)$. We shall transform this problem into a fixed point equation in $AP(\Omega)$ but first we go back to the examples. For Ω_0 the condition (6) says that f is independent of t and solutions in $AP(\Omega_0)$ are equilibria. For Ω_1 we go back to the T-periodic problem. For Ω_2 and Ω_3 we have genuine almost periodic problems.

3.3 The fixed point equation

We start with the composition property, which is important to make the problem treatable with the methods of Functional Analysis.

Lemma 1. *Assume that $f \in UAP$ and the condition (6) holds. Then,*

$$u \in AP(\Omega) \Rightarrow f(\cdot, u(\cdot)) \in AP(\Omega).$$

This property depends essentially on the algebraic structure of Ω. In principle one could define $AP(\Omega)$ for any non-empty subset of the real numbers and it would be a Banach space. The difference is that when Ω is not a group the space $AP(\Omega)$ is not an algebra and the composition property does not hold.

There is a proof of this Lemma using Fourier Analysis and we refer to [5]. We present a more direct approach.

Proof of Lemma 1. The space $AP(\Omega)$ is an algebra and so the composition property holds if f is a polynomial in u which is independent of t. Assume next that $f = f(u)$ is any continuous function, again independent of t. Given $u \in AP(\Omega)$ we approximate f by polynomials f_n converging uniformly to f in the range of u. Then $f_n \circ u$ converges in BC to $f \circ u$ and so $f \circ u \in AP(\Omega)$. It remains to prove that the composition property also holds when f depends on t.

Given $\epsilon > 0$ we shall find $p_\epsilon \in AP(\Omega)$ such that

$$||f(\cdot, u(\cdot)) - p_\epsilon||_\infty < \epsilon.$$

Fix $R > ||u||_\infty$. As $\{f(t,\cdot)\}_{t\in\mathbb{R}}$ is equicontinuous in $[-R,R]$, we find $\delta > 0$ such that

$$|f(t,u) - f(t,v)| < \epsilon \quad \text{if } t \in \mathbb{R}, |u-v| < \delta, |u|, |v| \leq R.$$

Next we find a partition $u_0 = -R < u_1 < \ldots < u_n = R$ such that $u_{i+1} - u_i < \delta$ for each i. We can construct a partition of unity on $[u_0, u_n]$ as follows. The functions $\chi_0, \chi_1, \ldots, \chi_n$ are continuous on $[-R,R]$ and satisfy

$$\sum_{i=1}^{n} \chi_i(u) = 1, \quad \chi(u) \geq 0 \quad \text{for each } u \in [-R,R],$$

$$\text{supp } \chi_i \subset [u_{i-1}, u_{i+1}], \quad \text{with the convention } u_{-1} = u_0, u_{n+1} = u_n.$$

Finally we define

$$p_\epsilon(t) = \sum_{i=0}^{n} f(t,u_i)\chi_i(u(t)).$$

From the above discussions we know that $\chi_i \circ u$ belongs to $AP(\Omega)$. The structure of algebra and the condition (6) imply that $p_\epsilon \in AP(\Omega)$. The proof is complete, for it is clear that $||f(\cdot,u(\cdot)) - p_\epsilon||_\infty$ is less than ϵ.

As in the case of second order equations we use an auxiliary linear equation. Given $k > 0$ we consider

$$\dot{u} = -ku + p(t), \quad p \in AP(\Omega).$$

This equation has a unique bounded solution given by

$$u(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,s)p(s)ds$$

where

$$\mathcal{G}(t,s) = \begin{cases} 0 & \text{if } t \leq s \\ e^{k(s-t)} & \text{if } s \leq t. \end{cases}$$

Now it is easy to prove that $u \in AP(\Omega)$. First we observe that if p_1, p_2 are two functions in $AP(\Omega)$ then

$$||u_1 - u_2||_\infty \leq \frac{1}{k}||p_1 - p_2||_\infty.$$

By direct computation we observe that if $p(t)$ is a trigonometric polynomial with frequencies in Ω then $u(t)$ is in the same class.

The equation (5) is rewritten as

$$\dot{u} = -ku + g(t,u), \quad g(t,u) := f(t,u) + ku,$$

and the Ω-problem becomes equivalent to

$$u = \mathcal{F}u,$$

with

$$\mathcal{F} : AP(\Omega) \to AP(\Omega), \;\; \mathcal{F}u(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,s)g(s,u(s))ds.$$

We know that degree theory can be applied to this operator when Ω is a cyclic group. Next we show that this is the only possible case.

Theorem 2. *Assume that Ω is not cyclic. Then there exist $f \in UAP$ satisfying (6) and $k > 0$ such that the associated operator $\mathcal{F}$ has no fixed points and*

$$\mathcal{F}(\overline{B}) \subset B$$

where

$$B = \{u \in AP(\Omega)/ \; ||u||_\infty < 1\}.$$

We will find a function f and numbers $\alpha < \beta$ such that

$$f(\cdot, \alpha) \gg 0 \gg f(\cdot, \beta)$$

and the equation (5) has no solutions in AP. From there the rest of the proof follows along the lines of Theorem 1. Yet we need to do some work to construct such an equation.

3.4 Primitives of functions in $AP(\Omega)$

Given a function $a \in C_T$ the primitive can be expressed as $A(t) = \overline{a}t + \tilde{A}(t)$, where $\overline{a}$ is the average and $\tilde{A}$ is T-periodic. The next result implies that such a result cannot be extended to any group Ω which is not cyclic.

Lemma 2. *Assume that Ω is not cyclic. Then there exists $a \in AP(\Omega)$ such that its primitives satisfy*

$$A(t) \to -\infty \quad \text{as } |t| \to \infty.$$

Proof. It follows along the lines of Bohr's example (see [15], page 157). As Ω is not cyclic it must be dense in $\mathbb{R}$. In particular, for each integer $n \geq 1$ it is possible to find $\omega_n \in \Omega$ with

$$n^{-2/3} \leq \omega_n \leq 2n^{-2/3}.$$

Define

$$a(t) = \sum_{n=1}^{\infty} \omega_n^2 \sin \omega_n t.$$

The function is in $AP(\Omega)$ and the primitive with $A(0) = 0$ is given by

$$A(t) = \sum_{n=1}^{\infty} \omega_n (1 - \cos \omega_n t) = 2 \sum_{n=1}^{\infty} \omega_n \sin^2 (\frac{\omega_n t}{2}).$$

The inequality $|\sin x| \geq \frac{1}{2}|x|$ if $|x| \leq 1$ imply that

$$A(t) \geq \frac{t^2}{8} \sum_{n \in I(t)} \omega_n^3,$$

where $I(t) = \{n \in \mathbb{N} / \ n \geq 1, \ \omega_n |t| \leq 2\}$. The numbers satisfying $n \geq |t|^{3/2}$ are in the set $I(t)$ and so

$$A(t) \geq \frac{t^2}{8} \sum_{n \geq |t|^{3/2}} \frac{1}{n^2} \geq \frac{t^2}{8} \int_{|t|^{3/2}+1}^{\infty} \frac{ds}{s^2} \to \infty.$$

3.5 Linear equations, homoclinic solutions and proof of Theorem 2

The function a constructed in the previous Lemma is such that all the solutions of

$$\dot{u} = a(t)u$$

are homoclinic to zero. This allows us to apply Theorem 2 in [14] and obtain the following result.

Proposition 1. *Assume that Ω is not cyclic. Then there exist functions $a, b \in AP(\Omega)$ such that for the linear equation*

$$\dot{u} = a(t)u + b(t)$$

all the solutions are bounded but none of them is almost periodic.

We can now apply the ideas of Section 5 in [13] and construct an equation of the type

$$\dot{u} = a(t)u + b(t) + D(u)$$

without almost periodic solutions. In this construction the function D is C^∞ and such that $-D(u)$ dominates the linear part at infinity. This implies that $h(t,u) := a(t)u + b(t) + D(u)$ satisfies

$$h(\cdot, -R) \gg 0 \gg h(\cdot, R)$$

for large R. Since h also satisfies (6) the proof of Theorem 2 follows.

References

1. P. Bohl, Sur certaines équations différentielles d'un type général utilizables en Mécanique, Bull. Soc. Math. France 38 (1910) 5-138.

2. C. De Coster, P. Habets, Two-point boundary value problems: lower and upper solutions, Elsevier 2006.
3. W.A. Coppel, Dichotomies in stability theory, Lecture Notes in Mathematics 629, Springer-Verlag 1978.
4. C. Corduneanu, Almost periodic functions, John Wiley 1968.
5. A.M. Fink, Almost periodic differential equations, Lecture Notes in Mathematics, Vol. 377, Springer 1974.
6. A.M. Fink, P. Frederickson, Ultimate boundedness does not imply almost periodicity, J. Differential Equations 9 (1971) 280-284.
7. R. Johnson, A linear almost periodic equation with an almost automorphic solution, Proc. Am. Math. Soc. 82 (1981) 199-205.
8. M.A. Krasnosels'kii, V.S. Burd, Y.S. Kolesov, Nonlinear almost periodic oscillations, John Wiley 1973.
9. M.A. Krasnosels'kii, P.P. Zabreiko, Geometrical methods of Nonlinear Analysis, Springer-Verlag 1984.
10. J. Mawhin, Topological degree methods in nonlinear boundary value problems (CBMS Series in Mathematics no. 40), American Math. Soc. 1979.
11. J. Moser, On the theory of quasiperiodic motions, SIAM Review 8 (1968) 145-172.
12. Z. Opial, Sur une équation différentielle presque-périodique sans solution presque-périodique, Bull. Acad. Polon. Sci. Ser. Math. Ast. Phys. IX (1961) 673-676.
13. R. Ortega, M. Tarallo, Almost periodic upper and lower solutions, J. Differential Equations 193 (2003) 343-358.
14. R. Ortega, M. Tarallo, Almost periodic linear differential equations with non-separated solutions, J. Functional Analysis 237 (2006) 402-426.
15. V.V. Zhikov, B.M. Levitan, Favard theory, Russian Math. Surveys 32 (1977) 129-180.