Linear motions in a periodically forced Kepler problem

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1 Introduction

Consider the differential equation

$$\ddot{u} = -\frac{1}{u^2} + p(t), \quad u > 0 \tag{1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a continuous and 2π-periodic function. The results by Lazer and Solimini in [8] imply that the condition

$$\int_0^{2\pi} p(t)dt > 0 \tag{2}$$

is necessary and sufficient for the existence of a 2π-periodic solution. The results by Campos and Torres in [6] are also applicable and the equation has a simple dynamics of saddle type. In particular the periodic solution is unique and unstable (hyperbolic). In both papers the solutions are understood in a classical sense and no collisions are admitted. The purpose of the present paper is to point out that the equation has a rich dynamics of twist type if one admits solutions with collisions. As it is typical in Celestial Mechanics for a binary collision, at an instant where $u = 0$ the velocity becomes infinity but the energy remains finite and has a well defined limit; that is,

$$u(t_0^\pm) = 0 \Longrightarrow \dot{u}(t_0^\pm) = \mp\infty \ \text{ and } \ h(t_0^\pm) = \lim_{t \to t_0^\pm} \{\frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)}\} \ \text{ is finite.}$$

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This observation leads to consider generalized solutions satisfying

$$\ddot{u} = -\frac{1}{u^2} + p(t), \quad u \geq 0 \tag{3}$$

and the additional condition

$$u(t_0) = 0 \implies h(t_0^-) = h(t_0^+). \tag{4}$$

It is easy to understand why the condition (2) is necessary for the existence of classical periodic solutions. Integrating the equation (1) over a period one obtains

$$0 = \dot{u}(2\pi) - \dot{u}(0) = \int_0^{2\pi} \ddot{u}(t)dt = -\int_0^{2\pi} \frac{dt}{u(t)^2} + \int_0^{2\pi} p(t)dt.$$

For a periodic solution with collisions the first two integrals are not convergent and the above identity does not produce any restriction on p. This is consistent with the following results on the existence of harmonic and sub-harmonic solutions.

Theorem 1 *Assume that $p(t)$ is 2π-periodic and of class C^1. Then (3)-(4) has at least two generalized periodic solutions of period 2π and having exactly one collision in the interval $[0, 2\pi[$.*

Theorem 2 *Assume that $p(t)$ is 2π-periodic and of class C^1. Then for each integer $N \geq 2$ the equation (3)-(4) has at least two periodic solutions of minimal period $2N\pi$, having exactly one collision in the interval $[0, 2\pi[$ and no collision on $[2\pi, 2N\pi[$.*

The basic tools for proving these results will be the regularization of binary collisions as presented by Sperling in [13] and an elementary version of the Poincaré-Birkhoff Theorem valid for twist maps. It seems reasonable to expect that the use of more sophisticated versions of the Poincaré-Birkhoff Theorem or KAM and Aubry-Mather theory could lead to more precise results on existence and stability of periodic solutions, as well as results on boundedness and recurrence.

The rest of the paper is organized in six sections. In Section 2 we follow [13] and discuss the behavior of a solution of (1) at a collision. This discussion leads to the concept of generalized or bouncing solution. In Section 3 we prove that the generalized Cauchy problem is well posed. This result shows that the notion of bouncing solution is meaningful. In Section 4 we present the version of the Poincaré-Birkhoff that will be employed. A sketch of the well known proof is given for completeness. In Sections 5, 6 and 7 we study different aspects of the successor map. This is a map in the plane sending each couple (t_0, h_0) into the next couple (t_1, h_1), where t_0 is the instant of collision and h_0 is the corresponding energy. This map is an exact symplectic twist map and the study of some of its periodic points leads to the proof of the results stated above.

2 Collisions and bouncing solutions

The periodicity of $p(t)$ will not play a role until Section 5. By now it is sufficient to assume that $p : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function with $||p||_\infty := \sup_{t\in\mathbb{R}} |p(t)|$.

Let $u(t)$ be a maximal solution of (1) defined in $]t_0, t_1[$ and assume that $t_0 > -\infty$. We will prove that the following limits exist,

$$\lim_{t\downarrow t_0} u(t) = 0 \tag{5}$$

$$\lim_{t\downarrow t_0}\{\frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)}\} = h_0 \tag{6}$$

with h_0 finite.

To prove (5) we notice that the general theory of Cauchy problems implies that one of the following statements must hold at $t = t_0^+$. Either the solution blows up,

$$\lim_{t\downarrow t_0}\{u(t)^2 + \dot{u}(t)^2\} = +\infty \tag{7}$$

or it touches the boundary,

$$\liminf_{t\downarrow t_0} u(t) = 0. \tag{8}$$

Indeed the second alternative always holds. Otherwise there should exist $\delta > 0$ and $\rho > 0$ with

$$u(t) \geq \delta \;\text{ if } t \in I :=]t_0, t_0 + \rho].$$

From the equation (1) we obtain

$$|\ddot{u}(t)| \leq \frac{1}{\delta^2} + ||p||_\infty \;\text{ on } I$$

and now it is easy to deduce that also $|\dot{u}(t)|$ and $u(t)$ are bounded on I. This is against the first alternative (7) and so none of the two alternatives would hold. This contradiction shows the validity of (8) and we can apply the Lemma in Section 3 of [13] and deduce that the stronger assertion (5) is also valid.

To prove the existence of the limit in (6) we apply the results of Section 6 in [13] and conclude that the "energy function"

$$h(t) = \frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)}$$

is bounded in a neighborhood of t_0^+. The next Section in the same paper leads to the asymptotic expansions

$$u(t) = (\frac{9}{2})^{1/3}(t-t_0)^{2/3} + O((t-t_0)^{4/3}), \quad t \downarrow t_0, \tag{9}$$

$$\dot{u}(t) = \frac{2}{3}(\frac{9}{2})^{1/3}(t-t_0)^{-1/3} + O((t-t_0)^{1/3}), \quad t \downarrow t_0. \tag{10}$$

The obtention of these expansions has some subtleties and we add the details. From (7.4) in [13] it can be deduced that $R(t) := u(t)^2$ solves

$$\dot{R} = (8R^{1/2} + b(t)R)^{1/2}$$

on some interval of the type $I =]t_0, t_0 + \delta[$ with $b : I \to \mathbb{R}$ continuous and bounded. The change of unknown $z = R^{3/4}$ transforms the equation to

$$\dot{z} = \frac{3}{4}(8 + b(t)z^{2/3})^{1/2}.$$

The solution $z(t)$ is continuous at $t = t_0^+$ and we arrive at the integral equation

$$z(t) = \frac{3}{4}\int_{t_0}^{t}(8 + b(s)z(s)^{2/3})^{1/2}ds.$$

From the initial estimate $z(t) = O(t-t_0)$ as $t \downarrow t_0$ it is easy to deduce that

$$z(t) = \frac{3}{4}\int_{t_0}^{t}(8^{1/2} + O((s-t_0)^{2/3}))ds = \frac{3}{\sqrt{2}}(t-t_0)[1 + O((t-t_0)^{2/3})]$$

and (9) can be obtained from $u = z^{2/3}$. The expansion (10) follows from $\dot{u} = \frac{2}{3}z^{-1/3}\dot{z}$ and

$$\dot{z}(t) = \frac{3}{\sqrt{2}} + O((t-t_0)^{2/3}).$$

This last formula can be derived from the differential equation in z.

Once (9) and (10) have been proved we can go back to the proof of (6). Differentiating the energy function one gets $\dot{h} = p(t)\dot{u}$ and

$$h(t) = h(\tau) + \int_\tau^t p(s)\dot{u}(s)ds, \quad t, \tau \in]t_0, t_1[.$$

From (10) we deduce that $p(t)\dot{u}(t)$ is integrable (in the Lebesgue sense) in $]t_0, \tau[$. In particular h has a limit when t decreases to t_0.

Next we describe a procedure to regularize collisions that is standard in Celestial Mechanics. Given a maximal solution in the previous conditions, the Sundman integral is defined as

$$S(t) = \int_{t_0}^t \frac{ds}{u(s)}, \quad t \in [t_0, t_1[.$$

The asymptotic expansion (9) guarantees that it is convergent. As a function S has a continuous inverse $T = T(s)$ defined in some interval $[0, \sigma[$ and taking values on $[t_0, t_1[$. The function T is of class C^2 in $]0, \sigma[$ and the triplet

$$U(s) = u(T(s)),\ T = T(s),\ H(s) = h(T(s))$$

is a solution of the autonomous system

$$U'' = 1 + 2UH + p(T)U^2, \quad T' = U, \quad H' = p(T)U'. \tag{11}$$

This can be proved by straightforward differentiation or following the discussions of Section 8 in [13]. A nice feature of the new equation is that it defines a continuous vector field on the phase space $\mathbb{R}^4$ with coordinates (U, U', T, H). The definition of T and the limits (5) and (6) imply that $T(0) = t_0$ and $U(0) = 0, H(0) = h_0$. Also, from the Lemma in Section 5 of [13],

$$U'(s) = \dot{u}(T(s))T'(s) = \dot{u}(T(s))u(T(s)) \to 0 \ \text{ as } s \downarrow 0.$$

In consequence the functions U, U', T, H are well defined and continuous on some interval of the type $[0, \sigma[$ and so they can be extended to the left of $s = 0$ as a solution of (11) satisfying the initial conditions

$$U(0) = U'(0) = 0,\ T(0) = t_0,\ H(0) = h_0. \tag{12}$$

This process can be reversed. To this end it is useful to notice that the quantity

$$\mathcal{I} = U^2H - \frac{1}{2}(U')^2 + U \tag{13}$$

is a first integral of the system (11). Assume now that we are given a solution of (11), (12) and notice that $U''(0) = 1$. This implies that $U(s) > 0$ if s is positive and small enough. Going to the second equation in (11) it is observed that T' is positive on the same interval. Thus it is possible to construct a local inverse of T, say S, that is defined on an interval $[t_0, t_0+\delta[$. A direct computation shows that the function $u(t) = U(S(t))$ satisfies

$$\ddot{u} = -\frac{1}{u^2} + \frac{2\mathcal{I}}{u^3} + p(t), \quad t_0 < t < t_0 + \delta.$$

From the initial conditions (12) we deduce that $\mathcal{I} = 0$ and the limits (5) and (6) holds. Hence $u(t)$ is a solution of (1), (5), (6) in some neighborhood of t_0^+.

The previous construction leads to a result on the existence and uniqueness of solution with a prescribed collision.

Lemma 3 *Given two numbers t_0 and h_0 there exists a maximal solution of (1), (5), (6) defined on $]t_0, t_1[$ with $t_0 < t_1 \leq +\infty$. Moreover this solution is unique as soon as $p(t)$ is Lipschitz-continuous.*

Proof. The existence is a consequence of Cauchy-Peano Theorem applied to (11). For the uniqueness we assume that $u_1(t)$ and $u_2(t)$ are solutions of (1), (5), (6). The corresponding solutions of (11) will satisfy the same initial conditions at $s = 0$. Hence they must coincide. In particular $T_1 = T_2$ and so the inverse functions $S_1(t)$ and $S_2(t)$ will coincide in a neighborhood of t_0^+. Differentiating the identity $T_1 = T_2$ one obtains $U_1 = U_2$ and so $u_1 = U_1 \circ S_1$ coincides with $u_2 = U_2 \circ S_2$ in a neighborhood of t_0^+. This argument proves only the local uniqueness around t_0^+, but this is enough. Once we know that u_1 and u_2 coincide in some interval, we can invoke the standard uniqueness result for the initial value problem.

The previous result motivates a notion of generalized solution with a long tradition in Celestial Mechanics. A *generalized or bouncing solution* of (1) is a continuous function $u : \mathbb{R} \to [0, \infty[$ satisfying

- $Z = \{t \in \mathbb{R} : \ u(t) = 0\}$ is discrete
- For any open interval $I \subset \mathbb{R} \setminus Z$ the function u is in $C^2(I)$ and satisfies (1) on I
- For each $t_0 \in Z$ the limit

  $$\lim_{t \to t_0} \{\frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)}\}$$

  exists.

Let us stress that the above limit is taken from both sides of t_0. Hence the energy function $h(t)$ has a well defined value at t_0 and $h(t_0+0) = h(t_0-0)$. This means that the energy must be preserved at the collision. In the rest of the paper we will prefer the terminology bouncing solution. The reason is that the term generalized solution is employed in the literature with many different meanings. Notice that our concept of bouncing solution is more demanding than that employed in [12] and [14], where the only conditions at collisions are $u(t_0) = 0$, $\dot{u}(t_0 - 0) = -\infty$, $\dot{u}(t_0 + 0) = +\infty$. The related notion of collision solution was introduced in [4], see also [7]. These collision solutions can be obtained by juxtaposing maximal classical solutions and discontinuities of the energy are admissible. We also refer to [2] for some interesting remarks on the meaning of the notion of collision solutions for systems. I thank Pedro Torres for informing me on these definitions. In the recent paper [5] there is an interesting notion of generalized solution guaranteeing the continuity of the energy in the non-autonomous case.

3 The generalized Cauchy problem

From now on solutions without collisions will be called classical solutions. Classical solutions of (1) which are defined on the whole line can be understood as bouncing solutions with $Z = \emptyset$. A less trivial example of bouncing solution is the function

$$u(t) = (\frac{9}{2})^{1/3}(t - t_0)^{2/3},$$

solving (1) for $p \equiv 0$ and having a unique collision, $Z = \{t_0\}$.

Next we obtain a global result on the existence of a bouncing solution when the instant and energy of the collision are prescribed.

Proposition 4 *Assume that $p(t)$ is Lipschitz-continuous and t_0, h_0 are given real numbers. Then there exists a unique bouncing solution satisfying (5) and (6).*

Remark. A consequence of this result is the global extendibility of all solutions of (3)-(4). Given a classical solution $u(t)$ defined on a maximal interval $]t_0, t_1[$ we distinguish two cases. If $t_0 = -\infty$ and $t_1 = +\infty$ then $u(t)$ is also a bouncing solution. If one of the extremes is finite, say $t_0 > -\infty$, then we know that (5) and (6) hold and so, once collisions are admitted, it is possible to extend u to a larger interval using Lemma 3. The case $t_1 < +\infty$ is similar and can be reduced to the previous situation by the change reflecting the time, $u = u(s), s = -t$.

In view of the above Remark the proof of this Proposition looks as an immediate consequence of Lemma 3. Indeed we could apply this Lemma recursively and juxtapose the resulting classical solutions at the collision instants. The objection is that at this point we have no reasons to discard a situation where the length between successive collisions shrinks to zero very fast. In such a case the resulting function would not be defined on the whole line and the set of bouncing instants could have an accumulation point. The proof will be complete if we are able to obtain an uniform lower estimate for the distance between successive collisions. To obtain this estimate we need several preliminary results which will be presented in two subsections.

3.1 Continuous dependence

Let $u(t)$ be a classical solution of (1) with maximal interval $]t_0, t_1[$ and t_0 finite. We know that (5), (6) hold for an appropriate h_0. Given $\epsilon > 0$ with $h_0 + \frac{1}{\epsilon} > 0$, the solution of (1) satisfying the initial conditions

$$u(t_0) = \epsilon, \quad \dot{u}(t_0) = +\sqrt{2(h_0 + \frac{1}{\epsilon})} \tag{14}$$

is denoted by $u^\epsilon(t)$. Next we present a result on the convergence of $u^\epsilon(t)$ to $u(t)$.

Lemma 5 *In the previous notations assume that $p(t)$ is Lipschitz-continuous and J is a compact interval contained in $]t_0, t_1[$. Then there exists $\epsilon_J > 0$ such that if $0 < \epsilon < \epsilon_J$ then the solution $u^\epsilon(t)$ is well defined and positive in J and*

$$u^\epsilon(t) \to u(t), \;\; \dot{u}^\epsilon(t) \to \dot{u}(t) \quad \text{as } \epsilon \to 0,$$

uniformly in J.

Proof. The procedure of regularization of collisions can be applied to $u(t)$, leading to the Sundman's integral $S(t)$ and the triplet U, T, H solving (11)-(12). The main idea of the proof will be to apply a similar procedure to the solutions $u^\epsilon(t)$ which do not have collision at $t = t_0$. Let U^ϵ, T^ϵ, H^ϵ be the solution of (11) with initial conditions

$$U^\epsilon(0) = \epsilon, \;\; (U^\epsilon)'(0) = \epsilon\sqrt{2(h_0 + \frac{1}{\epsilon})}, \;\; T^\epsilon(0) = t_0, \;\; H^\epsilon(0) = h_0. \tag{15}$$

Fix two numbers τ_1 and τ_2 with $t_0 < \tau_1 < \tau_2 < t_1$ and $J \subset]t_0, \tau_1[$. We define $\sigma_1 = S(\tau_1)$, $\sigma_2 = S(\tau_2)$ and observe that, by continuous dependence for (11), the solution $(U^\epsilon, (U^\epsilon)', T^\epsilon, H^\epsilon)$ is well defined on $[0, \tau_2]$ for small ϵ and converges to (U, U', T, H) uniformly on this interval. We claim that

$$U^\epsilon(s) > 0 \quad \text{if } s \in [0, \sigma_2]. \tag{16}$$

To prove this positivity we go back to (11) and observe that $U''(0) = 1$. Then we find $s_1 \in]0, \sigma_2]$ such that $U''(s) \geq \frac{1}{2}$ if $s \in [0, s_1]$. Since $(U^\epsilon)''$ converges uniformly to U'' on $[0, \sigma_2]$, we select ϵ small enough so that $(U^\epsilon)''(s) \geq \frac{1}{4}$ if $s \in [0, s_1]$. Using a Taylor expansion at the origin we find $\xi \in]0, s[$ such that

$$U^\epsilon(s) = \epsilon + \epsilon\sqrt{2(h_0 + \frac{1}{\epsilon})}s + (U^\epsilon)''(\xi)\frac{s^2}{2} \geq \epsilon \text{ if } s \in [0, s_1].$$

The positivity of U^ϵ on the interval $[s_1, \sigma_2]$ is straightforward since U is strictly positive on this interval.

Once (16) has been established we can go back to (11) and observe that $(T^\epsilon)' = U^\epsilon > 0$ on $[0, \sigma_2]$. In consequence the inverse function $S^\epsilon = (T^\epsilon)^{-1}$ is well defined and smooth on the interval $[t_0, \tau_1]$. Moreover it converges uniformly to S on this interval. Since $S^\epsilon([t_0, \tau_1]) \subset [0, \sigma_2]$ for small ϵ, we deduce that $v^\epsilon := U^\epsilon \circ S^\epsilon$ converges to $u = U \circ S$ uniformly on $[t_0, \tau_1]$. From the initial conditions (15) we observe that the first integral $\mathcal{I}$ vanishes for the solution $(U^\epsilon, (U^\epsilon)', T^\epsilon, H^\epsilon)$ and a computation shows that v^ϵ is a solution of (1) defined on the interval $[t_0, \tau_1]$. Since v^ϵ satisfies the initial conditions (14) we deduce that $u^\epsilon = v^\epsilon$ on $[t_0, \tau_1]$. This proves that u^ϵ converges to u uniformly in $J \subset [t_0, \tau_1]$. It remains to show that the derivative also converges. Since u is uniformly positive on J it is possible to find a number $\delta > 0$ such that $u^\epsilon(t) \geq \delta$ if ϵ is small enough and $t \in J$. It is now easy to verify that

$$\dot{u}^\epsilon(t) = \frac{(U^\epsilon)'(S^\epsilon(t))}{U^\epsilon(S^\epsilon(t))} \to \frac{U'(S(t))}{U(S(t))} = \dot{u}(t)$$

and this convergence is uniform in J.

3.2 Comparison principles

Imagine two identical particles P_1 and P_2 attracted by the sun S. Initially both have zero velocity but P_1 is closer to S than P_2. Then P_1 will arrive at the sun before P_2. This is an intuitive argument that can be made rigorous for the Kepler problem and even extended to non-autonomous equations.

Let $u_1(t)$ and $u_2(t)$ be classical solutions of (1) defined on maximal intervals $I_1 =]t_0, t_1[$ and $I_2 =]t_0^*, t_1^*[$. Assume that for some $\tau \in I_1 \cap I_2$,

$$u_1(\tau) \leq u_2(\tau), \quad \dot{u}_1(\tau) \leq \dot{u}_2(\tau), \tag{17}$$

then

$$t_1 \leq t_1^* \text{ and } u_1(t) \leq u_2(t), \;\; \dot{u}_1(t) \leq \dot{u}_2(t) \text{ for each } t \in [\tau, t_1[. \tag{18}$$

To prove this assertion it is convenient to employ the theory of quasi-monotone systems (see [15]). After transforming the equation in the first order system

$$\dot{u} = v =: f_1(t,u,v), \;\; \dot{v} = -\frac{1}{u^2} + p(t) =: f_2(t,u,v),$$

we observe that the conditions of quasi-monotonicity $\frac{\partial f_1}{\partial v} \geq 0$, $\frac{\partial f_2}{\partial u} \geq 0$ are satisfied. Therefore the standard ordering in $\mathbb{R}^2$ is preserved in the future and so (17) implies (18). The same conclusion can be obtained when $u_1(t)$, $u_2(t)$ are solutions of different equations, say

$$\ddot{u} = -\frac{1}{u^2} + p_i(t), \;\; i = 1,2 \tag{19}$$

with $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous and bounded functions satisfying

$$p_1(t) \leq p_2(t) \text{ for each } t \in \mathbb{R}.$$

The above discussion dealt with classical solutions but it admits an extension to collisions.

Lemma 6 *Assume that p_1 and p_2 are as before and $u_1(t)$, $u_2(t)$ are solutions of (19) for $i = 1,2$, having maximal intervals $I_1 =]t_0, t_1[$, $I_2 =]t_0, t_1^*[$ with t_0 finite. Define*

$$h_{0i} = \lim_{t \downarrow t_0} \{\frac{1}{2}\dot{u}_i(t)^2 - \frac{1}{u_i(t)}\}, \;\; i = 1,2.$$

If $h_{01} \leq h_{02}$ then $t_1 \leq t_1^*$ and

$$u_1(t) \leq u_2(t), \;\; \dot{u}_1(t) \leq \dot{u}_2(t) \text{ if } t \in]t_0, t_1[.$$

Proof. Let us fix any τ with $t_0 < \tau < \min\{t_1, t_1^*\}$. A combination of the previous discussion with Lemma 5 leads to the inequalities for $u_1(t)$ and $u_2(t)$ on the interval $]t_0, \tau[$. Hence $t_1 \leq t_1^*$ and the conclusion follows.

The following refinement of the above Lemma will be employed later.

Lemma 7 *In the conditions of Lemma 6 assume that $p_1 = p_2$ and $h_{01} < h_{02}$. Then $t_1 < t_1^*$.*

Proof. In principle we know that $t_1 \leq t_1^*$ and we are going to discard the equality by a contradiction argument. Assuming for the moment that $t_1 = t_1^*$, we observe that the function $w(t) = u_1(t) - u_2(t)$ satisfies

$$\ddot{w} = -\frac{1}{u_1^2} + \frac{1}{u_2^2}, \quad w(t_0) = w(t_1) = 0, \quad w > 0 \text{ on }]t_0, t_1[. \tag{20}$$

The asymptotic expansion (10) leads to

$$\lim_{t \to t_i^{\pm}} \dot{w}(t) = 0,$$

where the limit is understood to the right or to the left depending on whether $i = 0$ or $i = 1$. With this information we multiply the equation in (20) by w and integrate between t_0 and t_1. This integration is understood in the improper sense of Riemann. In principle the integrals could be divergent but this is not the case since an integration by parts shows that

$$-\int_{t_0}^{t_1} \dot{w}(t)^2 dt = \int_{t_0}^{t_1} w(t)\ddot{w}(t) dt = \int_{t_0}^{t_1} \left(-\frac{1}{u_1(t)^2} + \frac{1}{u_2(t)^2}\right)(u_1(t) - u_2(t)) dt.$$

This is a contradiction because the first term has to be negative and the last one should be positive.

3.3 Remarks on the autonomous equation

Next we analyze the equation (1) when $p(t) \equiv P$ is a non-zero constant. When P is negative a phase portrait analysis shows that each classical solution has a bounded maximal interval I and a unique critical point at the mid point of I. The maximum is reached at this instant and has the value

$$U(h_0, P) = (-h_0 - \sqrt{h_0^2 - 4P})/2P$$

where h_0 is defined via the first integral

$$\frac{1}{2}\dot{u}^2 - \frac{1}{u} - Pu = h_0.$$

Notice that h_0 can also be obtained as the limit

$$\frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)} \to h_0 \quad \text{as } t \uparrow t_1 \text{ or } t \downarrow t_0.$$

The length of I is given by the integral

$$\tau(h_0, P) = \sqrt{2} \int_0^{U(h_0,P)} \left(\frac{1}{\xi} + P\xi + h_0\right)^{-1/2} d\xi.$$

It is not hard to show that $\tau(\cdot, P)$ is a continuous positive function with

$$\lim_{h_0 \to -\infty} \tau(h_0, P) = 0, \quad \lim_{h_0 \to +\infty} \tau(h_0, P) = +\infty.$$

For positive P the maximal interval of a classical solution is bounded whenever $h_0 < -2P^{1/2}$ and $u < P^{-1/2}$. The above formulas for U and τ are still valid in this case. Again $\tau(h_0, P) \to 0$ as $h_0 \to -\infty$.

3.4 Proof of Proposition 4

We proceed by contradiction and assume that $u(t)$ is a solution obtained by successive juxtapositions at collisions accumulating at a finite time. Let us say that the collisions occur at a bounded and increasing sequence of instants t_n with

$$u(t_n) = 0, \ \lim_{t \to t_n} \{\frac{1}{2}\dot{u}(t)^2 - \frac{1}{u(t)}\} = h_n.$$

From Lemma 6 we know that

$$\tau(h_n, -||p||_\infty) \leq t_{n+1} - t_n,$$

and so the series $\sum_n \tau(h_n, -||p||_\infty)$ is dominated by the convergent series $\sum_n (t_{n+1} - t_n)$. In particular $\lim_{n \to \infty} \tau(h_n, -||p||_\infty) = 0$. The behavior of the function $\tau(\cdot, P)$ previously described allows us to conclude that $h_n \to -\infty$. From the derivative of the energy, $\dot{h}(t) = p(t)\dot{u}(t)$, we obtain

$$h_{n+1} - h_n = \int_{t_n}^{t_{n+1}} p(s)\dot{u}(s)ds = -\int_{t_n}^{t_{n+1}} \dot{p}(s)u(s)ds.$$

Notice that the integration by parts is possible. Indeed, since $p(t)$ is Lipschitz-continuous then it is also absolutely continuous and its derivative, defined almost everywhere, is bounded. Assuming that h_n is negative and large we apply again the comparison, now for $p(t)$ and $||p||_\infty$, and deduce that

$$\max_{[t_n, t_{n+1}]} u(t) \leq U(h_n, ||p||_\infty) \to 0 \text{ as } n \to \infty.$$

This implies that the series $\sum |h_{n+1} - h_n|$ is dominated by $||\dot{p}||_\infty \sum (t_{n+1} - t_n)$ and so it converges. We have arrived at a contradiction since we proved before that h_n goes to $-\infty$ and now we find that

$$\lim_{n \to \infty} h_n = h_0 + \sum_{n \geq 0} (h_{n+1} - h_n)$$

is finite.

4 Exact symplectic twist maps and the Poincaré-Birkhoff Theorem

Let us consider a plane with coordinates (θ, r) and a domain of the type

$$\Omega = \{(\theta, r) \in \mathbb{R}^2 : \ a < r < \psi(\theta)\}$$

where a is a fixed constant and $\psi : \mathbb{R} \to]a, +\infty]$ is a 2π-periodic function which is lower semi-continuous. We will work with a one-to-one map defined on the closure of Ω and denoted by $S : \overline{\Omega} \to \mathbb{R}^2$, $S(\theta, r) = (\theta_1, r_1)$. The coordinates of S are given by

$$\theta_1 = F(\theta, r), \quad r_1 = G(\theta, r),$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are functions of class C^1 satisfying

$$F(\theta + 2\pi, r) = F(\theta, r) + 2\pi, \ \ G(\theta + 2\pi, r) = G(\theta, r).$$

We say that S is exact symplectic if the differential form $r_1 d\theta_1 - r d\theta$ is exact in the cylinder. This means that there exists a function $V \in C^1(\overline{\Omega})$ such that

$$dV = r_1 d\theta_1 - r d\theta \ \text{ and } \ V(\theta + 2\pi, r) = V(\theta, r) \text{ for each } (\theta, r) \in \overline{\Omega}.$$

We say that S is a twist map if the function $r \in]a, \psi(\theta)[\mapsto F(\theta, r)$ is strictly increasing for each $\theta \in \mathbb{R}$. We present a simplified version of the Poincaré-Birkhoff Theorem for this class of maps. Notice that, in contrast to the most classical situations, the region Ω has not to be invariant under S. We refer to [9, 10] for recent related results.

Theorem 8 *Assume that S is an exact symplectic twist map in the above conditions. Let us fix an integer N and assume that for each $\theta \in \mathbb{R}$, there exists $r_\theta \in]a, \psi(\theta)[$ with*

$$F(\theta, a) < \theta + 2N\pi < F(\theta, r_\theta). \tag{21}$$

Then the system

$$F(\theta, r) = \theta + 2N\pi, \ \ G(\theta, r) = r, \ \ \theta \in [0, 2\pi[, \ \ (\theta, r) \in \Omega$$

has at least two solutions.

Proof. The quotient space $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ will be thought as a space of angles, denoted by $\overline{\theta} = \theta + 2\pi\mathbb{Z}$ with $\theta \in \mathbb{R}$. It will be convenient to work on the cylinder $C = \mathbb{T} \times \mathbb{R}$ with the covering map $\Pi(\theta, r) = (\overline{\theta}, r)$. The periodicity properties of the map S allow to define a new map, also denoted by S, mapping $\Pi(\overline{\Omega})$ into C. In particular this map is a topological embedding. Given a continuous and 2π-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$, the graph in the cylinder

$$\Gamma = \{(\overline{\theta}, \varphi(\theta)) : \theta \in \mathbb{R}\}$$

defines a Jordan curve that is not contractible in C and has positive orientation. We are going to prove that Γ and $\Gamma_1 = S(\Gamma)$ must intersect in at least two points of the cylinder. Let us fix two numbers $\alpha < \beta$ such that Γ and Γ_1 are contained in the finite cylinder $\{\alpha < r < \beta\}$. By the Jordan Curve Theorem on the cylinder there are two connected components of $\{\alpha < r < \beta\} \setminus \Gamma$ and, since the curve is not contractible, the circumferences $r = \alpha$ and $r = \beta$ cannot lie in the same component. We denote these components by $R_\alpha(\Gamma)$ and $R_\beta(\Gamma)$. Since S is a topological embedding we deduce that $\Gamma_1 = S(\Gamma)$ is also a non-contractible Jordan curve and denote by $R_\alpha(\Gamma_1)$, $R_\beta(\Gamma_1)$ the components of $\{\alpha < r < \beta\} \setminus \Gamma_1$. Notice that Γ_1 is not necessarily a graph but it has positive orientation. This is a consequence of the periodicity property of the function F, since $F(\theta + 2\pi, \varphi(\theta + 2\pi)) = F(\theta, \varphi(\theta)) + 2\pi$. We claim that

$$\mu(R_\alpha(\Gamma)) = \mu(R_\alpha(\Gamma_1)),$$

where $\mu = d\theta dr$ is the Haar measure in the cylinder. This result would be almost obvious if the function φ were smooth. In such a case the classical Green's formula implies that

$$\mu(R_\alpha(\Gamma)) = \int_\Gamma r d\theta - 2\pi\alpha, \quad \mu(R_\alpha(\Gamma_1)) = \int_{\Gamma_1} r d\theta - 2\pi\alpha = \int_\Gamma r_1 d\theta_1 - 2\pi\alpha.$$

Here it has been important that both curves are positively oriented. The conclusion follows since

$$\mu(R_\alpha(\Gamma)) - \mu(R_\alpha(\Gamma_1)) = \int_\Gamma (r_1 d\theta_1 - r d\theta) = \int_\Gamma dV = 0.$$

If φ is only continuous we approximate it by C^1 functions φ_n with period 2π and then pass to the limit. Let χ_n and $\chi_n^{(1)}$ be the characteristic functions of $R_\alpha(\Gamma_n)$ and $R_\alpha(S(\Gamma_n))$. Then χ_n converges to the characteristic function of $R_\alpha(\Gamma)$ outside Γ. Since Γ is the graph of a continuous function, Fubini's Theorem implies that it is of measure zero in the cylinder and so the convergence is almost everywhere. Similarly we conclude that $\chi_n^{(1)}$ converges almost everywhere to the characteristic function of $R_\alpha(\Gamma_1)$. This time one uses that S is a Lipschitz continuous map and so it preserves zero measure sets. Once we know that the sets $R_\alpha(\Gamma)$ and $R_\alpha(\Gamma_1)$ have the same measure we can conclude that either $\Gamma = \Gamma_1$ or $\Gamma_1 \cap R_\alpha(\Gamma) \neq \emptyset$ and $\Gamma_1 \cap R_\beta(\Gamma) \neq \emptyset$. In any case there are at least two intersection points.

For each $\theta \in \mathbb{R}$ the equation $F(\theta, r) = \theta + 2N\pi$ has a unique solution $r := \phi(\theta)$. This is a consequence of (21) and the twist condition. In particular the uniqueness implies that ϕ is continuous and 2π-periodic. The graph of ϕ in the cylinder and its image under S must intersect in at least two points of the cylinder. This will complete the proof since the solutions of the system can be obtained as lifts of these intersection points with argument θ in $[0, 2\pi[$.

5 A twist map associated to collisions

From now on it is assumed that $p(t)$ is 2π-periodic. Given $(t_0, h_0) \in \mathbb{R}^2$, the bouncing solution satisfying (5) and (6) will be denoted by $u(t; t_0, h_0)$. Throughout this Section it will be assumed that $p(t)$ is Lipschitz-continuous and so Proposition 4 implies that this solution is unique and globally defined. The number $t_1 > t_0$ will indicate the next instant of collision while h_1 will be the corresponding energy. It can happen that no collisions occur after t_0 and in that case $t_1 = +\infty$. The successor map is defined as

$$S : D \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad S(t_0, h_0) = (t_1, h_1)$$

with

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \ t_1 < +\infty\}.$$

The periodicity of $p(t)$ implies that $u(t; t_0 + 2\pi, h_0) = u(t - 2\pi; t_0, h_0)$ and so

$$S(t_0 + 2\pi, h_0) = S(t_0, h_0) + (2\pi, 0).$$

This identity leads to the interpretation of t_0 as an angle variable.

Given a bouncing solution $u(t; t_0, h_0)$ with successive collisions at times $t_0 < t_1 < \cdots < t_n < \cdots$ and corresponding energies $h_0, h_1, ..., h_n, ...$ each point (t_n, h_n) belongs to D and

$$(t_{n+1}, h_{n+1}) = S(t_n, h_n).$$

This fact explains why S plays an important role in the study of the dynamics of bouncing solutions. In particular the search of periodic solutions of period $2\pi m$ having n collisions in $[0, 2m\pi[$ is reduced to the equation

$$S^n(t_0, h_0) = (t_0 + 2\pi m, h_0).$$

The next result describes the geometry of the set D and shows that S has the twist property.

Proposition 9 *There exists a function $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that*

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \ h_0 < \psi(t_0)\}.$$

This function is 2π-periodic, lower semi-continuous and $\min_{\mathbb{R}} \psi \geq -2||p||_\infty^{1/2}$. The map $S : D \to \mathbb{R}^2$, $S(t_0, h_0) = (t_1, h_1)$, is one-to-one and such that, for each $t_0 \in \mathbb{R}$,

$$h_0 \in]-\infty, \psi(t_0)[\mapsto t_1(t_0, h_0)$$

is increasing.

Notice that the above properties imply that D is an open and connected subset of the plane. Before the proof we need two preliminary results on continuous dependence.

Lemma 10 *The Sundman's integral*

$$S(t; t_0, h_0) = \int_{t_0}^{t} \frac{d\tau}{u(\tau; t_0, h_0)}$$

is continuous as a function of three variables defined on the set

$$\mathcal{D} = \{(t; t_0, h_0) \in \mathbb{R}^3 : \ t_0 < t < t_1\}.$$

Proof. We go back to the process of regularization of collisions and consider the solution of (11) satisfying

$$U(0) = U'(0) = 0, \ T(0) = t_0, \ H(0) = h_0.$$

The Theorem on continuity with respect to initial conditions is applicable to (11) and so the functions

$$U(s; t_0, h_0), \ U'(s; t_0, h_0), \ T(s; t_0, h_0), \ H(s; t_0, h_0)$$

are continuous in the three variables. Assume that $\sigma > 0$ is a number such that

$$U(s; t_0, h_0) > 0 \ \text{ if } s \in]0, \sigma].$$

In particular it is assumed that the corresponding solution (U,T,H) of (11) is well defined on $[0,\sigma]$. Let $\{(t_{0n},h_{0n})\}$ be a sequence converging to (t_0,h_0), we claim that

$$U(s;t_{0n},h_{0n})>0 \ \text{ if } s\in]0,\sigma], \tag{22}$$

for large n. The argument to prove the positivity is analogous to one already employed in the proof of Lemma 5. By continuous dependence the solution of (11) satisfying $U(0)=U'(0)=0$, $T(0)=t_{0n}$, $H(0)=h_{0n}$, is well defined on $[0,\sigma]$ and $U''(\cdot;t_{0n},h_{0n})$ converges to $U''(\cdot;t_0,h_0)$ uniformly on $[0,\sigma]$. The positivity of $U(s;t_{0n},h_{0n})$ in a small but fixed interval of the type $]0,\epsilon]$ follows from Taylor's expansion. Outside this interval $U(s;t_0,h_0)$ is positive and (22) follows easily. The previous argument can also be employed to prove that $\mathcal{D}$ is open in $\mathbb{R}^3$ but we will not need this fact.

To prove the continuity of S on $\mathcal{D}$ we assume that $\{(t_n;t_{0n},h_{0n})\}$ is a sequence in $\mathcal{D}$ converging to $(t;t_0,h_0)\in\mathcal{D}$. We observe that $s_n = S(t_n;t_{0n},h_{0n})$ and $s=S(t;t_0,h_0)$ are such that

$$T(s_n;t_{0n},h_{0n})=t_n \ \text{ and } \ T(s;t_0,h_0)=t.$$

The convergence of s_n to s will be obtained in two steps.

Step 1: $\{s_n\}$ is bounded.

Since $T(\cdot;t_0,h_0)$ is strictly increasing as long as $U(\cdot;t_0,h_0)$ is positive, we find $\sigma>s$ and $\delta>0$ such that $t+\delta<t_1$ and

$$T(\sigma;t_0,h_0)=t+\delta.$$

By continuity we know that, for large n,

$$T(\sigma;t_{0n},h_{0n})>t+\frac{\delta}{2}>t_n.$$

From (22) we know that $T(\cdot;t_{0n},h_{0n})$ is strictly increasing on $[0,\sigma]$ and we deduce that $0<s_n<\sigma$.

Step 2: For every convergent subsequence $\{s_k\}$, $\lim_{k\to\infty}s_k=s$.

Assume that $\{s_k\}\to s^*\in[0,\sigma]$. Since T is continuous, $T(s^*;t_0,h_0)=t$ and the monotonicity of T with respect to s implies that $s=s^*$.

Lemma 11 *The map*

$$(t;t_0,h_0)\in\mathcal{D}\mapsto(u(t;t_0,h_0),\dot{u}(t;t_0,h_0))\in\mathbb{R}^2$$

is continuous.

Proof. We employ the identities

$$u(t;t_0,h_0) = U(s;t_0,h_0), \quad \dot{u}(t;t_0,h_0) = U'(s;t_0,h_0)/U(s;t_0,h_0),$$

where $s = S(t;t_0,h_0)$. The continuity is a consequence of the continuity of U, U' and the previous Lemma.

Proof of Proposition 9. We start with the following
Claim: *Let $u(t)$ be a solution of (1) such that for some $\tau \in \mathbb{R}$,*

$$u(\tau)^2||p||_\infty < 1, \;\; u(\tau) > 0, \;\; \dot{u}(\tau) < 0. \tag{23}$$

Then the first collision after τ occurs at some instant t^* satisfying

$$\tau < t^* < \tau - \frac{u(\tau)}{\dot{u}(\tau)}.$$

Let $t^* > \tau$ be the first instant of collision. In principle we admit the possibility $t^* = +\infty$. From the equation (1) and the conditions at τ it is easy to prove that

$$\ddot{u}(t) < 0, \;\; \dot{u}(t) < \dot{u}(\tau), \;\; \text{if } t \in]\tau, t^*[.$$

Hence,

$$0 < u(t) < u(\tau) + \dot{u}(\tau)(t-\tau), \;\; \tau < t < t^*$$

and this inequality proves the claim.

It is now easy to prove that D is open. If (t_0,h_0) is a point in D we know that $t_1 = t_1(t_0,h_0) < +\infty$ and $u(t_1) = 0$, $\dot{u}(t_1^-) = -\infty$ where $u(t) := u(t;t_0,h_0)$. This allows us to find some $\tau \in]t_0,t_1[$ in the conditions of the Claim (23). The previous Lemma on continuous dependence guarantees the existence of a neighborhood $\mathcal{U}$ of (t_0,h_0) such that if $(\hat{t}_0,\hat{h}_0) \in \mathcal{U}$ and $(\tau;\hat{t}_0,\hat{h}_0) \in \mathcal{D}$ then $\hat{u}(t) := u(t;\hat{t}_0,\hat{h}_0)$ satisfies the condition (23) at $t = \tau$ and so $\hat{t}_1$ is finite and $(\hat{t}_0,\hat{h}_0) \in D$. From the definition of $\mathcal{D}$ we can now deduce that the whole neighborhood $\mathcal{U}$ is contained in D and this proves that D is open.

Next we are going to describe D geometrically. For each $t_0 \in \mathbb{R}$ consider the set of energies producing a collision in the future; that is,

$$\mathcal{C}_{t_0} = \{h_0 \in \mathbb{R} : \; t_1 = t_1(t_0,h_0) < \infty\}.$$

By comparison of the equation (1) and an autonomous equation, $p_1(t) = p(t)$ and $p_2(t) = ||p||_\infty$, we deduce from Lemma 6 that $t_1(t_0,h_0) < \infty$ if $h_0 < -2||p||_\infty^{1/2}$. Hence $\mathcal{C}_{t_0}$ is non-empty, actually it contains the interval

$]-\infty,-2||p||_\infty^{1/2}[$. Again from Lemma 6 we deduce that $\mathcal{C}_{t_0}$ is an interval, now $p_1 = p_2 = p$. Define $\psi(t_0) = \sup \mathcal{C}_{t_0}$. Knowing that D is open we conclude that

$$D = \{(t_0, h_0) : \ h_0 < \psi(t_0)\}.$$

The lower semi-continuity of ψ is automatic. Given $\alpha < \psi(t_0)$, the point (t_0, α) is in D and so we can find a neighborhood of this point contained in D. In particular there exists $\delta > 0$ such that if $|\hat{t}_0 - t_0| < \delta$ then $(\hat{t}_0, \alpha) \in D$, and so $\alpha < \psi(\hat{t}_0)$.

To complete the proof we must discuss the properties of the map S. The uniqueness given by Proposition 4 implies that S is one-to-one. To prove the twist condition fix t_0 and $h_0 < h_0^* < \psi(t_0)$. Then Lemma 7 implies that $t_1 = t_1(t_0, h_0) < t_1^* = t_1(t_0, h_0^*)$.

6 The successor map is exact symplectic

In the variables time and energy the successor map can be interpreted as a return map associated to the differential equation (1) and the transversal section $\{u = 0\}$. There are standard methods to prove that certain return maps associated to Hamiltonian flows are exact symplectic. The general theory can be seen in Chapter 9 of [3] and some related examples can be found in [1, 11]. However our situation is more delicate because the section $u = 0$ coincides with the set of singularities of the equation. To overcome this difficulty we will approximate S by the return map associated to the section $\{u = \epsilon\}$ with $\epsilon > 0$. To make precise this idea consider $(t_0, h_0) \in \mathbb{R}^2$ and $\epsilon > 0$ with $h_0 + \frac{1}{\epsilon} > 0$. The solution of (1) satisfying

$$u(t_0) = \epsilon, \ \ \dot{u}(t_0) = +\sqrt{2h_0 + \frac{2}{\epsilon}}$$

will be denoted by $u(t; t_0, h_0, \epsilon)$. These initial data have been chosen so that

$$h_0 = \frac{1}{2}\dot{u}(t_0)^2 - \frac{1}{u(t_0)}$$

and it seems reasonable to extend this family of solutions to $\epsilon = 0$. From now on the family $u(t; t_0, h_0)$ appearing in the previous Section will be interpreted as $u(t; t_0, h_0, 0)$.

Proposition 12 *Assume that the forcing $p(t)$ is of class C^1 and let (t_0^*, h_0^*) be a given point of D. Then there exists $\epsilon^* > 0$, a neighborhood $\mathcal{V}$ of (t_0^*, h_0^*) and two functions $\tau, \mathcal{H} : \mathcal{V} \times [0, \epsilon^*] \to \mathbb{R}$ of class $C^{1,0}$ satisfying* *(i)* $S(t_0, h_0) = (\tau(t_0, h_0, 0), \mathcal{H}(t_0, h_0, 0))$ *for each* $(t_0, h_0) \in \mathcal{V}$

(ii) Given $\epsilon \in]0, \epsilon^*]$, $\tau = \tau(t_0, h_0, \epsilon)$ *is such that* $\tau > t_0$, $u(t; t_0, h_0, \epsilon) = \epsilon$, $u(t; t_0, h_0, \epsilon) > \epsilon$ *if* $t \in]t_0, \tau[$, $\mathcal{H}(t_0, h_0, \epsilon) = \frac{1}{2}\dot{u}(\tau; t_0, h_0, \epsilon)^2 - \frac{1}{\epsilon}$.

Remark. A function $f : \mathcal{V} \times [0, \epsilon^*] \to \mathbb{R}$, $f = f(t_0, h_0; \epsilon)$ is of class $C^{1,0}$ if it is continuous, $(t_0, h_0) \in \mathcal{V} \mapsto f(t_0, h_0; \epsilon)$ is of class C^1 for each $\epsilon \in [0, \epsilon^*]$ and the partial derivatives $\frac{\partial f}{\partial t_0}$, $\frac{\partial f}{\partial h_0}$ are continuous as functions of the three variables. In particular it follows from the previous Proposition that the successor map S is C^1 in D.

Proof. Given $(t_0, h_0) \in \mathbb{R}^2$ and $\epsilon \geq 0$ with $\epsilon h_0 + 1 > 0$, there is a unique solution of (11) with initial conditions

$$U(0) = \epsilon,\ U'(0) = +\sqrt{2\epsilon^2 h_0 + 2\epsilon},\ T(0) = t_0,\ H(0) = h_0, \tag{24}$$

denoted by $U(s; t_0, h_0, \epsilon)$, $T(s; t_0, h_0, \epsilon)$, $H(s; t_0, h_0, \epsilon)$. The value of $U'(0)$ has been adjusted so that the first integral $\mathcal{I}$ given by (13) vanishes for this solution. The function $(h_0, \epsilon) \mapsto \sqrt{2\epsilon^2 h_0 + 2\epsilon}$ is of class $C^{1,0}$ on $\epsilon \geq 0$, $\epsilon h_0 + 1 > 0$, and this regularity is inherited by U, U', T and H as functions of (s, t_0, h_0) and ϵ. At this point the regularity of $p(t)$ is important to guarantee the applicability of the Theorem of differentiability with respect to initial conditions.

The point $(t_0^*, h_0^*) \in D$ has been fixed and t_1^* is the first zero to the right of t_0^* for the solution $u(t; t_0^*, h_0^*, 0)$. For the associated solution of (11) we can find $\sigma_0^* > 0$ such that $t_1^* = T(\sigma_0^*; t_0^*, h_0^*, 0)$ and

$$U(s; t_0^*, h_0^*, 0) > 0, \ \text{ if } s \in]0, \sigma_0^*[, \quad U(\sigma_0^*; t_0^*, h_0^*, 0) = 0.$$

In particular $[0, \sigma_0^*]$ in contained in the maximal interval of this solution.

We would like to obtain a function $\sigma = \sigma(t_0, h_0, \epsilon)$ by an application of the Implicit Function Theorem to the problem

$$U(\sigma; t_0, h_0, \epsilon) = \epsilon, \ \ \sigma(t_0^*, h_0^*, 0) = \sigma_0^*. \tag{25}$$

This is not possible since $\mathcal{I} = 0$ at this solution and so $U'(s; t_0^*, h_0^*, 0)$ has to vanish at $s = \sigma_0^*$. However the equation for U in (11) shows that

$$U''(\sigma_0^*; t_0^*, h_0^*, 0) = 1$$

and this allows to apply the Implicit Function Theorem to the problem

$$U'(\sigma; t_0, h_0, \epsilon) = -\sqrt{2\epsilon + 2\epsilon^2 H(\sigma; t_0, h_0, \epsilon)}, \ \ \sigma(t_0^*, h_0^*, 0) = \sigma_0^*. \tag{26}$$

Indeed we need a slight variant of this Theorem because we do not have differentiability in the parameter ϵ. Given a problem of the type

$$F(x, \lambda; \mu) = 0, \ \ x(\lambda_0, \mu_0) = x_0$$

with F of class $C^{1,0}$ and $\det\{\frac{\partial F}{\partial x}(x_0, \lambda_0; \mu_0)\} \neq 0$, then the solution $x = x(\lambda, \mu)$ is also of class $C^{1,0}$. In our case we obtain a function $\sigma = \sigma(t_0, h_0, \epsilon)$ with differentiability in t_0 and h_0. The equation appearing in (26) has been deduced from the first integral (13) with $\mathcal{I} = 0$, $U = \epsilon$, $U' > 0$. Next we prove that the function solving (26) is also a solution of (25), at least in a small neighborhood of $(t_0^*, h_0^*, 0)$, say $\mathcal{V}_1 \times [0, \epsilon]$. To prove this we fix numbers $\nu > 0$ and $\eta > 0$ with $2\eta\nu < 1$ and $\nu > |h_1^*|$. Here h_1^* is the energy of $u(t; t_0^*, h_0^*, 0)$ at $t = t_1^*$. The neighborhood can be chosen so small that, for $\sigma = \sigma(t_0, h_0, \epsilon)$,

$$|H(\sigma; t_0, h_0, \epsilon)| \leq \nu, \ \ |U(\sigma; t_0, h_0, \epsilon)| \leq \eta \ \text{ if } (t_0, h_0, \epsilon) \in \mathcal{V}_1 \times [0, \epsilon]. \tag{27}$$

and the derivative of U satisfies $U'(\sigma; t_0, h_0, \epsilon) \leq 0$. The identity $\mathcal{I} = 0$ for $s = \sigma$ leads to

$$U'(\sigma; t_0, h_0, \epsilon) = -\sqrt{2U(\sigma; t_0, h_0, \epsilon) + 2U(\sigma; t_0, h_0, \epsilon)^2 H(\sigma; t_0, h_0, \epsilon)}. \tag{28}$$

The estimates (27) are now useful to deduce that the function

$$\xi \in [-\eta, \eta] \mapsto 2\xi + 2\xi^2 H(\sigma; t_0, h_0, \epsilon)$$

is one-to-one. The identities in (26) and (28) imply that $U(\sigma; t_0, h_0, \epsilon) = \epsilon$.

Our next task is to prove that

$$U(s; t_0, h_0, \epsilon) > \epsilon \ \ \text{ if } s \in]0, \sigma(t_0, h_0, \epsilon)[\tag{29}$$

where $(t_0, h_0) \in \mathcal{V}_2$ and $\epsilon \in [0, \epsilon_2]$ for new and smaller neighborhoods. If the above statement were false, there should exist sequences $\{(t_{0n}, h_{0n})\} \to (t_0^*, h_0^*)$ and $\epsilon_n \downarrow 0$ with

$$U(\hat{\sigma}_n; t_{0n}, h_{0n}, \epsilon_n) = \epsilon_n \ \text{ for some } \hat{\sigma}_n, \ 0 < \hat{\sigma}_n < \sigma_n := \sigma(t_{0n}, h_{0n}, \epsilon_n).$$

The sequence σ_n converges to $\sigma(t_0^*, h_0^*, 0) = \sigma_0^*$ and we extract a convergent subsequence of $\{\hat{\sigma}_n\}$, say $\hat{\sigma}_k$, with $\lim_{k\to\infty} \hat{\sigma}_k = \ell$, $0 \leq \ell \leq \sigma_0^*$. The continuity of U implies that ℓ is a zero of $U(\cdot; t_0^*, h_0^*, 0)$ and so either $\ell = 0$ or $\ell = \sigma_0^*$. The function $U''(s; t_0^*, h_0^*, \epsilon)$ is continuous in all its variables as long as it is defined. This is a consequence of continuous dependence and the first equation in (11). Moreover

$$U''(0; t_0^*, h_0^*, 0) = U''(\sigma_0^*; t_0^*, h_0^*, 0) = 1$$

and therefore it is possible to find $k_0 > 0$ and $\delta > 0$ such that

$$U''(s; t_{0k}, h_{0k}, \epsilon_k) \geq \frac{1}{2} \text{ if } s \in [0, \delta] \cup [\sigma_0^* - \delta, \sigma_0^*],\ k \geq k_0.$$

In particular $U'(\cdot; t_{0k}, h_{0k}, \epsilon_k)$ is strictly increasing in $[0, \delta]$ and $[\sigma_0^* - \delta, \sigma_0^*]$. From (26) and (24),

$$U'(\sigma_k; t_{0k}, h_{0k}, \epsilon_k) \leq 0 \leq U'(0; t_{0k}, h_{0k}, \epsilon_k).$$

Hence $U(\cdot; t_{0k}, h_{0k}, \epsilon_k)$ is increasing in $[0, \delta]$ and decreasing in $[\sigma_k - \delta, \sigma_k]$. Thus

$$U(s; t_{0k}, h_{0k}, \epsilon_k) > \epsilon_k \quad s \in]0, \delta] \cup [\sigma_k - \delta, \sigma_k[.$$

Since $U(\cdot; t_0^*, h_0^*, 0)$ is positive on $[\delta, \sigma_0^* - \delta]$ this is not compatible with the existence of $\hat{\sigma}_k$. Once we know that (29) holds we define

$$\tau(t_0, h_0; \epsilon) = T(\sigma(t_0, h_0, \epsilon); t_0, h_0, \epsilon),\ \ \mathcal{H}(t_0, h_0, \epsilon) = H(\sigma(t_0, h_0, \epsilon); t_0, h_0, \epsilon).$$

By chain rule we observe that these functions are in the class $C^{1,0}$. The properties (i) and (ii) are a consequence of the known connections between the original equation (1) and the solutions with $\mathcal{I} = 0$ of (11).

Proposition 13 *Assume that $p(t)$ is of class C^1. Then the differential form $h_1 dt_1 - h_0 dt_0$ is exact in the cylinder. This means that $dG = h_1 dt_1 - h_0 dt_0$ for some function $G = G(t_0, h_0)$ in $C^1(D)$ which is 2π-periodic in t_0. Notice that $(t_1, h_1) = S(t_0, h_0)$.*

Proof. Consider the differential form $\omega = hdt$ defined in the plane with coordinates (t, h) or in the cylinder with coordinates $(\bar{t}, h)$, $\bar{t} = t + 2\pi\mathbb{Z}$. It is enough to prove that

$$\int_\Gamma \omega = \int_{\Gamma_1} \omega$$

for each smooth Jordan curve in the cylinder with lift contained in D and $\Gamma_1 = S(\Gamma)$. From now on we assume that Γ is not contractible to a point, the contractible case can be treated similarly. To start with we fix a parametrization of the curve given by

$$t_0 = t_0(\xi),\ \ h_0 = h_0(\xi)$$

satisfying

$$t_0(\xi + 2\pi) = t_0(\xi) + 2\pi,\ \ h_0(\xi + 2\pi) = h_0(\xi).$$

By a compactness argument applied to Proposition 12 we can extend the functions τ, $\mathcal{H}$ to $\mathcal{W} \times [0, \epsilon_1]$, where $\mathcal{W}$ is a neighborhood of Γ and $\epsilon_1 > 0$. Next we consider a three dimensional space with coordinates u, v, t containing the surface S_ϵ with parametric equations

$$u = u(t'; t_0(\xi), h_0(\xi), \epsilon), \; v = \dot{u}(t'; t_0(\xi), h_0(\xi), \epsilon), \; t = t',$$

where $\xi \in [0, 2\pi], \; t_0(\xi) \leq t' \leq \tau(t_0(\xi), h_0(\xi), \epsilon)$. This is a surface contained in $\{u \geq \epsilon\}$ that is smooth everywhere excepting at four corner points at the boundary. These are the points corresponding to $\xi = 0$ or 2π and $t = t_0(\xi)$ or $\tau(t_0(\xi), h_0(\xi), \epsilon)$. Notice that, rigorously speaking, we must prove that the map $X : (\xi, t') \mapsto (u, v, t)$ is a chart for S_ϵ. This means that X is one-to-one, of class C^1 and such that the Jacobian matrix $DX(\xi, t')$ has rank two. The most delicate point is the computation of the rank. Notice that

$$DX(\xi, t') = \begin{pmatrix} u_\xi & \dot{u} \\ \dot{u}_\xi & \ddot{u} \\ 0 & 1 \end{pmatrix}, \quad \text{where} \begin{pmatrix} u_\xi \\ \dot{u}_\xi \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial t_0} & \frac{\partial u}{\partial h_0} \\ \frac{\partial \dot{u}}{\partial t_0} & \frac{\partial \dot{u}}{\partial h_0} \end{pmatrix} \begin{pmatrix} t_0'(\xi) \\ h_0'(\xi) \end{pmatrix}$$

and $u = u(t; t_0(\xi), h_0(\xi), \epsilon)$. The functions $y_1 = \frac{\partial u}{\partial t_0}$, $y_2 = \frac{\partial u}{\partial h_0}$ are solutions of the linearized equation

$$\ddot{y} = \frac{2}{u(t; t_0(\xi), h_0(\xi), \epsilon)^3} y \tag{30}$$

with respective initial conditions $y_1(t_0) = -\sqrt{2h_0 + \frac{2}{\epsilon}}$, $\dot{y}_1(t_0) = \frac{1}{\epsilon^2} - p(t_0)$ and $y_2(t_0) = 0$, $\dot{y}_2(t_0) = \frac{1}{\sqrt{2h_0 + \frac{2}{\epsilon}}}$. The Jacobi-Liouville formula applied to (30) implies that the Wronskian of y_1 and y_2 is constant, namely

$$\det \begin{pmatrix} \frac{\partial u}{\partial t_0} & \frac{\partial u}{\partial h_0} \\ \frac{\partial \dot{u}}{\partial t_0} & \frac{\partial \dot{u}}{\partial h_0} \end{pmatrix} = -1.$$

Since Γ is a regular curve, the velocity vector $(t_0'(\xi), h_0'(\xi))$ is not zero and so the vector $(u_\xi, \dot{u}_\xi)$ does not vanish. This implies that $DX(\xi, t')$ has rank two.

We intend to apply Stokes Theorem on S_ϵ. Following [3] we consider the Poincaré-Cartan differential form

$$\Omega = v du - E dt, \;\; E = \frac{1}{2} v^2 - \frac{1}{u} + p(t) u.$$

This form is defined in $\mathbb{R}^3$, with coordinates u, v, t and we are going to restrict it to S_ϵ. This means that we consider $\Sigma = i^* \Omega$, pull-back form associated to the inclusion $i : S_\epsilon \to \mathbb{R}^3$. Using ξ and t as coordinates in S_ϵ we express Σ in terms of $d\xi$ and dt,

$$\Sigma = v(u_\xi d\xi + vdt) - Edt.$$

Analogously we express $d\Sigma$ in terms of $d\xi \wedge dt$,

$$d\Sigma = (vv_\xi - \dot{v}u_\xi - E_u u_\xi - E_v v_\xi) d\xi \wedge dt.$$

By construction S_ϵ is composed by trajectories of the Hamiltonian system $\dot{u} = E_v, \quad \dot{v} = -E_u$ and therefore $d\Sigma = 0$. By Stokes Theorem $\int_{\partial S_\epsilon} \Sigma = \int_{S_\epsilon} d\Sigma = 0$ and the integral on the boundary is split in

$$\int_{\partial S_\epsilon} = \int_{\Gamma_\epsilon} + \int_{\tilde{\gamma}_\epsilon} - \int_{\tilde{\Gamma}_\epsilon} - \int_{\gamma_\epsilon},$$

where

$$\Gamma_\epsilon : \begin{cases} u = \epsilon, \\ v = +\sqrt{2(h_0(\xi) + \frac{1}{\epsilon})}, \\ t = t_0(\xi), \end{cases} \qquad \hat{\Gamma}_\epsilon : \begin{cases} u = \epsilon, \\ v = +\sqrt{2(\mathcal{H}(t_0(\xi), h_0(\xi), \epsilon) + \frac{1}{\epsilon})}, \\ t = \tau(t_0(\xi), h_0(\xi), \epsilon) \end{cases}$$

with $\xi \in [0, 2\pi]$ and γ_ϵ, $\tilde{\gamma}_\epsilon$ are the trajectories associated to $\xi = 0$ and $\xi = 2\pi$. The periodicity of the curve and the differential equation imply that

$$u(t; t_0(2\pi), h_0(2\pi), \epsilon) = u(t - 2\pi; t_0(0), h_0(0), \epsilon).$$

As a consequence

$$\tilde{\gamma}_\epsilon = \gamma_\epsilon + (0, 0, 2\pi) \text{ and } \int_{\tilde{\gamma}_\epsilon} \Sigma = \int_{\gamma_\epsilon} \Sigma.$$

In this way we arrive at $\int_{\Gamma_\epsilon} \Sigma = \int_{\tilde{\Gamma}_\epsilon} \Sigma$, equivalent to

$$\int_{\Gamma_\epsilon} Edt = \int_{\tilde{\Gamma}_\epsilon} Edt$$

because $u = \epsilon$ on $\Gamma_\epsilon \cup \tilde{\Gamma}_\epsilon$ and so du vanishes on these curves. These integrals can be expressed as

$$\int_0^{2\pi} \{h_0(\xi) + \epsilon p(t_0(\xi))\} t_0'(\xi) d\xi = \int_{\Gamma_\epsilon} Edt = \int_{\tilde{\Gamma}_\epsilon} Edt =$$

$$\int_0^{2\pi} \{\mathcal{H}(t_0(\xi), h_0(\xi), \epsilon) + \epsilon p(\tau(t_0(\xi), h_0(\xi), \epsilon))\}\tau'(\xi)d\xi,$$

with

$$\tau'(\xi) = \frac{\partial \tau}{\partial t_0}(t_0(\xi), h_0(\xi), \epsilon)t_0'(\xi) + \frac{\partial \tau}{\partial h_0}(t_0(\xi), h_0(\xi), \epsilon)h_0'(\xi).$$

Letting ϵ to go to zero and using the property (i) stated in Proposition 12,

$$\int_0^{2\pi} h_0(\xi)t_0'(\xi)d\xi = \int_0^{2\pi} h_1(t_0(\xi), h_0(\xi))\frac{d}{d\xi}t_1(t_0(\xi), h_0(\xi))d\xi$$

where $t_1 = t_1(t_0, h_0)$, $h_1 = h_1(t_0, h_0)$, are the components of S. This is precisely the identity we were looking for.

7 An asymptotic expansion for the successor map and the completion of the proofs

In this Section we discuss the behavior of the map $(t_1, h_1) = S(t_0, h_0)$ when the energy is negative and tends to infinity. The main result is

Proposition 14 *Assume that $p(t)$ is Lipschitz-continuous and 2π-periodic. Then*

$$\begin{cases} t_1 = t_0 + \frac{\pi}{\sqrt{2}|h_0|^{3/2}} + O(\frac{1}{|h_0|^{5/2}}), \\ h_1 = h_0 + O(\frac{1}{|h_0|^{5/2}}), \end{cases}$$

when $h_0 \to -\infty$, uniformly in $t_0 \in \mathbb{R}$.

The method of proof will be comparison with the autonomous equation for $P_+ = ||p||_\infty$ and $P_- = -||p||_\infty$. In the notations of Subsection 3.3 and according to Lemma 6,

$$\tau(h_0, P_-) \le t_1 - t_0 \le \tau(h_0, P_+).$$

A repetition of some of the arguments in the proof of Proposition 4 leads to

$$|h_1 - h_0| \le ||\dot{p}||_\infty U(h_0, P_+)\tau(h_0, P_+).$$

The proof of the above result is a direct consequence of these inequalities and the following expansions.

Lemma 15 *For any $P \neq 0$,*

$$U(h_0, P) = \frac{1}{|h_0|} + O(\frac{1}{|h_0|^2}), \;\; \tau(h_0, P) = \frac{\pi}{\sqrt{2}|h_0|^{3/2}} + O(\frac{1}{|h_0|^{5/2}})$$

as $h_0 \to -\infty$.

Proof. The first assertion is almost automatic since $U(h_0, P)$ can be expressed as

$$U(h_0, P) = \frac{2}{-h_0 + \sqrt{h_0^2 - 4P}}.$$

For the expansion of τ we assume $P > 0$. The case of negative P is similar. The polynomial $1 + Px^2 + h_0 x$ can be factorized as $P(\alpha - x)(\beta - x)$ with $0 < \alpha = U(h_0, P) < \beta = \frac{-h_0 + \sqrt{h_0^2 - 4P}}{2P}$. Then

$$\tau(h_0, P) = \sqrt{\frac{2}{P}} \int_0^\alpha \frac{\sqrt{\xi}}{\sqrt{(\alpha - \xi)(\beta - \xi)}} d\xi,$$

and the change of variables $\xi = \alpha v$ leads to

$$\tau(h_0, P) = \sqrt{\frac{2}{P}} \alpha \int_0^1 \frac{\sqrt{v}}{\sqrt{(1 - v)(\beta - \alpha v)}} dv.$$

Next we claim that

$$\frac{1}{\sqrt{\beta - \alpha v}} = \frac{1}{\sqrt{\beta}} + O(\frac{1}{|h_0|^{5/2}}), \;\; \text{as } h_0 \to \infty, \; \text{uniformly in } v \in [0, 1].$$

Indeed

$$0 \leq \frac{1}{\sqrt{\beta - \alpha v}} - \frac{1}{\sqrt{\beta}} \leq \frac{1}{\sqrt{\beta - \alpha}} - \frac{1}{\sqrt{\beta}} = \frac{\alpha}{\sqrt{\beta - \alpha}\sqrt{\beta}(\sqrt{\beta} + \sqrt{\beta - \alpha})}$$

and the claim follows because $\beta |h_0|^{-1} \to \frac{1}{P}$ and $\alpha |h_0| \to 1$ as $h_0 \to -\infty$. From $\alpha\beta = \frac{1}{P}$ we observe that

$$\frac{\alpha}{\sqrt{\beta}} = \sqrt{P}\alpha^{3/2} = \sqrt{P}(\frac{1}{|h_0|} + O(\frac{1}{|h_0|^2}))^{3/2} = \frac{\sqrt{P}}{|h_0|^{3/2}} + O(\frac{1}{|h_0|^{5/2}}).$$

Going back to the integral

$$\tau(h_0, P) = \sqrt{\frac{2}{P}} \frac{\alpha}{\sqrt{\beta}} \int_0^1 \frac{\sqrt{v}}{\sqrt{1 - v}} dv + O(\frac{1}{|h_0|^{5/2}}) = \frac{\pi}{\sqrt{2}} \alpha^{3/2} + O(\frac{1}{|h_0|^{5/2}}),$$

and the expansion for t_1 follows from the expansion for $\alpha = U$.

Proof of Theorems 1 and 2. The discussions of Section 5 show that it is enough to prove the existence of two solutions of the system

$$S(t_0, h_0) = (t_0 + 2\pi N, h_0), \;\; t_0 \in [0, 2\pi[, \;\; (t_0, h_0) \in D$$

for each $N \geq 1$. To this end we are going to apply Theorem 8 with $\theta = t_0$, $r = h_0$, and

$$\Omega = \{(t_0, h_0) \in \mathbb{R}^2 : \; a < h_0 < \psi(t_0)\}$$

with $a < -2||p||_\infty^{1/2}$. The constant a is chosen so that $t_1 - t_0 < 2\pi N$ whenever $t_1 = t_1(t_0, h_0)$ and $h_0 = a$. This is possible thanks to Proposition 14. It remains to prove that for each $t_0 \in \mathbb{R}$ there exists h_0^* with $a < h_0^* < \psi(t_0)$ such that $t_1 - t_0 > 2\pi N$ with $t_1 = t_1(t_0, h_0^*)$. We shall distinguish two cases depending on whether $\psi(t_0)$ is finite or infinite.

Case i) $\psi(t_0) < \infty$.

We take an increasing sequence $\{h_{0n}\}$ converging to $\psi(t_0)$ and prove that, for some n,

$$t_{1n} = t_1(t_0, h_{0n}) > t_0 + 2\pi N.$$

By a contradiction argument assume that $t_{1n} - t_0 \leq 2\pi N$ for all n. Then we extract a convergent subsequence $t_{1n} \to \eta \in [t_0, t_0 + 2\pi N]$. The use of Lemma 6 with $p_1(t) = -||p||_\infty$, $p_2(t) = p(t)$ implies that $t_{1n} - t_0 \geq \tau(h_{0n}, -||p||_\infty)$. Passing to the limit, $t_0 + 2\pi N \geq \eta \geq t_0 + \tau(\psi(t_0), -||p||_\infty) > t_0$. For large n, $t_0 < \hat{\eta} < t_{1n}$ with $\hat{\eta} = \frac{t_0+\eta}{2}$. From here we deduce that the triplet $(\hat{\eta}; t_0, h_{0n})$ belong to $\mathcal{D}$ for large n. Here $\mathcal{D}$ is the set introduced in Lemma 11. Since $t_1 = +\infty$ if $h_0 = \psi(t_0)$ we deduce that also $(\hat{\eta}; t_0, \psi(t_0)) \in \mathcal{D}$. In consequence

$$(u(\hat{\eta}; t_0, h_{0n}), \dot{u}(\hat{\eta}; t_0, h_{0n})) \to (u(\hat{\eta}; t_0, \psi(t_0)), \dot{u}(\hat{\eta}; t_0, \psi(t_0)))$$

as $n \to \infty$. The solution $u(t; t_0, \psi(t_0))$ has no collisions after t_0 and so it is well defined and positive on the compact interval $[\hat{\eta}, t_0 + 2\pi N + 1]$. The standard Theorem on continuous dependence applied to (1) says that, for large n, also $u(t; t_0, h_{0n})$ is well defined and positive on this interval. This is contradictory with the assumption $t_{1n} \to \eta \leq t_0 + 2\pi N$.

Case ii) $\psi(t_0) = +\infty$.

We go back to Subsection 3.3 and select $h_0^* > 0$ large enough so that $\tau(h_0^*, P) > 2\pi N$ with $P = -||p||_\infty$. By comparison $t_1 - t_0 \geq \tau(h_0^*, P) > 2\pi N$.

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