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A difference equation arising in Mechanics

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Abstract

A family of second order difference equations is presented. They have a variational structure and appear often in Mechanics.

1 The equation

Let us consider the second order difference equation

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0, \tag{1}$$

where $h = h(\theta, \theta')$ is a given function. Here $\partial_1 = \frac{\partial}{\partial \theta}$ and $\partial_2 = \frac{\partial}{\partial \theta'}$. This equation appears in some physical problems, see [1, 4]. The prototype of generating function h will be

$$h_p(\theta, \theta') = (\theta - \theta')^p,$$

defined on $\theta' > \theta$. The exponent p can be any real number excepting 0 and 1. When $h = h_p$ the equation becomes

$$p(\theta_n - \theta_{n-1})^{p-1} - p(\theta_{n+1} - \theta_n)^{p-1} = 0,$$

and this is equivalent to

$$\theta_{n+1} - \theta_n = constant = \omega > 0.$$

Finally we find the solutions $\theta_n = \theta_0 + n\omega$. Results on the equation (1) for functions h close to h_p find applications in conservative Mechanics of low dimension. There are several reasons explaining why this equation is useful in Mechanics and we will present two of them.

2 Discrete Lagrangian systems

Assume that $\theta = \theta(t)$ models the motion of a particle on a circle. The Lagrangian function $L = L(\theta, \dot{\theta})$ is defined as

$$L = T - V,$$

where T is the kinetic energy and V is the potential. The motions can be obtained as the critical points of the action functional

$$\mathcal{A}[\theta] = \int_{t_0}^{t_1} L(\theta(t), \dot{\theta}(t))dt.$$

They satisfy Euler-Lagrange equation

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = 0.$$

Assume now that we want to model the motion with a discrete sequence $\Theta = (\theta_n)$. By formal analogy we can replace the derivative $\dot{\theta}$ by the finite difference $\Delta\theta = \theta_{n+1} - \theta_n$ and consider a Lagrangian

$$L = L(\theta_n, \theta_{n+1} - \theta_n) \equiv h(\theta_n, \theta_{n+1}).$$

The integral in the functional is replaced by a sum,

$$\mathcal{A}[\Theta] = \sum_n h(\theta_n, \theta_{n+1})$$

and the "motions" are obtained as critical points of $\mathcal{A}$. The variable θ_n only appears in two terms of the sum defining $\mathcal{A}$,

$$\mathcal{A}[\Theta] = \cdots + h(\theta_{n-1}, \theta_n) + h(\theta_n, \theta_{n+1}) + \cdots$$

and so the equation $\frac{\partial \mathcal{A}}{\partial \theta_n} = 0$ leads to (1).

3 Symplectic twist maps

Let us consider a cylinder with coordinates (θ, r) where $\theta \equiv \theta + 2\pi$. A diffeomorphism of the cylinder $M : (\theta, r) \mapsto (\theta', r')$ is called symplectic if the differential form $d\theta \wedge dr$ is preserved,

$$d\theta_1 \wedge dr_1 = d\theta \wedge dr.$$

This is equivalent to $\det M' = 1$. M has twist if the derivative $\frac{\partial \theta_1}{\partial r}$ does not vanish. This last condition has a simple geometrical interpretation. Assume for instance that

$$\frac{\partial \theta_1}{\partial r} > 0.$$

Given a segment $\Gamma = \{\theta = \mathrm{constant}\}$, the image $\Gamma_1 = M(\Gamma)$ will be a twisted arc, meaning that the angle θ_1 goes forward as r increases. Already Birkhoff found that these maps play an important role in Hamiltonian dynamics, see [2]. Some considerations on differential forms together with the implicit function theorem show that every symplectic twist map M has an associated generating function $h = h(\theta, \theta')$, see [6, 7]. This means that the map can be expressed in the form

$$M : \begin{cases} r = \partial_1 h(\theta, \theta') \\ r' = -\partial_2 h(\theta, \theta'), \end{cases} \tag{2}$$

which formally resembles the structure of Hamiltonian systems.

The function h satisfies,

$$|\partial_{12} h(\theta, \theta')| > 0, \tag{3}$$

where $\partial_{12} = \frac{\partial^2}{\partial\theta\partial\theta'}$. This property reflects the twist condition at the level of the generating function. It allows to solve the equation $r = \partial_1 h(\theta, \theta')$ with respect to $\theta' = \theta'(\theta, r)$. This is important to recover the map M from h via the formulas in (2).

The equation (1) is crucial for the understanding of the dynamics of M. Given a solution $(\theta_n)_{n\in\mathbb{Z}}$ we can produce an M-orbit with the definition $r_n = \partial_1(\theta_n, \theta_{n+1})$. To illustrate the previous discussion we go back to the prototype $h_p(\theta, \theta') = (\theta - \theta')^p$ and compute the associated map M_p. From (2) we obtain,

$$r = p(\theta' - \theta)^{p-1} = r'$$

equivalent to

$$M_p : \begin{cases} \theta' = \theta + (\frac{r}{p})^{\frac{1}{p-1}}, \\ r' = r. \end{cases}$$

This is an integrable twist map having the invariant circles $r = \text{constant}$. Notice that the map M_p becomes a rotation on each of these circles and the rotation number changes with r. In the case $p < 1$ or $p > 2$ there is small twist at infinity, meaning that $\frac{\partial\theta'}{\partial r} \to 0$ as $r \to \infty$.

Going back to a general M, we notice that the notion of symplectic map can be reformulated in terms of the differential form $\eta = r'd\theta' - rd\theta$. Actually M is symplectic whenever this form is closed, that is $d\eta = 0$. The map M is called exact symplectic when the differential form η is exact in the cylinder. This means that there is a function $V = V(\theta, r)$, 2π-periodic in θ and such that $dV = \eta$. In such a case the generating function satisfies the periodicity condition

$$h(\theta + 2\pi, \theta' + 2\pi) = h(\theta, \theta'). \tag{4}$$

The classical theory of exact symplectic twist maps on the cylinder can be viewed as collection of results on the equation (1) when the function h satisfies (3) and (4). In this connection we mention the version on Moser's invariant curve theorem formulated in [5] or [6, 7] for presentations of Aubry-Mather theory in terms of this difference equation. In the recent paper [3], by Markus Kunze and the present author, some results on the equation (1) have been obtained. They do not assume (3) or (4) but h must be close to h_p with $p < -1$. As could be expected from the previous discussions they have found several applications in classical Mechanics.

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