Periodic solutions of Liénard second order differential equations with one or two weak singularities

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Abstract

In this paper we study the existence and asymptotic stability of periodic solutions of the differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = h(t),$$

where $f(x)$ is positive and $g(x)$ is strictly monotonically increasing and has one or two weak singularities. The method of proof for existence is based on original arguments by M. Nagumo.

Keywords: Periodic solution, Liénard equation, weak singularity.

1 Introduction

In this paper we deal with the existence and stability of T-periodic solutions of a second order differential equation of Liénard-type

$$\ddot{x} + f(x)\dot{x} + g(x) = h(t), \tag{1}$$

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where $h(t)$ is a continuous and T-periodic function, $f, g\colon]l_1, l_2[\to \mathbb{R}$ are locally Lipschitz continuous functions.

In principle, $-\infty \le l_1 < l_2 \le +\infty$ but we are interested in the case where at least one of them is finite.

The study of scalar second order equations with singularities can be traced back to a paper by Nagumo [9] published in 1944. It is important to remark this fact because it seems to be little known, and up to our knowledge it is not recorded in the related literature. The available reviews [8, 11, 12] register as early references some papers by Forbat, Huaux and Derwindué in the sixties [4, 5, 2]. Although the first application of topological degree is due to Fauré [3], the papers [7, 6] constitute the landmarks on this topic.

In the study of equations with singularities, the so-called strong force assumption has played a prominent role. To explain it, let us define $\overline{x}$ as the minimum of the values of $]l_1, l_2[$ such that $g(\overline{x}) = 0$ and let us define the potential

$$G(x) := \int_{\overline{x}}^{x} g(s)ds.$$

Then, it is said that g has a strong singularity at l_1 (resp. l_2) if

$$\lim_{x \to l_1^+} G(x) = +\infty. \quad (\text{ resp. } \lim_{x \to l_2^-} G(x) = +\infty)$$

On the other hand, when such a limit is finite, we speak about a weak singularity. All the classical papers mentioned before assume a strong force condition. In fact, in the seminal paper of Lazer and Solimini [7] it is shown that such a condition can not be dropped without further assumptions. On this basis, such a condition became standard in the related works. However, in the latter years the interest on weak singularities has increased and some conditions for existence of periodic solutions [1, 10, 13, 14, 15, 16, 17] can be found. Our purpose in this paper is to recover the original method of Nagumo developed in [9] for strong singularities and show that it is suitable to deal also with weak singularities.

In order to formulate our main result, let us define the following functions

$$F(x) := \int_{\bar{x}}^{x} f(s)ds, \tag{2}$$

$$W(x) := \frac{F^2(x)}{4} + 2G(x), \tag{3}$$

$$Z(x) := \frac{F^2(x)}{2} + 2G(x). \tag{4}$$

In the following, $\|.\|_\infty$ stands for the usual supremum norm.

Theorem 1. *Let us assume that*

(H1) $C^* := \inf\{W(l_1), W(l_2)\} < \infty$.

(H2) *There exists* $f_0 > 0$ *such that* $f(x) \geq f_0$ *for every* $x \in]l_1, l_2[$.

(H3) $g(x)$ *is strictly increasing and there exists* $\overline{x} \in]l_1, l_2[$ *such that* $g(\overline{x}) = 0$.

Fix $0 < C < C^*$ *and let* $l_1' < l_2'$ *be the solutions of* $Z(x) = C$. *Define the function*

$$R(x) = -\frac{1}{2}|F(x)| + \sqrt{C^* - W(x)}, \tag{5}$$

and fix the following positive constants

$$\begin{aligned} K_1 &= \min_{i=1,2} \left\{ \frac{|g(l_i')|}{f_0}, \sqrt{\frac{2g(l_i')F(l_i')}{f_0}} \right\}, \\ K_2 &= \min_{i=1,2} \left\{ \frac{1}{2}R(l_i'), \frac{1}{|F(l_i')|}R^2(l_i') \right\}. \end{aligned}$$

Then, under the assumption

$$\|h\|_\infty < \frac{f_0}{2}\min\{K_1, K_2\}, \tag{6}$$

there exists at least one T*-periodic solution* $\varphi(t)$ *of eq.* (1). *Such solution verifies*

$$l_1' \leq \varphi(t) \leq l_2' \qquad \textit{for all } t. \tag{7}$$

Some comments are pertinent here. Condition $(H1)$ is a weak force assumption. $(H2)$ is a dissipative condition. Finally, $(H3)$ implies that the statement is consistent. In fact, if $g(x)$ and $f(x)$ satisfy $(H2) - (H3)$ then functions $W(x)$, $Z(x)$ satisfy

$$\begin{aligned} &W'(x),\ Z'(x) < 0, \quad x \in]l_1, \bar{x}[\\ &W'(x),\ Z'(x) > 0, \quad x \in]\bar{x}, l_2[. \end{aligned}$$

Hence $Z(x)$ and $W(x)$ have an absolute minimum at $\overline{x}$ and $W(\overline{x}) = Z(\overline{x}) = 0$. Moreover, for all $0 < C \leq C^*$ the equation $Z(x) = C$ has exactly two solutions l_1', l_2' with $l_1' < \overline{x} < l_2'$. Finally, the function $R(x)$ is positive for $x \in]l_1', l_2'[$ (see Lemma 1) and therefore the constant K_2 is in fact positive.

2 Proof of the main result

We begin with a preliminary lemma.

Lemma 1. *The function* $R(x)$ *is positive for* $x \in]l_1', l_2'[$.

Proof. If $x \in]l_1', l_2'[$, we have $Z(x) = W(x) + \frac{1}{4}F^2(x) < C$ and $C^* - W(x) > C^* - C + \frac{1}{4}F^2(x) > 0$. Then

$$\sqrt{C^* - W(x)} > \sqrt{C^* - C + \frac{1}{4}F^2(x)} > \frac{1}{2}|F(x)|.$$

□

With this lemma, $K_2 > 0$ and the assumptions of Theorem 1 are consistent. Now, let us rewrite (1) as a system

$$\begin{cases} \dot{x} = & y - F(x) \\ \dot{y} = & -g(x) + h(t) \end{cases}. \tag{8}$$

The proof of Theorem 1 will consist on establish a positively invariant region for system (8). To this purpose, let us define the energy functional

$$P(x, y) = \left(y - \frac{F(x)}{2}\right)^2 + W(x). \tag{9}$$

Obviously, the inequality $P(x, y) \leq C^*$ is equivalent to

$$\frac{F(x)}{2} - \sqrt{C^* - W(x)} \leq y \leq \frac{F(x)}{2} + \sqrt{C^* - W(x)}. \tag{10}$$

Moreover, it is easy to realize that $P(x, y) = C^*$ is a simple closed curve. The goal is to prove that the simply connected set defined by

$$D = \{(x, y) : l_1' \leq x \leq l_2', P(x, y) \leq C^*\},$$

is positively invariant for system (8).

The following auxiliary result will be useful. For convenience, in the rest of the section we will call $H = \|h\|_\infty$.

Lemma 2. *Under the conditions of Theorem 1, if $x \in]l_1', l_2'[$ and $P(x, y) \geq C^*$ then the following inequalities are fulfilled*

$$|y - F(x)| \geq \frac{4H}{f_0}, \tag{11}$$

$$(y - F(x))^2 > \frac{2H}{f_0}|F(x)|. \tag{12}$$

Proof. In order to prove (11), we must meet that

$$y > \frac{4H}{f_0} + F(x),$$

or

$$y < -\frac{4H}{f_0} + F(x).$$

Since $P(x, y) \geq C^*$, from (10) we have

$$y \geq \frac{F(x)}{2} + \sqrt{C^* - W(x)},$$

or

$$y \leq \frac{F(x)}{2} - \sqrt{C^* - W(x)}.$$

Therefore, it is sufficient to prove

$$F(x) + \frac{4H}{f_0} < \frac{F(x)}{2} + \sqrt{C^* - W(x)},$$
$$F(x) - \frac{4H}{f_0} > \frac{F(x)}{2} - \sqrt{C^* - W(x)},$$

for all $x \in]l_1', l_2'[$, that is,

$$H < \frac{f_0}{4}\left(-\frac{|F(x)|}{2} + \sqrt{C^* - W(x)}\right). \tag{13}$$

Note that from the definition of l_1', l_2' we have $\sqrt{C^* - W(x)} \geq \frac{1}{2}|F(x)|$ for all $x \in]l_1', l_2'[$. On the other hand, it is easy to verify that $R'(x) > 0$ for $x \in]l_1', \bar{x}[$ and $R'(x) < 0$ for $x \in]\bar{x}, l_2'[$. Therefore $\min\limits_{x \in [l_1', l_2']} R(x) = \min\limits_{i=1,2} R(l_i')$. By using the condition (6),

$$H < \frac{f_0}{2}K_2 \leq \frac{f_0}{4}\min_{i=1,2} R(l_i') \leq \frac{f_0}{4}R(x),$$

which is just inequality (13), thus (11) is proved.

Similarly we have that H satisfies (12) if

$$0 < H < \frac{f_0}{2|F(x)|}\left(-\frac{|F(x)|}{2} + \sqrt{C^* - W(x)}\right)^2. \tag{14}$$

Note that

$$\frac{d}{dx}\left(\frac{1}{|F(x)|}R^2(x)\right) > 0 \quad x \in]l_1', \bar{x}[$$
$$\frac{d}{dx}\left(\frac{1}{|F(x)|}R^2(x)\right) < 0 \quad x \in]\bar{x}, l_2'[.$$

Therefore $\min\limits_{x \in [l_1', l_2']} \frac{R^2(x)}{|F(x)|} = \min\limits_{i=1,2} \frac{R^2(l_i')}{|F(l_i')|}$. By using (6),

$$H < \frac{f_0}{2}K_2 \leq \frac{f_0}{2}\frac{R^2(l_i')}{|F(l_i')|} \leq \frac{f_0}{2}\frac{R^2(x)}{|F(x)|},$$

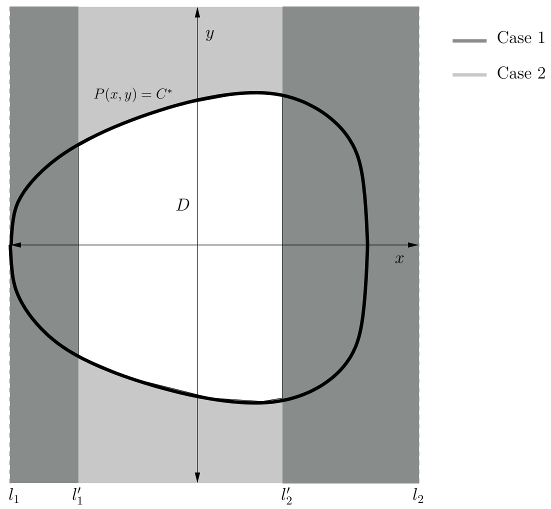

Figure 1: The white region D is positively invariant for system (8).

which is just inequality (14). □

Now, we are ready to prove the main theorem.

Proof of Theorem 1. Let us define the set

$$D = \{(x,y) : l_1' \le x \le l_2', P(x,y) \le C^*\}.$$

We are going to that the region D is positively invariant. It is sufficient to prove that

$$\dot{P}(x,y) < 0, \quad (x,y) \notin D, \tag{15}$$

where $\dot{P}$ is the total derivative of P.

Taking into account (9) and (8),

$$\begin{aligned}\dot{P} &= [y + (y - F(x))]\dot{y} + [-f(x)(y - F(x)) + 2g(x)]\dot{x} \\ &= [2y - F(x)](h(t) - g(x)) + [2g(x) - f(x)(y - F(x))](y - F(x)) \\ &= h(t)[2y - F(x)] - g(x)[2y - F(x)] - f(x)(y - F(x))^2 + 2g(x)(y - F(x)) \\ &= -f(x)(y - F(x))^2 + 2h(t)(y - F(x)) - F(x)g(x) + F(x)h(t).\end{aligned} \tag{16}$$

We distinguish two cases (see Figure 1):

- **Case 1.** Let be $x \in]l_1, l_1'] \cup [l_2', l_2[$. Note $F(x)g(x)$ and $|g(x)|$ are nonnegative functions with an absolute minimum in $\bar{x}$ and such that they are decreasing in $]l_1, l_1']$ and increasing in $[l_1', l_2[$. By (6), we have that $H < \dfrac{f_0}{2}K_1$, and considering the definition of K_1 and the previous argument, the following inequalities hold

$$F(x)g(x) > \frac{2H^2}{f_0}, \quad |g(x)| > 2H$$

for all $x \in]l_1, l_1'] \cup [l_2', l_2[$. In consequence,

$$\begin{aligned}
-f(x)(y-F(x))^2 &\leq -f_0(y-F(x))^2 \\
2h(t)(y-F(x)) &\leq 2H|y-F(x)| \\
F(x)h(t) - F(x)g(x) &< |F(x)|H - F(x)g(x) \\
&< |F(x)|\frac{|g(x)|}{2} - F(x)g(x), \\
&\leq -\frac{1}{2}F(x)g(x) < -\frac{H^2}{f_0}.
\end{aligned}$$

Introducing the above inequalities in (16), it follows

$$\dot{P} < -\left(\sqrt{f_0}|y-F(x)| - \frac{H}{\sqrt{f_0}}\right)^2 \leq 0,$$

for all $x \in]l_1, l_1'] \cup [l_2', l_2[$ and $y \in \mathbb{R}$.

- **Case 2.** Let be $x \in]l_1', l_2'[$ and $P(x,y) \geq C^*$. By Lemma 2, the inequalities (11)-(12) are satisfied and therefore

$$\begin{aligned}
-f(x)(y-F(x))^2 &\leq -f_0(y-F(x))^2, \\
F(x)h(t) - F(x)g(x) &\leq H|F(x)| < \frac{f_0}{2}(y-F(x))^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\dot{P} &< |y-F(x)|\left(-\frac{f_0}{2}|y-F(x)| + 2h(t)\right) \\
&< |y-F(x)|\left(-2H + 2h(t)\right) < 0,
\end{aligned}$$

for all $x \in]l_1', l_2'[$ and $y \in \mathbb{R}$ such that $P(x,y) \geq C^*$.

Therefore, combining Case 1 and Case 2 we get (15).

Finally using (15), equation (1) has a T-periodic solution by a basic application of Brouwer's fixed point theorem. Denote by $(x(t;t_0,x_0,y_0), y(t;t_0,x_0,y_0))$ the unique solution of the Cauchy problem for system (8). Using the fact that D is positively invariant if $(x_0,y_0) \in D$, we have that the solution $x(T) := x(T;0,x_0,y_0), y(T) := y(T;0,x_0,y_0)$ belongs also to D. Considering that D is compact and simply connected, the Poincaré map has a fixed point, which of course is the initial condition of a T-periodic solution $\varphi(t)$ of system (8), and such solution verifies (7). □

3 Asymptotically stability of periodic solutions

In this section we combine the bounds obtained in the proof for existence with the results in [18] in order to get a uniqueness and stability criterion.

Theorem 2. *Under the condition of Theorem 1, assume moreover that g has a continuous derivative in its domain. Let us call*

$$m = \min_{x\in[l_1',l_2']} f(x), \quad M = \max_{x\in[l_1',l_2']} f(x)$$

and fix the constants

$$\beta = (M-m)/2, \quad \gamma = (M+m)/2, \quad \alpha = (\pi/T)^2 + \gamma^2/4.$$

If the following condition holds

$$0 < \max_{x\in[l_1',l_2']} g'(x) \leq \alpha - \beta(\gamma + \alpha^{1/2}), \tag{17}$$

then the periodic solution found in Theorem 1 is unique and asymptotically stable.

Proof. The uniqueness follows by using the argument of the proof of [18, Proposition 4.3]. Concerning the asymptotic stability, the solution found in Theorem 1 has index 1 because it comes for Brouwer's fixed point theorem, then we can apply [18, Proposition 1.2] (see also Remark 1 in this paper).

4 Examples and comparison with related results

It is important to remark that the equation with weak singularities and nonlinear friction term has been scarcely explored until now. A recent reference is [17]. In order to compare the respective results, let us take the model equation

$$\ddot{x} + f(x)\dot{x} + x^\mu - \frac{1}{x^\delta} = a\cos(wt), \tag{18}$$

where $a, \mu, \delta, w > 0$. This corresponds to eq. (1) with $g(x) = x^\mu - \dfrac{1}{x^\delta}$ and $h(t) = a\cos(wt)$. Of course $T = \frac{2\pi}{w}$. If $(H2)$ holds, [17, Theorem 1.1] establishes the existence of a positive T-periodic solution under the condition

$$f_0 > \frac{\sqrt{T}\,\|h\|_2}{g^{-1}(-a)} = \frac{\pi\sqrt{2}a}{g^{-1}(-a)w}. \tag{19}$$

If compared with this condition, our main assumption (6) has the advantage that it does not depend on the frequency w. Therefore both assumptions are independent and it is easy to construct examples verifying (6) but not (19), just taking w small enough. For instance, fix $f(x) = 1, \mu = 1, \delta = \frac{1}{2}$, if we take $C = 1$ in Theorem 1, l_1', l_2' can be computed numerically giving $l_1' = 0.340298, l_2' = 1.73068$. As a consequence, $K_1 = 0.970549$, $K_2 = 0.589254$. Then, condition (6) reads $a \leq \frac{1}{2}\min\{K_1, K_2\} = 0.294627$. Under such condition, (18) has a T-periodic solution for all w. If $a = \frac{1}{4}$, a numerical computation shows that (19) does not hold if $w < 0.189215$. As a further example, if we take $\mu = 2, \delta = \frac{1}{2}, f(x) \equiv 4, l_1' = \frac{1}{2}$, and $C = Z(l_1')$, then $l_2' = 1.49107$, $K_1 = 0.698446$, $K_2 = 0.291053$, condition (6) reads $a \leq 2\min\{K_1, K_2\} = 0.582107$ and condition (17) reads

$$\max_{x \in [l_1', l_2']} g'(x) = 3.25676 < 4 + \frac{w^2}{4},$$

for any frequency $w > 0$. Therefore, if we choose $a = \frac{1}{2}$, Theorems 1 and 2 guarantee the existence of a unique T-periodic solution which is asymptotically stable for all w, but a numerical computation shows that (19) does not hold if $w < 0.698132$.

Other interesting reference is [10]. In this paper, the authors study the equation with linear friction term ($f(x) \equiv c$ constant) by a combination of the method of upper and lower solutions and the ideas from [18]. In particular, for the equation

$$\ddot{x} + c\dot{x} - \frac{1}{x^\delta} = -1 + a\cos(wt), \tag{20}$$

with $a, c > 0$, Theorem 1.2 (see also Example 3.1) gives the existence of a unique T-periodic solution which is asymptotically stable under the condition

$$1 + a \leq \left(\frac{1}{4\delta}\left(w^2 + c^2\right)\right)^{\delta/(\delta+1)}. \tag{21}$$

Again, the condition depends explicitly on the frequency w, hence it is essentially independent from condition (6), being not difficult to derive explicit examples to illustrate this fact. We omit further details.

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