Location of Fixed Points in the Presence of Two Cycles

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Abstract

Any orientation-preserving homeomorphism of the plane having a two cycle has also a fixed point. This well known result does not provide any hint on how to locate the fixed point, in principle it can be anywhere. Campos and Ortega in *Location of fixed points and periodic solutions in the plane* consider the class of Lipschitz-continuous maps and locate a fixed point in the region determined by the ellipse with foci at the two cycle and eccentricity the inverse of the Lipschitz constant. It will be shown that this region is not optimal and a sub-domain can be removed from the interior. A curious fact is that the ellipse mentioned above is relevant for the optimal location of fixed point in a neighborhood of the minor axis but it is of no relevance around the major axis.

Key words and phrases: Ellipse, Lipschitz-continuous homeomorphism, Two cycle, Fixed Point.

1 Introduction

Given a continuous map of the real line $h : \mathbb{R} \longrightarrow \mathbb{R}$ and a two cycle $Q \neq P$ with $h(P) = Q, h(Q) = P$, there exists a fixed point lying between P and Q. This inequality is linked to the last inequality $2 \triangleright 1$ in the Sharkovsky ordering. Brouwer's theory of planar maps leads to a partial extension of this result to two dimensions. If we now assume that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orientation preserving

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homeomorphism having two cycles, then a fixed point always exists. However no information on the location of this point can be provided, it can be anywhere in the plane. In [4], Campos and Ortega obtained a result on the location of a fixed point for Lipschitz-continuous maps.

Theorem 1.1 *([4]) Assume that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orientation preserving homeomorphism having a two cycle $P \neq Q$. In addition assume that f is Lipschitz-continuous. Then f has a fixed point x satisfying*

$$\|x - P\| + \|x - Q\| \leq L\|P - Q\|, \tag{1.1}$$

with $\|\cdot\|$ the euclidean norm and $[f]_{Lip} \leq L$ where $[f]_{Lip}$ denotes the best Lipschitz constant of f.

The inequality (1.1) describes the domain determined by the ellipse with foci at P and Q and eccentricity $\frac{1}{L}$.
In this paper we are going to study the optimality of the previous domain in the theorem 1.1. We will see that the previous theorem can be refined and the fixed point can be found in a subregion of the interior of the ellipse having several holes around the major axis. Also we will prove that close to the ellipse, the major axis is irrelevant for the location and the minor axis is optimum. The proofs of our results combine elementary geometric constructions and subtler topological facts. The ideas of M. Brown in [3] on planar maps with a two cycles are crucial. Mainly we will use the following result.

Theorem 1.2 *Suppose that h is an orientation preserving homeomorphism of the plane with a two cycle at $\{P, Q\}$. If A is an arc from P to Q then h has a fixed point either in A or in some bounded connected component of $\mathbb{R}^2 \backslash (h(A) \cup A)$.*

For a partial extension of this result to n-cycles see [2]. For other interesting results of location of a fixed point for maps of the type Identity + contraction, see [5], [1].

2 A refinement of the Ellipse Theorem

Firstly, we are going to fix the notation of the ellipse elements. Consider $P \neq Q$ two points and $L > 1$. The ellipse with foci at P and Q and eccentricity $\frac{1}{L}$ will be denoted by $\mathcal{E}$. The bounded component of $\mathbb{R}^2 \backslash \mathcal{E}$ will be E. The intersection of this ellipse with the minor and major axes is composed by four points: A^-, A^+, B^-, B^+.

Definition 2.1 *A map $f = f(x)$ is in the class $\mathcal{F}_L$ if it satisfies:*

- *f is an orientation-preserving homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$,*
- *$[f]_{Lip} \leq L$,*
- *$f(P) = Q$, $f(Q) = P$.*

We know from the above mentioned theorem that every map in $\mathcal{F}_L$ has a fixed points lying in $\mathcal{E} \cap E$. In this section we will show that this set can be reduced with respect to the location of fixed points. To this end we consider the open discs D^- and D^+ given by the equations

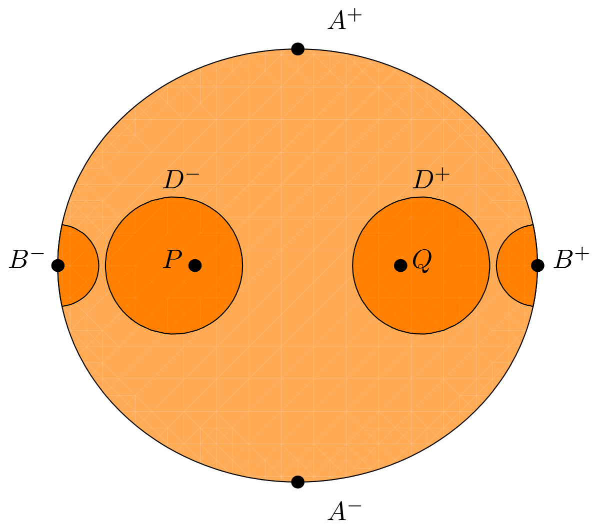
$$\|Q - x\| > L\|P - x\| \tag{1.2}$$

$$\|P - x\| > L\|Q - x\| \tag{1.3}$$

respectively. A straightforward computation shows that these discs shrink to P and Q when L goes to infinity and becomes very large for L decreasing to 1.

Theorem 2.1 *For each $L > 1$ there exist neighborhoods V^+ and V^- of B^+ and B^- such that every map in $\mathcal{F}_L$ has a fixed point lying in $E \backslash (V^+ \cup V^- \cup D^+ \cup D^-)$.*

The next figure illustrates the region where the fixed point is found for $L = 2.7$, $P = (-1, 0)$ and $Q = (1, 0)$. Notice that the ellipse and the discs have been exactly computed but the neighborhoods V^+ and V^- are just hypothetical.



Proof. Take $f \in \mathcal{F}_L$. We know in advance that there is a fixed point in $E \cup \mathcal{E}$ and so we must exclude the sets $\mathcal{E}$, $D^+ \cup D^-$ and $V^+ \cup V^-$. We proceed by steps.
Step 1: A metric obstruction.

$$Fix(f) \cap (D^+ \cup D^-) = \emptyset.$$

Assume that x is a fixed point of f. Then

$$\|Q - x\| = \|f(P) - x\| \leq L\|P - x\|$$

and (1.2) does not hold. In consequence x is not in the disc D^+. The argument for D^- is analogous.
Step 2: Exclusion of $\mathcal{E}$.

$$Fix(f) \cap E \neq \emptyset.$$

Before proving this claim we recall two basic geometrical facts:

- Assume that $\gamma = f([A, B])$ is the image of the segment joining two points $A \neq B$. Then γ is rectifiable and its length satisfies

$$l(\gamma) \leq L\|A - B\|.$$

- Assume that γ is a rectifiable arc with end points at the foci P and Q. In addition assume that $\gamma \cap \mathcal{E} \neq \emptyset$. Then

$$l(\gamma) \geq L\|P - Q\|$$

  and the inequality is strict excepting for the piecewise linear arcs of the type $\gamma = [P, R] * [R, Q]$ with $R \in \mathcal{E}$.
  The notation $*$ is employed for the juxtaposition of arcs. Namely, given arcs $\alpha, \beta : [0, 1] \longrightarrow \mathbb{R}^2$ with $\alpha(1) = \beta(0)$, $\alpha * \beta(t) = \alpha(2t)$ if $t \in [0, \frac{1}{2}]$ and $\alpha * \beta(t) = \beta(2t - 1)$ if $t \in [\frac{1}{2}, 1]$.

We are now ready to prove the assertion of Step 2. It is not restrictive to assume that f has no fixed points on the segment $[P, Q]$, for otherwise the result is already proved. From the previous comments, it is clear that the loop $\Gamma = [P, Q] * f([P, Q])$ remains inside E or touches $\mathcal{E}$ in at the most one point. In any case all the bounded connected components of $\mathbb{R}^2 \backslash \Gamma$ are included in E. Hence the result follows from theorem 1.2
Step 3: Exclusion of V^+ and V^- .
Firstly we are going to construct V^+, V^-.
We fix a positive number $\epsilon \leq \frac{\|P-Q\|}{2L}$ and consider the strip around the major axis

$$\Sigma = \{x \in E : dist(x, [B^+, B^-]) \leq \epsilon\}.$$

Next we find connected neighbourhoods V^+ and V^- of B^+ and B^- respectively with the following property: any rectifiable arc γ joining P and Q and satisfying

$$\gamma \cap (V^+ \cup V^-) \neq \emptyset, \quad l(\gamma) \leq L\|P - Q\|$$

must be contained in Σ. Notice that such neighbourhoods exist because the length of the part $[P, B^+] * [B^+, Q]$ is precisely $L\|P - Q\|$. Moreover, it satisfies that if $\gamma \cap V^+ \neq \emptyset$ then $\gamma \cap V^- = \emptyset$ or if $\gamma \cap V^- \neq \emptyset$ then $\gamma \cap V^+ = \emptyset$. Next we are going to prove an implication that will complete the proof. Namely,

$$Fix(f) \cap (V^+ \cup V^-) \neq \emptyset \Longrightarrow Fix(f) \cap [E \backslash (V^- \cup V^+)] \neq \emptyset.$$

We can assume that f does not have a fixed point in $[P, Q]$. After that, we can distinguish two cases:

- $f([P,Q]) \cap (V^- \cup V^+) = \emptyset$.
In this case, it is clear that the bounded connected components of $\mathbb{R}^2 \backslash ([P,Q] * f([P,Q]))$ are contained in $E \backslash (V^+ \cup V^-)$ and the proof follows from theorem 1.2.

- $f([P,Q]) \cap (V^- \cup V^+) \neq \emptyset$.
From the previous observation, we deduce that either $f([P,Q]) \cap V^+ \neq \emptyset$ and $f([P,Q]) \cap V^- = \emptyset$ or $f([P,Q]) \cap V^+ = \emptyset$ and $f([P,Q]) \cap V^- \neq \emptyset$. We are going to concentrate on the first case. We can assume that there exists a fixed point R of f that belongs to V^+ and a bounded connected components of $\mathbb{R}^2 \backslash [P,Q] * f([P,Q])$ for otherwise the searched conclusion already holds.
Denote by r the line perpendicular to $[P,Q]$ passing through Q. This line splits the plane in two half-planes, one of them contains R, B^+ and will be denoted H_1 and the other half plane is denoted by H_2. Next, define $C = D_1 \cap H_1$, $K = D_2$ where D_1 is the open disc with center at Q and radius $\frac{\|P-Q\|}{L}$ and D_2 is another open disc with center at P and radius $\|P-Q\|$. Since $f(P) = Q$, $f(Q) = P$ and f has Lipschitz-constant not greater than L, it follows that $K \cap C = \emptyset$ and $f(C) \subset K$. Since the loop $[P,Q] * f([P,Q])$ is contained in Σ, we can take $p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a contraction toward r along the orthogonal direction so that $p(R)$ and $p(B^+)$ belong to C and thus $p(f([P,Q]) \cap H_1) \subset C$. For instance, if r is the y axis, then $p(x_1, x_2) = (\delta x_1, x_2)$ with $\delta > 0$ small enough. Next, we define the orientation preserving homeomorphism:

$$h(x) = \begin{cases} p(x) & \text{if } x \in H_1 \\ x & \text{if } x \in H_2 \end{cases} . \tag{1.4}$$

Finally it is clear that $\widehat{f} = f \circ h$ is an orientation preserving homeomorphism with a two cycle in $\{P,Q\}$. Then the Brown's results is applicable and $\widehat{f}$ must have a fixed point lying on D, where D is the union of the bounded components of the complement of the loop $[P,Q] * \widehat{f}([P,Q]) = [P,Q] * f([P,Q])$.
We can deduce that $\widehat{f}$ has not a fixed point in $D \cap H_1$ since $h(D \cap H_1) \subset C$ and $f(C) \subset K$. Therefore $\widehat{f}$ has a fixed point in $H_2 \cap D$ but in this case $\widehat{f} = f$ and so the conclusion is reached.

3 Non-removable points

The elements introduced in the previous section clearly depend on L. In this section, we will make this dependence explicit. For example, $\mathcal{E}_L$ is the ellipse with foci at P, Q and eccentricity $\frac{1}{L}$.
We say that a point $x \in \mathbb{R}^2 \backslash \{P,Q\}$ is non-removable if there exists $h \in \mathcal{F}_L$ such that x is the unique fixed point of h. Notice that the number L plays an important role in the above definition. The results in the previous section imply that x must belong to $E_L \backslash \{D_L^+ \cup D_L^- \cup V_L^+ \cup V_L^-\}$.
For all $L > 1$, the simplest non-removable point is the midpoint between P, Q. After

a simple change of variables we can assume that $P = -Q$. Then the map $h = -id$ belongs to $\mathcal{F}_L$ and the only fixed point is the origin. The rest of the paper will be devoted to find other non-removable points.

3.1 The amenable set

In this section we will realize the importance of the ellipse in the location since the non-removable points "touch" to the ellipse in a neighbourhood of A_L^+ and A_L^-.

Proposition 3.1 *Consider $P \neq Q$ two points of $\mathbb{R}^2$. Then, given $R \in \mathbb{R}^2 \backslash \{P, Q\}$ there exists an unique point R_* in the segment $]P, Q[$ such that*

$$\frac{\|P - R_*\|}{\|Q - R\|} = \frac{\|Q - R_*\|}{\|P - R\|}. \tag{1.5}$$

Moreover, the map

$$\mathbb{R}^2 \backslash \{P, Q\} \longrightarrow [P, Q]$$
$$R \mapsto R_*$$

is Lipschitz-continuous. Notice that the number $\epsilon = \frac{\|P-R_*\|}{\|Q-R\|}$ is precisely the eccentricity of the ellipse passing through R and having P and Q as foci.

Proof. It is clear that R belongs to the ellipse with eccentricity

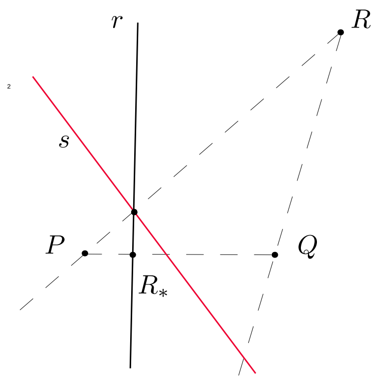
$$\epsilon = \frac{\|P - Q\|}{\|P - R\| + \|Q - R\|}.$$

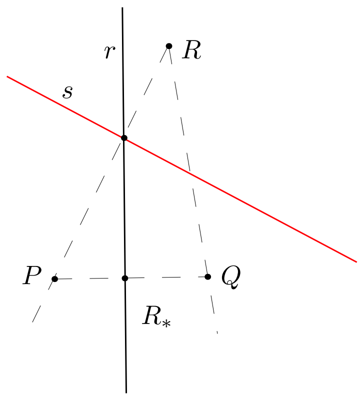
Firstly, we are going to concentrate on proving $\epsilon = \frac{\|P-R_*\|}{\|Q-R\|}$. We look for a point $R_* = tQ + (1-t)P$ with $t \in]0, 1[$ such that the previous identity holds. A straightforward computation shows that R_* is unique and $t(R) = \frac{\|Q-R\|}{\|Q-R\|+\|P-R\|}$. Again a direct computation shows that the identity in (1.5) holds. Here we are using the equation of the ellipse. Therefore R_* is the searched point. The function $t = t(R)$ is Lipschitz-continuous and the same property holds for the map $R \mapsto R_*$.

It will be useful to get some geometric insights on this map. If we consider an arc of the ellipse going from B_L^- to B_L^+ then the image through the map $R \mapsto R_*$ is the segment going from $(B_L^-)_*$ to $(B_L^+)_*$. Notice also that $(B_L^-)_*$ is closer to B_L^+ than $(B_L^+)_*$. Moreover $(A_L^{\pm})_* = \frac{P+Q}{2}$ holds.
This map is helpful for the following geometric construction. Given a point $R \in \mathbb{R}^2 \backslash [P, Q]$ we draw the line r passing through R_* and perpendicular to the segment $[P, Q]$. We say that R is a right point (resp. left point) if the line r intersects $]P, R]$ (resp. $]Q, R]$). Notice that the points of the mediatrix of P and Q are simultaneously left and right points. For R a right (resp. left) point, we denote by S the point of intersection between r and $]P, R]$ (resp. $]Q, R]$). Finally we consider the line s passing through S and perpendicular to $[P, R]$ (resp. $[Q, R]$).

Definition 3.1 *A right point R in $\mathbb{R}^2 \backslash [P, Q]$ will be amenable if the line s cuts the segment $]Q, R]$.*

We can define analogously amenable point for left point. We illustrate this definition with two figures, in the second case R is amenable but not in the first one.





The set of amenable points will be denoted by $\mathcal{A}(P, Q)$. The next aim is to study this set.

Proposition 3.2 *Given $P \neq Q$, let C denote the midpoint. Then the set $\mathcal{A}(P, Q)$ is not empty and there exists $\rho > 0$ such that $\mathcal{A}(P, Q) \cap \{\|x - C\| > \rho\}$ is an open subset of $\mathbb{R}^2$.*

Proof. Firstly, the amenable set is not empty since it always contains the mediatrix of the segment $[P, Q]$, excepting the midpoint. It is clear that there are no amenable points on the line passing through P and Q. The points in the segment $[P, Q]$ are excluded by definition. For the remaining points R on the line, we observe that the line s is perpendicular to $[P, Q]$ and passes through R_*. Thus s cannot intersect the

segment $[Q, R]$.
Let H^+ denote the open half-plane above the line passing through P and Q. By symmetry it is enough to prove that $\mathcal{A}(P, Q) \cap H^+ \cap \{\|x - C\| > \rho\}$ is open. First, we pick $\rho > 0$ large enough so that the angle determined by the vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ is small whenever R is outside the disc of center C and radius ρ. To be precise,

$$\sphericalangle(\overrightarrow{RP}, \overrightarrow{RQ}) \leq \frac{\pi}{4} \quad \text{if } \|R - C\| > \rho. \tag{1.6}$$

Consider now the map

$$\Psi : H^+ \cap \{\|x - C\| > \rho\} \longrightarrow \mathbb{R}^2$$

$$R \mapsto \Psi(R)$$

where $\Psi(R)$, for a right point, is the intersection point between s and the line passing through R and Q. For left points the definition of $\Psi(R)$ is analogous.
Notice that the condition (1.6) says that these two lines are far from being parallel and so they intersect at an unique $\Psi(R)$. Notice also that $\Psi(R) = R$ on the mediatrix and therefore it is easy to check that Ψ is well defined and continuous.
From the definition of amenable point,

$$\mathcal{A}(P, Q) \cap H^+ \cap \{\|x - C\| > \rho\} = \Psi^{-1}(H^+)$$

and so we conclude that it is an open set.

Theorem 3.1 *Let $\mathcal{S}$ be the open strip between P and Q determined by the lines perpendicular to the segment $[P, Q]$ passing through P and Q. Then every point in $E_L \cap \mathcal{A}(P, Q) \cap \mathcal{S}$ is non-removable.*

We are going to need the following definition and results.

Definition 3.2 *Given points $z_1, z_2, z_3, ..., z_n$ in the unit circle $\mathbb{S}^1$, we say that they are cyclically ordered if they can be represented as $z_j = e^{i\theta_j}$ with $\theta_1 < \theta_2 < ... < \theta_1 + 2\pi$. We employ the notation*

$$z_1 \prec z_2 \prec ... \prec z_n.$$

The set of rays emanating from a point R on the plane are in an one-to-one correspondence with $\mathbb{S}^1$ and we will employ the cyclic ordering on this set of rays. Notice that we are ordering the rays in the counter-clockwise sense.

Remark 3.1 *Let $r_1 \prec r_2 \prec r_3 ... \prec r_n$ be rays emanating from a point R and consider the closed sectors $A_1, A_2, ..., A_n$ which are determined by $r_1, ..., r_n$. Assume that $f_i : A_i \longrightarrow \mathbb{R}^2$ is a Lipschitz-continuous with $[f_i]_{Lip} \leq L_i$. Moreover $f_1 = f_n$ on r_1 and $f_i = f_{i+1}$ on r_{i+1}, $1 \leq i \leq n-1$. Then the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as $f(x) = f_i(x)$, when x belongs to A_i, is well defined and Lipschitz-continuous with $[f]_{Lip} \leq \max\{L_i : i = 1, 2, ..., n\}$.*

Lemma 3.1 *Let A be a linear map of $\mathbb{R}^2$ satisfying $A(v_1) = w_1$, $A(v_2) = 0$ where v_1, v_2 are two linearly independent vectors. Then*

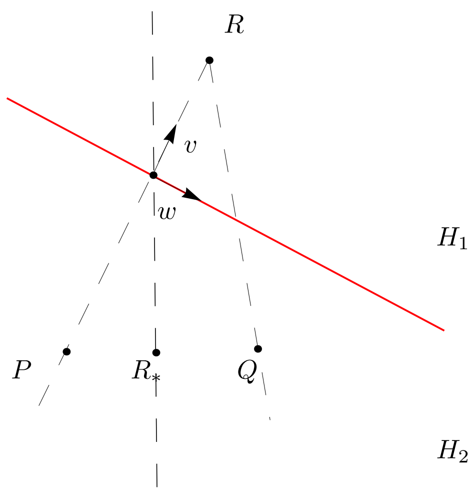
$$\|A\| = \frac{\|w_1\|}{\|v_1\| |\sin\alpha|}$$

where α is the angle between v_1 and v_2 and $\|A\|$ refers to the matrix norm associated to the euclidean norm in the plane.

Proof. Given a rotation R, it is clear that $\|A \circ R\| = \|R \circ A\| = \|A\|$ and so, after a rescaling, it is not restrictive to assume that $v_1 = (1, 0)$ and $w_1 = \lambda v_1$ with $\lambda > 0$. Thus $v_2 = \|v_2\|(\cos\alpha, \sin\alpha)$. Now, we have just to compute the norm of the following matrix

$$A = \lambda \begin{pmatrix} 1 & -\frac{\cos\alpha}{\sin\alpha} \\ 0 & 0 \end{pmatrix}.$$

Proof of the theorem 3.1. We are going to prove that given $\mathcal{L} > 1$ and $R \in \mathcal{E}_{\mathcal{L}} \cap \mathcal{A}(P, Q) \cap \mathcal{S}$ then for all $\mathcal{L}_* > \mathcal{L}$ there exists $F \in \mathcal{F}_{\mathcal{L}_*}$ such that $Fix(F) = \{R\}$. The previous claim proves the theorem since $E_L = \bigcap_{1 \leq \mathcal{L} < L} \mathcal{E}_{\mathcal{L}}$.
In the rest of the construction we will assume that R is an amenable right point. The homeomorphism, which we are going to construct, is the composition of two homeomorphism:
Construction of the first homeomorphism.
The line s splits the plane into two half-planes, one of them contains the point R and will be denoted by H_1 and the other contains the segment $[P, Q]$ and we will be denoted by H_2. To fix the notation we assume that they are closed so that $H_1 \cap H_2$ is the line s. We choose an orthonormal basis $\{v, w\}$ of $\mathbb{R}^2$, such that w is in the direction of s and and v enters into H_1.



Next we consider a contraction on H_1 parallel to v. To be more precise take $\delta \in]0,1]$ and define (for simplicity assume that $S = 0$)

$$h_\delta(x) = \begin{cases} \delta < x, v > v + < x, w > w & \text{if x} \in \mathrm{H}_1 \\ x & \text{if x} \in \mathrm{H}_2. \end{cases}$$

This map has the following properties:

- h_δ is a Lipschitz-continuous homeomorphism with $[h_\delta]_{Lip} = 1$,
- $[P, Q] \subset Fix(h_\delta)$,
- $h_\delta(R) \longrightarrow S$ as $\delta \searrow 0$.

Before the construction of the second map, we need some preliminaries. For a fixed δ in $]0,1]$ we employ the notation $R_\delta = h_\delta(R)$. Denote by t_1 and t_3 the rays emanating from R_δ and passing through P, Q respectively. The sets $a_\delta = h^{-1}(t_1)$ and $b_\delta = h_\delta^{-1}(t_3)$ will play a role in what follows. Notice that a_δ is just the ray emanating from R and passing through P while b_δ ia a piecewise linear set.
Finally we select an arbitrary ray j_2 emanating from R and lying in the sector determined by the rays passing through P, Q. This ray is chosen so that it does not intersect a_δ and b_δ.
Construction of the second homeomorphism.
Consider an ordered sequence of rays

$$t_1 \prec t_2 \prec t_3 \prec t_4 \prec t_5$$

emanating from R_δ and having the following properties: t_1, t_2, t_3 pass through P, R_*, Q respectively, t_4 and t_5 are perpendicular to t_3 and t_1 respectively. Notice that this construction is possible because the angle determined by the rays t_1 and t_3 is less than π. Consider A_1, A_2, A_3, A_4, A_5 the sectors determined by the previous rays so that the boundaries of A_1 and A_5 are $t_1 \cup t_2$ and $t_5 \cup t_1$ respectively.
Now, we consider other configuration $j_1 \prec j_2 \prec j_3 \prec j_4 \prec j_5$. These rays emanate from R and have following properties: j_1, j_4 pass through P, Q respectively, j_2 is defined previously, j_3 is an arbitrary ray between j_2 and j_4, and finally j_5 is the ray bisecting the exterior of $\widehat{PRQ}$.
We distinguish between points P, Q, R, R_δ... lying in the affine space and vectors $\overrightarrow{v}$ in the underlying vector space. Let $\overrightarrow{v_{t_i}}$ and $\overrightarrow{v_{j_i}}$ be the vectors in the direction of the rays t_i, j_i having norm 1.
Fix $\epsilon > 0$, we are going to define a continuous map f_ϵ which is affine on each sector $A_1, A_2, ..., A_5$. In A_1, f_ϵ is the unique affine map such that $R_\delta \mapsto R$, $P \mapsto Q$, $\overrightarrow{v_{t_2}} \mapsto \epsilon\overrightarrow{v_{j_5}}$. In A_2 it is sufficient to define f_ϵ on t_3, namely $Q \mapsto P$. Analogously in A_3, $\overrightarrow{v_{t_4}} \mapsto \epsilon\overrightarrow{v_{j_2}}$. In A_4, $\overrightarrow{v_{t_5}} \mapsto \epsilon\overrightarrow{v_{j_3}}$. In A_5, it is defined by continuity. The figure 1 describes the behaviour of the map.

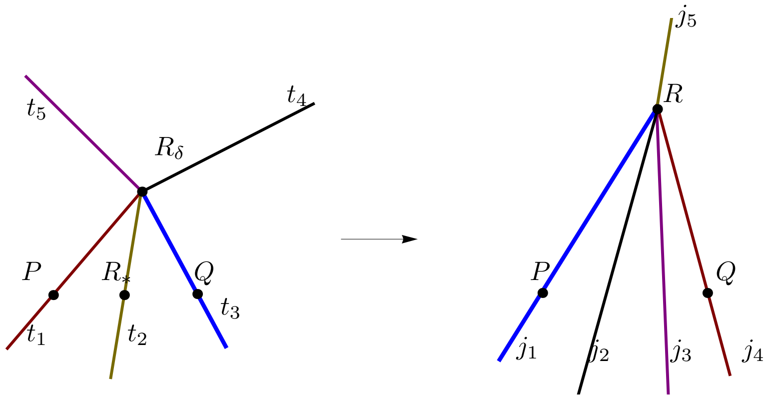


Figure 1: Behaviour of f_ϵ

For $\epsilon > 0$, f_ϵ is an orientation preserving homeomorphism such that $f_\epsilon(P) = Q$, $f_\epsilon(Q) = P$, $f_\epsilon(R_\delta) = R$.
In each sector A_i we employ the notation

$$f_\epsilon(x) = M_{\epsilon,i}x + b_i \quad i = 1, 2, ...5.$$

As $\epsilon \searrow 0$ we notice that $M_{\epsilon,1}$ converges to linear map of the type given by the lemma 3.1 with $\overrightarrow{v_1} = P - R_\delta$, $\overrightarrow{w_1} = Q - R$, $\overrightarrow{v_2} = R_* - R_\delta$. The same happens in the sectors A_2, A_3 and A_5 with $\overrightarrow{v_1} = Q - R_\delta$, $\overrightarrow{w_1} = P - R$, $\overrightarrow{v_2} = R_* - R_\delta$;$\overrightarrow{v_1} = Q - R_\delta$, $\overrightarrow{w_1} = P - R$, $\overrightarrow{v_2} = \overrightarrow{v_{t_4}}$; $\overrightarrow{v_1} = P - R_\delta$, $\overrightarrow{w_1} = Q - R$, $\overrightarrow{v_2} = \overrightarrow{v_{t_5}}$ respectively. Finally we observe that $M_{\epsilon,4}$ converges to the matrix 0. The continuity of the norm, the lemma 3.1 and the remark 3.1 imply that

$$\lim_{\epsilon \searrow 0} [f_\epsilon]_{Lip} = \max\{\frac{\|R - Q\|}{\|R_\delta - P\| \sin\beta}, \frac{\|R - P\|}{\|R_\delta - Q\| \sin\gamma}\} = \widetilde{L}$$

where β is the angle between t_1, t_2 and γ is the angle between t_2 and t_3. When R_δ is S, $\widetilde{\mathcal{L}} = \mathcal{L}$ because

$$\|P - S\| \sin\beta = \|P - R_*\|,$$

$$\|Q - S\| \sin\gamma = \|Q - R_*\|$$

and using the proposition 1.5, we know that

$$\mathcal{L} = \frac{\|R - Q\|}{\|P - R_*\|} = \frac{\|R - P\|}{\|Q - R_*\|}.$$

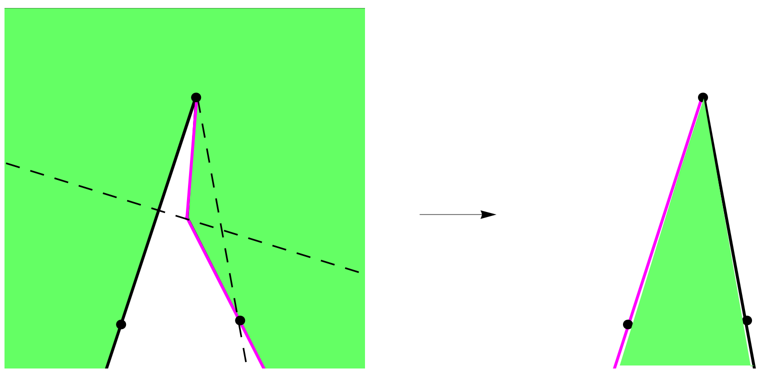
Therefore, we can achieve $\epsilon_0 > 0$ and $\delta_0 > 0$ such that $[f_{\epsilon_0}]_{Lip} \leq \mathcal{L}_*$.
Finally, consider $F = f_{\epsilon_0} \circ h_{\delta_0}$. It is clear that F is an orientation-preserving homeomorphism and verifies

$$F(P) = f_\epsilon(h_\delta(P)) = f_\epsilon(P) = Q,$$

$$F(Q) = f_\epsilon(h_\delta(Q)) = f_\epsilon(Q) = P,$$

$$F(R) = f_\epsilon(h_\delta(R)) = f_\epsilon(R) = R.$$

Since $[h_\delta]_{Lip} = 1$, it is clear that $[F]_{Lip} \leq \mathcal{L}_*$.
Now, we have just to prove that the uniqueness of fixed point. We denote by W the closed region limited by a_δ and b_δ which does not contain the segment $[P, Q]$. We recall that a_δ is the ray emanating from R passing through P and b_δ is a piecewise linear set. By construction, we know that that $F(a_\delta)$ is the ray j_4 where j_4 is the ray emanating from R and passing through Q and $F(b_\delta)$ is the ray j_1 where j_1 is the ray emanating from R and passing through P. As $a_\delta \cap j_4 = \{R\}$ and $b_\delta \cap j_1 = \{R\}$, we deduce that $\{R\}$ is the unique fixed point for F in the boundary of W. In addition, we know that $F(W)$ is the closure of the sector $\widehat{PRQ}$.



From the previous comments, we deduce easily that $\{R\}$ is the unique fixed point for F in the following regions:

- Region 1: $\mathcal{R}_1$ is the closed region determined by the intersection between the sector $\widehat{PRQ}$ and the complement of W.
- Region 2: $\mathcal{R}_2$ is the closed region determined by the complement of the sector $\widehat{PRQ}$ and W.

To conclude the uniqueness of fixed point for F we need to study the following regions:

- Region 3: $\mathcal{R}_3$ is the closed region determined by the intersection between the sector $\widehat{PRQ}$ and W. By construction, we deduce that $h_\delta(\mathcal{R}_3)$ is contained in the sector determined by j_1 and j_2. From the definition j_2, we deduce that $F(\mathcal{R}_3) \cap \mathcal{R}_3 = \{R\}$.
- Region 4: $\mathcal{R}_4$ is the closed region determined by the complement of $\widehat{PRQ}$ and the complement of W. From the definition of $\mathcal{R}_4$ we deduce that $h_\delta(\mathcal{R}_4) = \mathcal{R}_4$. Hence $F(\mathcal{R}_4) = f_\epsilon(\mathcal{R}_4)$. We are going to show that $\mathcal{R}_4 \subset A_2$ and so

$F(\mathcal{R}_4) \cap \mathcal{R}_4 \subset f_\epsilon(A_2) \cap A_2 = \{R\}$. To verify that $\mathcal{R}_4$ is contained in A_2 it is sufficient to check that the ray t_2 and $j_4$do not intersect. This holds because R is an amenable right point lying in the strip $\mathcal{S}$.

3.2 Non-removable points in the minor axis

In this subsection we prove that , for large values of L, the ellipse $\mathcal{E}_L$ is optimal in the small neighborhood of the minor axis.

Theorem 3.2 *There exists L_* such that for $L > L_*$ there exists an open set U_L such that*

$$[A_L^-, A_L^+] \subset U_L$$

and every point in $U_L \cap E_L$ is non-removable.

This result is obtained as a direct consequence of Proposition 3.2 and Theorem 3.1 and the result stated below. From the proofs it is possible to obtain a more or less explicit description of U_L.

Proposition 3.3 *For each $L > 1$ there exists an open set $V_L \subset E_L$ such that $]A_L^-, A_L^+[\subset V_L$ and every point of V_L is non-removable.*

Proof. For simplicity, suppose that $Q = (1, 0)$ and $P = (-1, 0)$. Also we fix a point (x_0, y_0) with $0 \leq x_0 < 1, 0 < y_0$. We are going to construct a family of maps $\{F_\lambda\}$ having a two cycle in $\{P, Q\}$ and an unique fixed point in (x_0, y_0). Given $\lambda > 0$, we define

$$F_\lambda(x, y) = (\varphi(x), \psi(x) + \tau(y))$$

where $\varphi, \psi, \tau : \mathbb{R} \longrightarrow \mathbb{R}$ are the simplest piecewise linear functions that can be constructed in the following way. First, we fix $\mu > y_0$ close enough to y_0 so that the line joining $(0, \mu)$ and (y_0, y_0) has slope dominated by λ. In other words, $\frac{\mu - y_0}{y_0} < \lambda$ and so μ tends to y_0 if λ tends to 0. Then we impose the conditions:

- $\phi(-1) = 1$, $\phi(x_0) = x_0$, $\phi(1) = -1$, ϕ has a corner point at (x_0, x_0).
- $\tau(0) = \mu$, $\tau(y_0) = y_0$ and $\tau'(y_0) = -\lambda$, τ has a corner point at (y_0, y_0).
- $\psi(-1) = -\mu$, $\psi(x_0) = 0$, $\psi(1) = -\mu$, ψ has a corner point at x_0.

It is easy to prove that F_λ is an orientation-preserving homeomorphism with $F_\lambda(P) = Q$, $F_\lambda(Q) = P$, $Fix(F_\lambda) = \{(x_0, y_0)\}$. Moreover F_λ is Lipschitz-continuous and it is possible to compute its Lipschitz-constant via the Jacobian matrix, defined almost every (x, y) and using the following observation of the norm of a triangular matrix. Given a matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ the norm is given by

$$\|A\| = \sqrt{\frac{a^2 + b^2 + c^2 + |a^2 + b^2 - c^2|}{2}}.$$

Letting λ to tend to 0 we notice that:

$$\lim_{\lambda \searrow 0} [F_\lambda]_{Lip} = \sqrt{(\frac{1+x_0}{1-x_0})^2 + (\frac{y_0}{1-x_0})^2}$$

since the possible values of c are $0 < \frac{\mu - y_0}{y_0} < \lambda$. From this construction we conclude that the points (x_0, y_0) satisfying

$$(\frac{1+x_0}{1-x_0})^2 + (\frac{y_0}{1-x_0})^2 < L^2, \ 0 \leq x_0 < 1, y_0 > 0$$

are non-removable.
Also the points $(x_0, 0)$ with $(\frac{1+x_0}{1-x_0})^2 < L^2$ are non-removable. This is easily achieved with a map of the type $F_\lambda(x, y) = (\phi(x), \tau(y))$. Repeating the previous argument on the other quadrants one concludes the proof.

The search of non-removable points is not finished. For instance, using a similar homeomorphism to the second homeomorphism in the Theorem 3.1, it is possible to construct, apart from D^+ and D^-, a strip of the major axis between P, Q of non-removable points.

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