Complete orbits for twist maps on the plane

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Abstract

In this paper twist maps $(\theta_1, r_1) = f(\theta, r)$ are considered with no assumption on the periodicity of the map in θ. Under appropriate assumptions the existence of infinitely many bounded (in r) complete orbits is proven. In particular our results apply to the class of maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin \omega_1(\theta + r) + \sin \omega_2(\theta + r)),$$

where $\lambda > 0$ and no arithmetic condition has to be imposed on ω_1/ω_2.

1 Introduction

By now the mathematical investigation of twist maps has a long tradition. Originating in the work of Poincaré, a major relevance of these maps lies in their relation to continuous time dynamical systems models. From the very beginning of the theory the main focus has been in maps $f = f(\theta, r) : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ that are periodic in the θ-variable. In this case f should be properly viewed as the lift of the actual map to the universal covering of the cylinder. The topic of periodic twist maps turned out to be very fruitful, leading in particular to the development of KAM theory and Aubry-Mather theory for Hamiltonian systems; see [4, 2] for an overview and many relevant references. There are also some papers dealing with twist maps which are quasi-periodic in the angle (see in particular [5, 11, 7]). In the present paper we consider general twist maps $f = f(\theta, r)$ on the plane without imposing any assumption of periodicity or almost periodicity in the θ-variable. To illustrate the type of results which we obtain, we remain for a moment in the quasi-periodic context and consider the family of maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin \omega_1(\theta + r) + \sin \omega_2(\theta + r)), \tag{1.1}$$

where $\lambda, \omega_1, \omega_2 > 0$. The KAM method is applicable when λ is small and ω_1/ω_2 satisfies a diophantine condition. It leads to the existence of invariant curves and, as a consequence, to the boundedness of all orbits. In contrast our results apply to arbitrary parameters λ, ω_1, and ω_2, and they lead to a more modest conclusion: The existence of infinitely many complete orbits $(\theta_n, r_n)_{n\in\mathbb{Z}}$ which are bounded and which have finite upper and lower rotation numbers,

$$\sup_{n\in\mathbb{Z}} |r_n| < \infty, \quad -\infty < \liminf_{|n|\to\infty} \frac{\theta_n}{n} \le \limsup_{|n|\to\infty} \frac{\theta_n}{n} < \infty.$$

Our original motivation to study aperiodic twist maps was due to the so-called Littlewood problem for oscillatory differential equations. The basic question there is to decide on the boundedness or unboundedness of the solutions to

$$\ddot{x} + F(x) = p(t), \tag{1.2}$$

where F is a nonlinear function satisfying $F(x) \to \pm\infty$ as $x \to \pm\infty$ and the forcing p is bounded. Since the first definitive answer by Morris [8] for $F(x) = 2x^3$ and periodic p (in which case all solutions are bounded), the method of proof has been to describe the dynamics of the differential equation by means of a certain twist map f. This is followed by the application of a suitable invariant curve theorem. After Morris' result there were many others for the periodic case and also some in the quasi-periodic case with diophantine frequencies [5]. If the forcing p is only bounded, Littlewood himself already noticed in [6] that unbounded motions could appear even for the nonlinearity $F(x) = 2x^3$. However for such p bounded and unbounded motions must coexist, as was recently shown in [10, 9]. In the aperiodic case it is still possible to associate a twist map to equation (1.2). Assume that we are given a solution satisfying $x(\tau) = 0$ and $\dot{x}(\tau) = v$ with $v > 0$ large enough. Then we compute the next positive zero $\tau_1 > \tau$ such that $x(\tau_1) = 0$ and $\dot{x}(\tau_1) = v_1 > 0$ and consider the map $(\tau, v) \mapsto (\tau_1, v_1)$. The role of the angle θ is played by the variable τ and, if the forcing p is not periodic, then also τ will not be periodic. Therefore it seems reasonable to expect that the study of general twist maps should be helpful in understanding the dynamics of the non-autonomous equation (1.2). The results of this paper are not directly useful for (1.2), and we preferred to consider the quasi-periodic map (1.1) as a first application. We plan to give some further applications, including ones the Littlewood problem, in a follow-up to this paper.

We finish this introduction with an outline of the main contents. In Section 2 we consider a Lagrangian formulation and work only with the sequence of angles $(\theta_n)_{n\in\mathbb{Z}}$. They satisfy the equations

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z},$$

where h is a generating function. Assuming that h grows quadratically we prove the existence of solutions with bounded upper and lower rotation numbers. This is reminiscent of Aubry-Mather theory but the map h does not have any periodicity property, which leads to some complications. The rest of the paper is devoted to maps of the form

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r). \tag{1.3}$$

In Section 3 the theorem for quadratic generating functions is applied to (1.3), in the case where (1.3) leaves invariant the boundary of the strip where it is defined. In Section 4 the latter hypothesis is dropped. Then it is not enough to assume that f is exact symplectic, in the sense that $r_1 d\theta_1 - r d\theta = d\hat{h}$ for some C^2-function $\hat{h} = \hat{h}(\theta, r)$: Even if f is periodic in θ, in order that f be exact on the cylinder one additionally needs that $\hat{h}$ is periodic in θ. Having this observation in mind, we introduce a natural generalized notion of 'exact symplectic' in Definition 4.1. Under the sole (technical) restriction that $|\partial_r F| \le \frac{1}{43}$, it is then shown in Theorem 4.3 that (1.3) admits infinitely many complete orbits. In addition, the r-component of these orbits can be controlled, as is the case for the upper and lower rotation numbers of the orbits. Section 4 also contains an easy application of Theorem 4.3 to (1.1), leading to the results mentioned at the beginning of this introduction.

2 Complete orbits for quadratic generating functions

The main result of this section is about the existence of a complete orbit under the assumption that the generating function $h = h(\theta, \theta')$ on the plane grows quadratically. Although no periodicity condition is imposed on h, we will use θ and θ' to denote the variables.

Theorem 2.1 *Let $\Delta > \delta > 0$. Suppose that $h : \Omega = \{(\theta, \theta') \in \mathbb{R}^2 : \delta \le \theta' - \theta \le \Delta\} \to \mathbb{R}$ is C^1 and such that*

$$\underline{\alpha}(\theta' - \theta)^2 \le h(\theta, \theta') \le \overline{\alpha}(\theta' - \theta)^2, \quad (\theta, \theta') \in \Omega, \tag{2.1}$$

for some constants $\overline{\alpha} \ge \underline{\alpha} > 0$ so that $\overline{\alpha} < \frac{3}{2}\underline{\alpha}$. Then there is a constant $\sigma_{**} \ge 1$ (depending only on $\overline{\alpha}/\underline{\alpha} \in [1, \frac{3}{2}[$) with the following property. If

$$\sigma_{**}\delta < \sigma_{**}^{-1}\Delta, \tag{2.2}$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ such that $|\theta_0^*| \le \Delta$, $\delta \le \theta_{n+1}^* - \theta_n^* \le \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}^*, \theta_n^*) + \partial_1 h(\theta_n^*, \theta_{n+1}^*) = 0, \quad n \in \mathbb{Z}.$$

Moreover,

$$\delta \le \liminf_{n\to\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to\infty} \frac{\theta_n^*}{n} \le \Delta, \quad \delta \le \liminf_{n\to-\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to-\infty} \frac{\theta_n^*}{n} \le \Delta. \tag{2.3}$$

Before we go on to the proof of Theorem 2.1, we include an example.

Example 2.2 Consider

$$h(\theta, \theta') = \frac{1}{2}(\theta - \theta')^2 - V(\theta)$$

for some potential $V \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\|V\|_\infty \le \frac{1}{42}$. Defining $\overline{\alpha} = \frac{1}{2} + \|V\|_\infty$ and $\underline{\alpha} = \frac{1}{2} - \|V\|_\infty$, then $\overline{\alpha}/\underline{\alpha} \in [1, \frac{11}{10}]$. Therefore Corollary 2.8 below shows that $\sigma_{**} = 10$ can be taken in (2.2). Thus, for instance choosing $\delta = 1$ and $\Delta = 101$, it is found that the difference equation (discretization of a Newtonian equation)

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = -V'(\theta_n), \quad n \in \mathbb{Z},$$

has a solution $(\theta_n^*)_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that (2.3) is satisfied for $\delta = 1$ and $\Delta = 101$, $|\theta_0^*| \le 101$, and $1 \le \theta_{n+1}^* - \theta_n^* \le 101$ for $n \in \mathbb{Z}$.

Returning to the proof of Theorem 2.1, it is split into several parts. First we need to construct $(\theta_n^*)_{n\in\mathbb{Z}}$ on finite segments $-N \le n \le N$. To do this, for fixed $A > 0$, $N \in \mathbb{N}$, and $\Delta > \delta > 0$, put

$$\Sigma^{(N)} = \Big\{\Theta = (\theta_n)_{-N\le n\le N} : \theta_{\pm N} = \pm A,\ \delta \le \theta_{n+1} - \theta_n \le \Delta \text{ for } n = -N, \dots, N-1\Big\}.$$

Since later $A = A_N$ will be chosen to depend on N, the dependence of $\Sigma^{(N)}$ on A is suppressed in our notation.

Lemma 2.3 *If $\delta \le \frac{A}{N} \le \Delta$, then $\Sigma^{(N)} \ne \emptyset$, and $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is compact.*

Proof: Defining $\hat\Theta$ by $\hat\theta_n = -A + \frac{A}{N}(n+N)$, we see that $\hat\Theta \in \Sigma^{(N)}$, due to $\hat\theta_{n+1} - \hat\theta_n = \frac{A}{N} \in [\delta, \Delta]$, and in particular, $\Sigma^{(N)} \neq \emptyset$. Also $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is closed and bounded. For the latter, we have

$$(N+n)\delta - A \le \theta_n \le (N+n)\Delta - A \quad \text{for} \quad -N \le n \le N,$$

as follows from $\theta_{-N} = -A$ and $\delta \le \theta_{n+1} - \theta_n \le \Delta$. Hence $\Sigma^{(N)} \subset [-A, 2N\Delta - A]^{2N+1}$. □

If $\Theta = (\theta_n)_{-N \le n \le N} \in \Sigma^{(N)}$, then by the assumptions of Theorem 2.1, $h(\theta_n, \theta_{n+1})$ is defined for $-N \le n \le N-1$. Put

$$S(\Theta) = \sum_{n=-N}^{N-1} h(\theta_n, \theta_{n+1}), \quad \Theta = (\theta_n)_{-N \le n \le N} \in \Sigma^{(N)}. \tag{2.4}$$

Since $S : \Sigma^{(N)} \to \mathbb{R}$ is continuous, there exists a minimizer, i.e.,

$$S(\Theta^{(N)}) = \min_{\Theta \in \Sigma^{(N)}} S(\Theta) \tag{2.5}$$

for a suitable $\Theta^{(N)} = (\theta_n^{(N)})_{-N \le n \le N} \in \Sigma^{(N)}$, which henceforth we consider to be fixed.

First we need to derive some N-independent estimates for the minimizers, along the lines of [9, Lemmas 6.1.& 6.2].

Lemma 2.4 *Suppose that $\overline{\alpha} < \frac{3}{2}\,\underline{\alpha}$. There exists a constant $\sigma_* = \sigma_*(\overline{\alpha}/\underline{\alpha}) \ge 1$ such that for all $N \in \mathbb{N}$,*

$$\sigma_*^{-1}(\theta_n^{(N)} - \theta_{n-1}^{(N)}) \le \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}), \quad -N+1 \le n \le N-1.$$

Proof: First we derive the upper bound. Write $\theta_n^{(N)} - \theta_{n-1}^{(N)} = L$ and $\theta_{n+1}^{(N)} - \theta_n^{(N)} = \sigma L$ for $L, \sigma > 0$. We consider

$$\tilde\Theta = (\tilde\theta_k)_{-N \le k \le N} = \Big(\theta_{-N}^{(N)}, \dots, \theta_{n-1}^{(N)}, s, \theta_{n+1}^{(N)}, \dots, \theta_N^{(N)}\Big),$$

where $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_{n-1}^{(N)})$. Then $\tilde\theta_{\pm N} = \theta_{\pm N}^{(N)} = \pm A$ and $s - \theta_{n-1}^{(N)} = \theta_{n+1}^{(N)} - s = \frac{1}{2}(\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)}) \in [\delta, \Delta]$, due to $\theta_{n+1}^{(N)} - \theta_n^{(N)}, \theta_n^{(N)} - \theta_{n-1}^{(N)} \in [\delta, \Delta]$. Therefore $\tilde\Theta \in \Sigma^{(N)}$ in conjunction with (2.5) leads to $S(\Theta^{(N)}) \le S(\tilde\Theta)$. Using the definition of S from (2.4), this can be rewritten as

$$h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \le h(\theta_{n-1}^{(N)}, s) + h(s, \theta_{n+1}^{(N)}),$$

since all the other terms cancel. Therefore assumption (2.1) leads to

$$\begin{aligned} \underline{\alpha}(1+\sigma^2)L^2 &= \underline{\alpha}(\theta_n^{(N)} - \theta_{n-1}^{(N)})^2 + \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^2 \le \overline{\alpha}(s - \theta_{n-1}^{(N)})^2 + \overline{\alpha}(\theta_{n+1}^{(N)} - s)^2 \\ &= \frac{1}{2}\,\overline{\alpha}(\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)})^2 = \frac{1}{2}\,\overline{\alpha}(1+\sigma)^2 L^2. \end{aligned} \tag{2.6}$$

The function $\varphi(\sigma) = \frac{2(1+\sigma^2)}{(1+\sigma)^2} : [1, \infty[\to [1, 2[$ is strictly increasing. As $q = \overline{\alpha}/\underline{\alpha} \in [1, \frac{3}{2}[$, there is a unique $\sigma_* \in [1, \infty[$ so that $\varphi(\sigma_*) = q$; explicitly, we have

$$\sigma_* = (2-q)^{-1}(q + 2\sqrt{q-1}), \quad q = \overline{\alpha}/\underline{\alpha}. \tag{2.7}$$

Since (2.6) says that $\varphi(\sigma) \le q$, we must have $\sigma \le \sigma_*$, proving the upper bound.

For the lower bound, we consider the function

$$h_0(\theta,\theta') = h(-\theta',-\theta), \quad (\theta,\theta') \in \Omega.$$

Since $\delta \le \theta' - \theta \le \Delta$ iff $\delta \le -\theta - (-\theta') \le \Delta$, h_0 is well-defined and satisfies (2.1). Put

$$S_0(\Phi) = \sum_{n=-N}^{N-1} h_0(\varphi_n, \varphi_{n+1}), \quad \Phi = (\varphi_n)_{-N \le n \le N} \in \Sigma^{(N)}.$$

Then S_0 has a minimizer, i.e., $S_0(\Phi^{(N)}) = \min_{\Phi \in \Sigma^{(N)}} S_0(\Phi)$ for some $\Phi^{(N)} = (\varphi_n^{(N)})_{-N \le n \le N} \in \Sigma^{(N)}$. According to the upper bound, for every such minimizer we have

$$\varphi_{n+1}^{(N)} - \varphi_n^{(N)} \le \sigma_*(\varphi_n^{(N)} - \varphi_{n-1}^{(N)}), \quad -N+1 \le n \le N-1. \tag{2.8}$$

By definition of h_0, a minimizer is obtained by taking $\varphi_n^{(N)} = -\theta_{-n}^{(N)}$ for $-N \le n \le N$. Using this minimizer in (2.8) and replacing $-n$ by n, the lower bound is obtained. □

For $N \in \mathbb{N}$, define

$$\delta^{(N)} = \min_{-N \le n \le N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \quad \text{and} \quad \Delta^{(N)} = \max_{-N \le n \le N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}). \tag{2.9}$$

Then $\delta \le \delta^{(N)} \le \Delta^{(N)} \le \Delta$ holds, since $\Theta^{(N)} = (\theta_n^{(N)})_{-N \le n \le N} \in \Sigma^{(N)}$.

Lemma 2.5 *Suppose that $\overline{\alpha} < \frac{3}{2}\underline{\alpha}$. There exists a constant $\sigma_{**} = \sigma_{**}(\overline{\alpha}/\underline{\alpha}) \ge 1$ such that for all $N \in \mathbb{N}$,*

$$\Delta^{(N)} \le \sigma_{**}\delta^{(N)}.$$

Proof: Put

$$\sigma_{**} = 5\sigma_*. \tag{2.10}$$

If $\delta^{(N)} \ge \frac{1}{2\sigma_*}\Delta$, then $\Delta \ge \Delta^{(N)}$ yields $\delta^{(N)} \ge \frac{1}{2\sigma_*}\Delta^{(N)} \ge \frac{1}{\sigma_{**}}\Delta^{(N)}$. Therefore we can assume that $\delta^{(N)} \le \frac{1}{2\sigma_*}\Delta$. Similarly, if $\Delta^{(N)} \le 2\delta$, then $\Delta^{(N)} \le 2\delta^{(N)} \le \sigma_{**}\delta^{(N)}$, recalling $\sigma_* \ge 1$. Thus we only have to consider the cases where

$$\delta^{(N)} \le \frac{1}{2\sigma_*}\Delta \quad \text{and} \quad \Delta^{(N)} \ge 2\delta. \tag{2.11}$$

Let $-N \le m, n \le N-1$ be such that

$$\delta^{(N)} = \theta_{m+1}^{(N)} - \theta_m^{(N)} \quad \text{and} \quad \Delta^{(N)} = \theta_{n+1}^{(N)} - \theta_n^{(N)}.$$

Without loss of generality, we can suppose that $m \le n$, since the argument is similar for $m \ge n$. We may further restrict ourselves to $m+2 \le n$. In fact, if $m = n$, then $\Delta^{(N)} = \delta^{(N)}$. If $m+1 = n$, then by Lemma 2.4, $\Delta^{(N)} = \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}) = \sigma_*\delta^{(N)} \le \sigma_{**}\delta^{(N)}$. Hence $m+2 \le n$ can be assumed. Now we put

$$\tilde{\Theta} = (\tilde{\theta}_k)_{-N \le k \le N} = \left(\theta_{-N}^{(N)}, \ldots, \theta_m^{(N)}, \theta_{m+2}^{(N)}, \ldots, \theta_n^{(N)}, s, \theta_{n+1}^{(N)}, \ldots, \theta_N^{(N)}\right),$$

where $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_n^{(N)})$. That is, we remove $\theta_{m+1}^{(N)}$ from $\Theta^{(N)}$, then shift the block $(\theta_{m+2}^{(N)}, \ldots, \theta_n^{(N)})$ one place to the left, and finally insert s for $\theta_n^{(N)}$.

First we check that $\tilde{\Theta} \in \Sigma^{(N)}$. To begin with, $\tilde{\theta}_N = \theta_N^{(N)} = A$. Also, since $m+1 \geq -N+1$, $\theta_{-N}^{(N)}$ is not removed, which means that $\tilde{\theta}_{-N} = \theta_{-N}^{(N)} = -A$. Furthermore,

$$\theta_{m+2}^{(N)} - \theta_m^{(N)} = (\theta_{m+2}^{(N)} - \theta_{m+1}^{(N)}) + (\theta_{m+1}^{(N)} - \theta_m^{(N)}),$$

so that $\theta_{m+2}^{(N)} - \theta_m^{(N)} \geq 2\delta > \delta$. In addition, by Lemma 2.4 and by (2.11), recalling $\sigma_* \geq 1$,

$$\theta_{m+2}^{(N)} - \theta_m^{(N)} \leq (\sigma_* + 1)(\theta_{m+1}^{(N)} - \theta_m^{(N)}) \leq 2\sigma_* \delta^{(N)} \leq \Delta. \tag{2.12}$$

The next observation is that $\gamma := s - \theta_n^{(N)} = \theta_{n+1}^{(N)} - s = \frac{1}{2}(\theta_{n+1}^{(N)} - \theta_n^{(N)}) = \frac{1}{2}\Delta^{(N)}$. In particular, $\gamma = \frac{1}{2}\Delta^{(N)} \leq \frac{1}{2}\Delta < \Delta$, and also $\gamma = \frac{1}{2}\Delta^{(N)} \geq \delta$ by (2.11). To summarize the preceding arguments, we have shown that $\tilde{\Theta} \in \Sigma^{(N)}$. Hence (2.5) yields $S(\Theta^{(N)}) \leq S(\tilde{\Theta})$, so that by definition of S, see (2.4),

$$\begin{aligned} &h(\theta_m^{(N)}, \theta_{m+1}^{(N)}) + h(\theta_{m+1}^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\leq h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}). \end{aligned}$$

Since $h \geq 0$, we can drop two terms on the left-hand side and only keep $h(\theta_n^{(N)}, \theta_{n+1}^{(N)})$. Using (2.1), (2.12), and $\overline{\alpha} < \frac{3}{2}\,\underline{\alpha}$, we get

$$\begin{aligned} \underline{\alpha}(\Delta^{(N)})^2 &= \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^2 \leq h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\leq h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}) \\ &\leq \overline{\alpha}(\theta_{m+2}^{(N)} - \theta_m^{(N)})^2 + 2\overline{\alpha}\gamma^2 \leq 4\overline{\alpha}\sigma_*^2(\delta^{(N)})^2 + \frac{1}{2}\,\overline{\alpha}(\Delta^{(N)})^2 \\ &\leq 4\overline{\alpha}\sigma_*^2(\delta^{(N)})^2 + \frac{3}{4}\,\underline{\alpha}(\Delta^{(N)})^2. \end{aligned}$$

Consequently,

$$(\Delta^{(N)})^2 \leq 16\Big(\frac{\overline{\alpha}}{\underline{\alpha}}\Big)\sigma_*^2(\delta^{(N)})^2 \leq 24\,\sigma_*^2(\delta^{(N)})^2,$$

which yields the claim. □

Corollary 2.6 *Suppose that the assumptions of Lemma 2.5 are satisfied. If*

$$\sigma_{**}\delta < \frac{A}{N} < \sigma_{**}^{-1}\Delta, \tag{2.13}$$

then for all $N \in \mathbb{N}$ and $-N \leq n \leq N-1$,

$$\delta < \delta^{(N)} \leq \theta_{n+1}^{(N)} - \theta_n^{(N)} \leq \Delta^{(N)} < \Delta.$$

Proof: If $\Delta^{(N)} = \Delta$, then by (2.9) and by Lemma 2.5,

$$2A = \theta_N^{(N)} - \theta_{-N}^{(N)} = \sum_{n=-N}^{N-1}(\theta_{n+1}^{(N)} - \theta_n^{(N)}) \geq 2N\delta^{(N)} \geq \frac{2N\Delta^{(N)}}{\sigma_{**}} = \frac{2N\Delta}{\sigma_{**}},$$

contradicting (2.13). Similarly, if $\delta^{(N)} = \delta$, then also

$$2A = \sum_{n=-N}^{N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \le 2N\Delta^{(N)} \le 2\sigma_{**} N \delta^{(N)} = 2\sigma_{**} N \delta$$

contradicts (2.13). □

Now we are ready for the

Proof of Theorem 2.1 : We define $\sigma_{**} \ge 1$ as in Lemma 2.5, and we suppose that $\sigma_{**}\delta < \sigma_{**}^{-1}\Delta$ is satisfied. For $N \in \mathbb{N}$, we take $A = A_N = \frac{1}{2}(\sigma_{**}^{-1}\Delta + \sigma_{**}\delta)N$. Then $\sigma_{**}\delta < \frac{A_N}{N} < \sigma_{**}^{-1}\Delta$ holds for every $N \in \mathbb{N}$. In particular, $\delta \le \sigma_{**}\delta < \frac{A_N}{N} < \sigma_{**}^{-1}\Delta \le \Delta$. Therefore all the preceding results apply, and we obtain from Corollary 2.6 that

$$\delta < \theta_{n+1}^{(N)} - \theta_n^{(N)} < \Delta$$

for $N \in \mathbb{N}$ and $-N \le n \le N-1$. Fix $-N+1 \le n \le N-1$. Then $\Theta(\varepsilon) = (\theta_k(\varepsilon))_{-N \le k \le N} \in \Sigma^{(N)}$ for all $\varepsilon \in \mathbb{R}$ with $0 \le |\varepsilon|$ sufficiently small, where

$$\Theta(\varepsilon) = \Big(\theta_{-N}^{(N)}, \dots, \theta_{n-1}^{(N)}, \theta_n^{(N)} + \varepsilon, \theta_{n+1}^{(N)}, \dots, \theta_N^{(N)}\Big).$$

Recalling

$$S(\Theta(0)) = S(\Theta^{(N)}) = \min_{\Theta \in \Sigma^{(N)}} S(\Theta) \le S(\Theta(\varepsilon)),$$

see (2.5), it follows from differentiating $S(\Theta(\varepsilon))$ w.r. to ε that

$$0 = \frac{d}{d\varepsilon} S(\Theta(\varepsilon))\Big|_{\varepsilon=0} = \partial_2 h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + \partial_1 h(\theta_n^{(N)}, \theta_{n+1}^{(N)}), \quad -N+1 \le n \le N-1.$$

Next we intend to pass to the limit $N \to \infty$, but before doing so, we have to normalize the $\Theta^{(N)} = (\theta_n^{(N)})_{-N \le n \le N}$ appropriately. Denoting $n_0(N)$ the last index for which $\theta_{n_0(N)}^{(N)} \le 0$, we get $-N \le n_0(N) \le N-1$, since $\theta_{\pm N}^{(N)} = \pm A_N$, and also $\theta_{n_0(N)+1}^{(N)} > 0$. As $\delta \le \theta_{n_0(N)+1}^{(N)} - \theta_{n_0(N)}^{(N)} \le \Delta$, this yields $|\theta_{n_0(N)}^{(N)}| = -\theta_{n_0(N)}^{(N)} \le \Delta$. In addition,

$$A_N - \Delta \le \theta_N^{(N)} - \theta_{n_0(N)}^{(N)} = \sum_{n=n_0(N)}^{N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \le (N - n_0(N))\Delta,$$

proves that

$$N - n_0(N) \ge \frac{A_N - \Delta}{\Delta} \to \infty \quad \text{as} \quad N \to \infty, \tag{2.14}$$

and we also have $-N - n_0(N) \to -\infty$ as $N \to \infty$. Defining

$$\Phi^{(N)} = (\varphi_k^{(N)})_{-N-n_0(N) \le k \le N-n_0(N)}, \quad \varphi_k^{(N)} = \theta_{n_0(N)+k}^{(N)},$$

we find $|\varphi_0^{(N)}| \le \Delta$ as well as

$$\delta < \varphi_{k+1}^{(N)} - \varphi_k^{(N)} < \Delta \tag{2.15}$$

for $N \in \mathbb{N}$ and $-N - n_0(N) \le k \le N - n_0(N) - 1$. As above, this allows us to deduce that

$$0 = \partial_2 h(\varphi_{k-1}^{(N)}, \varphi_k^{(N)}) + \partial_1 h(\varphi_k^{(N)}, \varphi_{k+1}^{(N)}), \quad -N - n_0(N) + 1 \le k \le N - n_0(N) - 1, \tag{2.16}$$

since $\Phi^{(N)}$ is a minimizer of the appropriately index-shifted functional S in the appropriately index-shifted set $\Sigma^{(N)}$. Also $|\varphi_0^{(N)}| \le \Delta$ in conjunction with (2.15) yields

$$|\varphi_k^{(N)}| \le k\Delta \ \ (0 \le k \le N - n_0(N)) \quad \text{and} \quad |\varphi_k^{(N)}| \le |k|\Delta \ \ (-N - n_0(N) \le k \le 0). \tag{2.17}$$

Furthermore, we extend $\Phi^{(N)}$ to all indices $k \in \mathbb{Z}$ by defining $\varphi_k^{(N)} = 0$ for $k < -N - n_0(N)$ and for $k > N - n_0(N)$. Hence if we fix $k \in \mathbb{Z}$, then the sequence $(\varphi_k^{(N)})_{N\in\mathbb{N}}$ is bounded, by (2.17). Therefore we may employ a diagonal sequence argument to find a subsequence of $N \in \mathbb{N}$, indexed by $N' \to \infty$, such that for all $k \in \mathbb{Z}$ the limit

$$\theta_k^* = \lim_{N'\to\infty} \varphi_k^{(N')} \tag{2.18}$$

does exist. If $k \in \mathbb{Z}$ is fixed, then by (2.14) there is $M_k \in \mathbb{N}$ such that $N' \ge M_k$ implies that $-N' - n_0(N') + 1 \le k \le N' - n_0(N') - 1$. Thus (2.16) yields

$$\partial_2 h(\varphi_{k-1}^{(N')}, \varphi_k^{(N')}) + \partial_1 h(\varphi_k^{(N')}, \varphi_{k+1}^{(N')}) = 0, \quad N' \ge M_k,$$

so that

$$\partial_2 h(\theta_{k-1}^*, \theta_k^*) + \partial_1 h(\theta_k^*, \theta_{k+1}^*) = 0$$

is obtained in the limit $N' \to \infty$. In addition, since $|\varphi_0^{(N)}| \le \Delta$, also $|\theta_0^*| \le \Delta$ by (2.18). Concerning (2.3), we first note that

$$\delta \le \theta_{k+1}^* - \theta_k^* \le \Delta, \quad k \in \mathbb{Z},$$

follows from (2.15) and (2.18). Hence if $n \ge 1$, then $\theta_n^* - \theta_0^* = \sum_{k=0}^{n-1}(\theta_{k+1}^* - \theta_k^*)$ shows that $n\delta + \theta_0^* \le \theta_n^* \le n\Delta + \theta_0^*$, implying the first part of (2.3). Similarly, if $n \le -1$, then we write $\theta_0^* - \theta_n^* = \sum_{k=n}^{-1}(\theta_{k+1}^* - \theta_k^*)$ and get $(-n)\delta \le \theta_0^* - \theta_n^* \le (-n)\Delta$. This gives the second part of (2.3) and completes the proof of Theorem 2.1. □

Remark 2.7 Readers familiar with Aubry-Mather theory will notice that the above proof is inspired by it. In Aubry-Mather theory, the generating function h satisfies the additional condition

$$h(\theta + 1, \theta' + 1) = h(\theta, \theta').$$

As a consequence, global minimals are invariant under shifts of the indices and under integer translations. That is, if $(\theta_n^*)_{n\in\mathbb{Z}}$ is minimal, so are $(\theta_{n+1}^*)_{n\in\mathbb{Z}}$ and $(\theta_n^* + 1)_{n\in\mathbb{Z}}$. The second invariance fails if the periodicity is lost. This fact prevents from a straightforward extension of Aubry-Mather theory to our present situation.

Corollary 2.8 *Under the assumptions of Theorem 2.1, an explicit expression for σ_{**} is*

$$\sigma_{**} = 5(2-q)^{-1}(q + 2\sqrt{q-1}), \quad q = \overline{\alpha}/\underline{\alpha}.$$

In particular, one can take $\sigma_{**} = 10$ in (2.2), if $\overline{\alpha}/\underline{\alpha} \in [1, \frac{11}{10}]$.

Proof: The form of σ_{**} is due to (2.10) and (2.7). The last statement follows from the fact that $\sigma_{**} = \sigma_{**}(q)$ is increasing as a function of $q \in [1, 2[$, and $\sigma_{**}(\frac{11}{10}) \le 10$. Hence if $10\delta < \frac{1}{10}\Delta$ holds and $q = \overline{\alpha}/\underline{\alpha} \in [1, \frac{11}{10}]$, then $\sigma_{**}(q)\delta \le 10\delta < \frac{1}{10}\Delta \le \frac{1}{\sigma_{**}(q)}\Delta$ is obtained and (2.2) is satisfied. □

Remark 2.9 Concerning the form of $\sigma_{**} = \sigma_{**}(q)$ as obtained in Corollary 2.8, it is not completely satisfactory that $\lim_{q\searrow 1} \sigma_{**}(q) = 5$, but not $\lim_{q\searrow 1} \sigma_{**}(q) = 1$, as would be expected from (2.1).

3 Nonperiodic twist maps: A special case

In this section we are going to apply Theorem 2.1 to twist maps which leave invariant the boundary of the strip where they are defined. This additional assumption will be removed in Section 4.

Definition 3.1 *Let $f = f(\theta, r) : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ be a C^1-map.*

(a) We say that f is symplectic, if there exists a C^2-function $\hat{h} : \mathbb{R} \times [a, b] \to \mathbb{R}$ such that

$$r_1 d\theta_1 - r d\theta = d\hat{h}, \quad \textit{where} \quad (\theta_1, r_1) = f(\theta, r).$$

(b) A sequence $(\theta_n, r_n)_{n\in\mathbb{Z}} \subset \mathbb{R} \times [a, b]$ is called a complete orbit for f, if $(\theta_{n+1}, r_{n+1}) = f(\theta_n, r_n)$ for $n \in \mathbb{Z}$.

Remark 3.2 In the following Theorem 3.3 we need to impose three kinds of assumptions:

(a) The map $f : (\theta, r) \mapsto (\theta_1, r_1) = (\theta + r + F(\theta, r), r + G(\theta, r))$ is close to the integrable map $(\theta, r) \mapsto (\theta + r, r)$, where 'close' only requires that $|\partial_r F| \le \frac{1}{22}$.

(b) The boundaries $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$ of the strip $\mathbb{R} \times [a, b]$ enjoy some invariance properties, as is expressed by (3.1) below.

(c) The strip $\mathbb{R} \times [a, b]$ is sufficiently large in the sense that $b/a > 100$ is sufficiently large. Note that in the present context the size of the strip should be measured in terms of b/a rather than $b - a$, since (as opposed to periodic twist maps) both r and θ can be scaled. Putting

$$\Theta = \lambda\theta \quad \text{and} \quad R = \lambda r$$

for some $\lambda > 0$, the map f becomes

$$\Theta_1 = \Theta + R + \mathcal{F}(\Theta, R), \quad R_1 = R + \mathcal{G}(\Theta, R)$$

on the strip $(\Theta, R) \in \mathbb{R} \times [\lambda a, \lambda b]$, where

$$\mathcal{F}(\Theta, R) = \lambda F(\Theta/\lambda, R/\lambda), \quad \mathcal{G}(\Theta, R) = \lambda G(\Theta/\lambda, R/\lambda).$$

Both conditions $|\partial_r F| \le \frac{1}{22}$ and $b/a > 100$ are invariant under this scaling.

Theorem 3.3 *Let $b > a > 0$ be such that*

$$10a < \frac{1}{10} b.$$

Suppose that $f : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ is a symplectic C^1-map given by

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r),$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, b] \to \mathbb{R}$ we assume that $F, G \in C^1$,

$$F(\theta, a) = F(\theta, b) = G(\theta, a) = G(\theta, b) = 0, \quad \theta \in \mathbb{R}, \tag{3.1}$$

as well as

$$|\partial_r F(\theta, r)| \le \frac{1}{22}, \quad \theta \in \mathbb{R}, \quad r \in [a, b]. \tag{3.2}$$

Then there exists a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ *for* f *such that* $|\theta_0| < b$ *and* $a < \inf_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) \le \sup_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) < b$ *for* $n \in \mathbb{Z}$. *In addition,*

$$a < \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} < b. \tag{3.3}$$

Proof: First we need to construct the generating function h for f and discuss its properties. Although this is standard, see [1, 3], we include some details adapted to the present setup. For any fixed $\theta \in \mathbb{R}$, the function $[a, b] \ni r \mapsto \theta_1(\theta, r) = \theta + r + F(\theta, r) \in \mathbb{R}$ satisfies $\partial_r\theta_1 = 1 + \partial_r F \ge \frac{21}{22}$ by (3.2). Hence the inverse function

$$[\theta + a, \theta + b] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [a, b] \tag{3.4}$$

is well-defined, strictly increasing, and onto; note that $\theta_1(\theta, a) = \theta + a$ and $\theta_1(\theta, b) = \theta + b$ by (3.1). With $\hat{h}$ as in Definition 3.1, let

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1)) + C, \quad (\theta, \theta_1) \in \Omega_0 := \Big\{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \theta + a \le \theta_1 \le \theta + b\Big\},$$

where $C = a^2/2 - \hat{h}(0, a)$ is a normalizing constant. Now, the fact that $r_1 d\theta_1 - r d\theta = d\hat{h}$ may be seen to be equivalent to

$$\partial_\theta \hat{h} = r_1\, \partial_\theta\theta_1 - r \quad \text{and} \quad \partial_r \hat{h} = r_1\, \partial_r\theta_1. \tag{3.5}$$

Since $\partial_\theta r = -(1 + \partial_\theta F)(1 + \partial_r F)^{-1} = -(\partial_\theta\theta_1)(\partial_r\theta_1)^{-1}$ and $\partial_{\theta_1} r = (\partial_r\theta_1)^{-1}$, this results in

$$\partial_\theta h = \partial_\theta \hat{h} + (\partial_r \hat{h})(\partial_\theta r) = r_1\, \partial_\theta\theta_1 - r - r_1(\partial_r\theta_1)(\partial_\theta\theta_1)(\partial_r\theta_1)^{-1} = -r \tag{3.6}$$

and

$$\partial_{\theta_1} h = (\partial_r \hat{h})(\partial_{\theta_1} r) = r_1(\partial_r\theta_1)(\partial_r\theta_1)^{-1} = r_1, \tag{3.7}$$

i.e., the usual relations for the generating function h. We also observe that by (3.6) and (3.2),

$$\begin{aligned} |\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| &= |1 - \partial_{\theta_1} r(\theta, \theta_1)| = |1 - (\partial_r\theta_1)^{-1}(\theta, \theta_1)| = |1 - (1 + \partial_r F(\theta, r))^{-1}| \\ &= (1 + \partial_r F(\theta, r))^{-1}|\partial_r F(\theta, r)| \le \frac{1}{21}, \quad (\theta, \theta_1) \in \Omega_0. \end{aligned} \tag{3.8}$$

Put

$$V(\theta, \theta_1) = h(\theta, \theta_1) - \frac{1}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.9}$$

Then the relations

$$\partial_\theta V = -r + \theta_1 - \theta = F \quad \text{and} \quad \partial_{\theta_1} V = r_1 - \theta_1 + \theta = G - F \tag{3.10}$$

are obtained. In addition, $r(0, a) = a$ yields

$$V(0, a) = h(0, a) - \frac{a^2}{2} = \hat{h}(0, a) + C - \frac{a^2}{2} = 0$$

by the definition of C. Moreover, (3.10) and (3.1) lead to

$$\partial_\theta V(\theta, \theta + a) = F(\theta, r(\theta, \theta + a)) = F(\theta, a) = 0, \quad \theta \in \mathbb{R}, \tag{3.11}$$

and in the same way,

$$\partial_{\theta_1} V(\theta, \theta + a) = G(\theta, a) - F(\theta, a) = 0, \quad \theta \in \mathbb{R}. \tag{3.12}$$

Noting that

$$V(\theta, \theta_1) = V(0, a) + \int_\gamma dV = \int_\gamma F \, d\theta + (G - F) \, d\theta_1$$

for any path γ in the simply connected Ω_0 which connects $(0, a) \in \Omega_0$ to $(\theta, \theta_1) \in \Omega_0$, it follows from (3.11) and (3.12) that

$$V(\theta, \theta + a) = 0, \quad \theta \in \mathbb{R}, \tag{3.13}$$

since γ may be chosen to be part of the lower boundary $\{\theta_1 = \theta + a : \theta \in \mathbb{R}\}$ of Ω_0. By (3.8),

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| = |\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| \le \frac{1}{21}, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.14}$$

Due to (3.13), (3.12), and (3.14), Lemma 3.5 below shows that

$$|V(\theta, \theta_1)| \le \frac{1}{42}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0.$$

Thus by (3.9),

$$\frac{10}{21}(\theta_1 - \theta)^2 \le h(\theta, \theta_1) \le \frac{11}{21}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.15}$$

Fix $\varepsilon^* > 0$ so small that

$$10(a + \varepsilon^*) < \frac{1}{10}(b - \varepsilon^*), \tag{3.16}$$

and put

$$\delta = a + \varepsilon^*, \quad \Delta = b - \varepsilon^*, \quad \underline{\alpha} = \frac{10}{21}, \quad \text{and} \quad \overline{\alpha} = \frac{11}{21}.$$

Then $\Delta > \delta > 0$ by (3.16) and

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\} \subset \Omega_0.$$

Also $\overline{\alpha}/\underline{\alpha} = \frac{11}{10}$. Thus (3.15) and (3.16) in conjunction with Corollary 2.8 shows that Theorem 2.1 applies. Hence there exists $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0| \le \Delta$, $\delta \le \theta_{n+1} - \theta_n \le \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_{\theta_1} h(\theta_{n-1}, \theta_n) + \partial_\theta h(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z}. \tag{3.17}$$

In addition,

$$\delta \le \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} \le \Delta$$

implies that (3.3) is satisfied. Putting $r_n = -\partial_\theta h(\theta_n, \theta_{n+1})$ in accordance with (3.6), it follows that $(\theta_n, r_n)_{n\in\mathbb{Z}}$ is a complete orbit for f. In fact, by (3.17) and (3.7),

$$r_{n+1} = -\partial_\theta h(\theta_{n+1}, \theta_{n+2}) = \partial_{\theta_1} h(\theta_n, \theta_{n+1}) = r_n + G(\theta_n, r_n).$$

Moreover, $r_n = -\partial_\theta h(\theta_n, \theta_{n+1}) = r(\theta_n, \theta_{n+1})$ by (3.6) is equivalent to

$$\theta_n + r_n + F(\theta_n, r_n) = \theta_1(\theta_n, r_n) = \theta_{n+1}.$$

Hence $f(\theta_n, r_n) = (\theta_n + r_n + F(\theta_n, r_n), r_n + G(\theta_n, r_n)) = (\theta_{n+1}, r_{n+1})$ is obtained. Thus it remains to verify that $r_n \in [a, b]$ for $n \in \mathbb{Z}$. However, due to $\theta_n + a \le \theta_{n+1} \le \theta_n + b$ it is a consequence of (3.4) that $r_n = r(\theta_n, \theta_{n+1}) \in [a, b]$. □

Remark 3.4 Observe that due to the conclusion

$$a < \inf_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) \le \sup_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) < b$$

in Theorem 3.3 the bounded orbit lies in the interior of the strip $\mathbb{R} \times [a, b]$ an does not accumulate on its boundary.

The following technical lemma has been used in the preceding proof.

Lemma 3.5 *Let $\Delta > \delta > 0$. Suppose that $V : \Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\} \to \mathbb{R}$ is C^2 and such that*

$$V(\theta, \theta + \delta) = \partial_{\theta_1} V(\theta, \theta + \delta) = 0, \quad |\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \le \varepsilon,$$

for $\theta \in \mathbb{R}$ and $(\theta, \theta_1) \in \Omega$, respectively. Then

$$|V(\theta, \theta_1)| \le \frac{\varepsilon}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega. \tag{3.18}$$

Proof: Note that this is a consequence of d'Alembert's formula. Since

$$\begin{aligned} -\int_{\theta+\delta}^{\theta_1} d\eta \int_{\theta}^{\eta-\delta} d\xi\, \partial^2_{\theta\theta_1} V(\xi, \eta) &= -\int_{\theta+\delta}^{\theta_1} d\eta \Big(\partial_{\theta_1} V(\eta - \delta, \eta) - \partial_{\theta_1} V(\theta, \eta)\Big) = \int_{\theta+\delta}^{\theta_1} d\eta\, \partial_{\theta_1} V(\theta, \eta) \\ &= V(\theta, \theta_1) - V(\theta, \theta + \delta) = V(\theta, \theta_1), \end{aligned}$$

it follows that $|V(\theta, \theta_1)| \le \varepsilon \int_{\theta+\delta}^{\theta_1} d\eta \int_{\theta}^{\eta-\delta} d\xi = \frac{\varepsilon}{2}(\theta_1 - \theta - \delta)^2 \le \frac{\varepsilon}{2}(\theta_1 - \theta)^2$, which is (3.18). □

4 Existence of infinitely many complete orbits

In this section we drop the assumption (3.1) on the invariance of the boundaries. This has to be compensated by a stronger and more qualitative hypothesis on the generating function $\hat{h}$.

Definition 4.1 *Let $f = f(\theta, r) : \mathbb{R} \times [a, \infty[\to \mathbb{R}^2$ be a C^1-map and suppose that $c : [a, \infty[\to \mathbb{R}$ is a function. We say that f is c-exact symplectic, if there exists a C^2-function $\hat{h} : \mathbb{R} \times [a, \infty[\to \mathbb{R}$ such that*

$$r_1 d\theta_1 - r d\theta = d\hat{h}, \quad \textit{where} \quad (\theta_1, r_1) = f(\theta, r),$$

and

$$\sup_{\theta_0\in\mathbb{R}} \sup_{r\in[a,\infty[} \min_{\theta\in[\theta_0, \theta_0+T]} |\hat{h}(\theta, r) - c(r)| < \infty$$

for some $T > 0$.

Before we turn to our next theorem we give some motivation for Definition 4.1, which may be viewed as a proper extension of the notion of exact symplectic maps from the periodic case. This is illustrated by the following

Lemma 4.2 *Let $f = f(\theta, r) : (\mathbb{R}/T\mathbb{Z}) \times [a, \infty[\to \mathbb{R}^2$ be a symplectic C^1-map. Then f is exact symplectic on the cylinder if and only if the condition from Definition 4.1 holds.*

Proof: First suppose that f is exact symplectic on the cylinder. Then $r_1 d\theta_1 - r d\theta = d\hat{h}$ for a C^2-function $\hat{h}$ which is T-periodic in θ. Defining $c(r) = \frac{1}{T}\int_0^T \hat{h}(\theta, r)\, d\theta$ as the average, it follows that

$$\hat{h}(\theta, r) = c(r) + \hat{V}(\theta, r),$$

where $\int_0^T \hat{V}(\theta, r)\, d\theta = 0$ for $r \in [a, \infty[$. Fix $\theta_0 \in \mathbb{R}$ and $r \in [a, \infty[$. Since $\hat{V}(\cdot, r)$ is T-periodic, also $\int_{\theta_0}^{\theta_0+T} \hat{V}(\theta, r)\, d\theta = 0$. Therefore $\hat{V}(\theta_*, r) = 0$ for some $\theta_* \in [\theta_0, \theta_0 + T]$ yields

$$\min_{\theta \in [\theta_0,\, \theta_0+T]} |\hat{h}(\theta, r) - c(r)| = \min_{\theta \in [\theta_0,\, \theta_0+T]} |\hat{V}(\theta, r)| = 0$$

for every $\theta_0 \in \mathbb{R}$ and $r \in [a, \infty[$. Conversely, suppose that

$$M = \sup_{\theta_0 \in \mathbb{R}} \sup_{r \in [a, \infty[} \min_{\theta \in [\theta_0,\, \theta_0+T]} |\hat{h}(\theta, r) - c(r)| < \infty.$$

We need to show that $\hat{h}(\cdot, r)$ is T-periodic for $r \in [a, \infty[$. Writing $(\theta_1, r_1) = f(\theta, r)$, the functions $\theta \mapsto \theta_1(\theta, r) - \theta$ and $\theta \mapsto r_1(\theta, r)$ are T-periodic by assumption for every $r \in [a, \infty[$. The relation $d\hat{h} = r_1 d\theta_1 - r d\theta$ implies that $\partial_\theta \hat{h} = r_1 \partial_\theta \theta_1 - r$ and $\partial_r \hat{h} = r_1 \partial_r \theta_1$. In particular, every $\partial_\theta \hat{h}(\cdot, r)$ is T-periodic. From $M < \infty$ it then follows that $\hat{h}(\cdot, r)$ is bounded. In fact, if $\theta_0 \in \mathbb{R}$ is fixed, then there is $\theta_* \in [\theta_0, \theta_0 + T]$ such that $|\hat{h}(\theta_*, r) - c(r)| \le M$. Thus

$$|\hat{h}(\theta_0, r)| = \left| \hat{h}(\theta_*, r) - \int_{\theta_0}^{\theta_*} \partial_\theta \hat{h}(\theta, r)\, d\theta \right| \le M + c(r) + \|\partial_\theta \hat{h}(\cdot, r)\|_\infty T, \quad \theta_0 \in \mathbb{R}.$$

Since $\hat{h}(\cdot, r)$ is bounded and $\partial_\theta \hat{h}(\cdot, r)$ is T-periodic, also $\hat{h}(\cdot, r)$ is T-periodic. □

In [11] the author uses a notion of exact symplectic for some quasiperiodic maps. In general, if we assume that the map and its derivatives of first order are almost periodic, uniformly in r, then the natural definition of exact symplectic would be to require that $\hat{h}$ and its first derivatives are also almost periodic, uniformly in r. With some more effort one can then prove a result similar to Lemma 4.2, showing that also in this case this notion is equivalent to the one introduced by Definition 4.1.

The main result of this section is

Theorem 4.3 *Suppose that $f : \mathbb{R} \times [a, \infty[\to \mathbb{R}^2$ is c-exact symplectic with $c(r) = \frac{1}{2} r^2$. Let f be given by*

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r),$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, \infty[\to \mathbb{R}$ we assume that $F, G \in C^1$ and

$$\|F\|_\infty < \infty, \quad \|G\|_\infty < \infty, \quad \text{and} \quad |\partial_r F(\theta, r)| \le \frac{1}{43} \tag{4.1}$$

for $\theta \in \mathbb{R}$ and $r \in [a, \infty[$. Then f has infinitely many complete orbits $(\theta_n^j, r_n^j)_{n \in \mathbb{Z}}$ such that $R_j \le r_n^j \le R_{j+1}$ for $n \in \mathbb{N}$, where $R_j < R_{j+1} \to \infty$ as $j \to \infty$. Moreover,

$$R_j \le \liminf_{n \to \pm\infty} \frac{\theta_n^j}{n} \le \limsup_{n \to \pm\infty} \frac{\theta_n^j}{n} \le R_{j+1}.$$

Proof: Define

$$M = \sup_{\theta_0 \in \mathbb{R}} \sup_{r \in [a,\infty[} \min_{\theta \in [\theta_0, \theta_0+T]} \left| \hat{h}(\theta, r) - \frac{1}{2} r^2 \right| < \infty \tag{4.2}$$

and

$$m = \max\{\|F\|_\infty, \|G\|_\infty\} < \infty.$$

Next fix $b > a$ such that $b \geq 1$ and

$$b \geq 205(a+m), \quad b \geq 235300800 \cdot (M + m + Tm), \tag{4.3}$$

hold. For $f : \mathbb{R} \times [a,b] \to \mathbb{R}^2$ we are going to argue similar to the proof of Theorem 3.3. However, since f does not preserve the boundaries, the technicalities are more involved. As before, we consider the strictly increasing maps $[a,b] \ni r \mapsto \theta_1(\theta, r) = \theta + r + F(\theta, r) \in \mathbb{R}$ and their inverses

$$[\theta_1(\theta, a), \theta_1(\theta, b)] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [a,b].$$

Taking $\hat{h}$ from Definition 4.1, let

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1)) \quad \text{and} \quad V(\theta, \theta_1) = h(\theta, \theta_1) - \frac{1}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0,$$

where

$$\Omega_0 = \Big\{ (\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \, \theta_1(\theta, a) \leq \theta_1 \leq \theta_1(\theta, b) \Big\}.$$

Then

$$\partial_\theta h = -r, \quad \partial_{\theta_1} h = r_1, \quad \partial_\theta V = F, \quad \partial_{\theta_1} V = G - F. \tag{4.4}$$

Also

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| = |\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| \leq \frac{\frac{1}{43}}{1 - \frac{1}{43}} = \frac{1}{42}, \quad (\theta, \theta_1) \in \Omega_0, \tag{4.5}$$

cf. (3.8) and (4.1). Put

$$\Omega_1 = \Big\{ (\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R}, \, \theta + a_1 \leq \theta_1 \leq \theta + b_1 \Big\}$$

for $a_1 = a + m$ and $b_1 = b - m$. Then $\theta_1(\theta, a) = \theta + a + F(\theta, a) \leq \theta + a_1$ and $\theta_1(\theta, b) = \theta + b + F(\theta, b) \geq \theta + b_1$ implies that $\Omega_1 \subset \Omega_0$. The qualitative assumption (4.2) on $\hat{h}$ results in the following estimate, see Lemma 4.4 below:

$$|V(\theta, \theta_1)| \leq M + \frac{22}{43}(a_1 + b_1 + 4m)m + \frac{3}{2}\Big(b_1 - a_1 + 2T\Big)m, \quad (\theta, \theta_1) \in \Omega_1. \tag{4.6}$$

Now put

$$a_2 = a_1 + \eta \quad \text{and} \quad b_2 = b_1 - \eta, \quad \text{where} \quad \eta = \frac{1}{200} b.$$

Then $a_2 < b_2$. Next we fix a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(s) = 1$ for $s \in [a_2, b_2]$, $\chi(s) = 0$ for $s \leq a_1$, $\chi(s) = 0$ for $s \geq b_1$, as well as

$$|\chi(s)| \leq 1, \quad |\chi'(s)| \leq \frac{4}{\eta}, \quad \text{and} \quad |\chi''(s)| \leq \frac{20}{\eta^2}, \quad s \in \mathbb{R},$$

are satisfied. Let $\tilde{V}(\theta,\theta_1)=\chi(\theta_1-\theta)V(\theta,\theta_1)$ for $(\theta,\theta_1)\in\Omega_1$. Then $\tilde{V}(\theta,\theta+a_1)=0$ for $\theta\in\mathbb{R}$. Moreover,

$$\begin{aligned}
\partial_{\theta_1}\tilde{V}(\theta,\theta_1) &= \chi'(\theta_1-\theta)V(\theta,\theta_1)+\chi(\theta_1-\theta)\partial_{\theta_1}V(\theta,\theta_1),\\
\partial^2_{\theta\theta_1}\tilde{V}(\theta,\theta_1) &= -\chi''(\theta_1-\theta)V(\theta,\theta_1)+\chi'(\theta_1-\theta)\Big[\partial_\theta V(\theta,\theta_1)-\partial_{\theta_1}V(\theta,\theta_1)\Big]\\
&\quad +\chi(\theta_1-\theta)\partial^2_{\theta\theta_1}V(\theta,\theta_1).
\end{aligned}$$

In particular, $\partial_{\theta_1}\tilde{V}(\theta,\theta+a_1)=0$ as well as

$$\begin{aligned}
|\partial^2_{\theta\theta_1}\tilde{V}(\theta,\theta_1)| &\le 20\,\eta^{-2}\Big(M+\frac{22}{43}(a+b+4m)m+\frac{3}{2}\Big(b-a-2m+2T\Big)m\Big)\\
&\quad +12\,\eta^{-1}m+\frac{1}{42}=:\varepsilon_1,\quad (\theta,\theta_1)\in\Omega_1,
\end{aligned}$$

by (4.6), (4.4), and (4.5). Hence Lemma 3.5 applies to yield

$$|\tilde{V}(\theta,\theta_1)|\le\frac{\varepsilon_1}{2}(\theta_1-\theta)^2,\quad (\theta,\theta_1)\in\Omega_1.$$

Thus defining $\tilde{h}(\theta,\theta_1)=\frac{1}{2}(\theta_1-\theta)^2+\tilde{V}(\theta,\theta_1)$ for $(\theta,\theta_1)\in\Omega_1$, it follows that

$$\underline{\alpha}(\theta_1-\theta)^2\le\tilde{h}(\theta,\theta_1)\le\overline{\alpha}(\theta_1-\theta)^2,\quad (\theta,\theta_1)\in\Omega_1, \tag{4.7}$$

where $\underline{\alpha}=\frac{1}{2}(1-\varepsilon_1)$ and $\overline{\alpha}=\frac{1}{2}(1+\varepsilon_1)$. Due to (4.3), $\varepsilon_1\le\frac{1}{21}$ is satisfied. In fact, $a+b+4m\le 4(a+m)+b\le\frac{209}{205}\,b$ and $b\ge 1$ yield

$$\begin{aligned}
\varepsilon_1 &\le 20\,\eta^{-2}\Big(M+\frac{3}{5}\,b\,m+3(2b+T)m\Big)+12\,\eta^{-1}m+\frac{1}{42}\\
&\le 140\,\eta^{-2}(M+(b+T)m)+12\,\eta^{-1}m+\frac{1}{42}\\
&\le 28000\,\eta^{-1}(M+Tm)+28000\,\eta^{-1}m+12\,\eta^{-1}m+\frac{1}{42}\\
&\le 28012\,\eta^{-1}(M+m+Tm)+\frac{1}{42}\le\frac{1}{21}
\end{aligned}$$

by (4.3). Since $\varepsilon_1\le\frac{1}{21}$, the bound $\overline{\alpha}/\underline{\alpha}=(1+\varepsilon_1)/(1-\varepsilon_1)\le\frac{11}{10}$ is obtained. In particular, on $\Omega_2=\{(\theta,\theta_1)\in\mathbb{R}^2 : a_2\le\theta_1-\theta\le b_2\}\subset\Omega_1$ the estimate (4.7) holds for h, since $\tilde{h}=h$ on Ω_2 by definition of χ. Next note that

$$100a_2=100(a+m)+\frac{1}{2}\,b<\frac{199}{200}\,b-m=b-m-\eta=b_2$$

by (4.3). Consequently we can use Corollary 2.8 and Theorem 2.1 for h on Ω_2 to get a sequence $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0|\le b_2$, $a_2\le\theta_{n+1}-\theta_n\le b_2$ for $n\in\mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1},\theta_n)+\partial_1 h(\theta_n,\theta_{n+1})=0,\quad n\in\mathbb{Z}.$$

In addition, $a_2\le\liminf_{n\to\pm\infty}\frac{\theta_n}{n}\le\limsup_{n\to\pm\infty}\frac{\theta_n}{n}\le b_2$. If we put $r_n=-\partial_\theta h(\theta_n,\theta_{n+1})$, then the argument from the proof to Theorem 3.3 shows that $(\theta_n,r_n)_{n\in\mathbb{Z}}\subset\mathbb{R}\times[a,b]$ is a complete orbit for the map f.

Hence there is a first complete orbit for f. In the next step we replace $R_1 = a$ by $R_2 = b+1$ in the preceding argument and select a new R_3 which satisfies $R_3 \geq 205(R_2 + m)$. As the conditions (4.3) hold for R_3, a further complete orbit for $f : \mathbb{R} \times [R_2, R_3] \to \mathbb{R}^2$ is obtained. Continuing this way, we get infinitely many complete orbits for f which have the desired properties. □

We add one more technical result that has been used before.

Lemma 4.4 *Using the notation from the proof to Theorem 4.3,*

$$|V(\theta,\theta_1)| \leq M + \frac{22}{43}(a_1 + b_1 + 4m)m + \frac{3}{2}\Big(b_1 - a_1 + 2T\Big)m, \quad (\theta,\theta_1) \in \Omega_1.$$

Proof: Consider a fixed $(\tilde{\theta},\tilde{\theta}_1) \in \Omega_1$, i.e., $a_1 \leq \tilde{\theta}_1 - \tilde{\theta} \leq b_1$ holds. As a consequence of (4.2), for $\tilde{\theta} \in \mathbb{R}$ and $c_1 = (a_1 + b_1)/2 \geq a$ there is $\theta \in [\tilde{\theta}, \tilde{\theta} + T]$ such that $|\hat{h}(\theta, c_1) - c_1^2/2| \leq M$. Next, $\theta + c_1 = \theta + r + F(\theta, r)$ for $r = r(\theta, \theta + c_1)$ yields $|r(\theta, \theta + c_1) - c_1| \leq \|F\|_\infty \leq m$. Since by (3.5), $|\partial_r \hat{h}(\theta, r)| = |r_1\, \partial_r \theta_1| = (r + G)(1 + \partial_r F) \leq \frac{44}{43}(c_1 + 2m)$ for $|r - c_1| \leq m$, it follows that

$$\begin{aligned} |V(\theta,\theta+c_1)| &= |\hat{h}(\theta, r(\theta,\theta+c_1)) - c_1^2/2| \leq |\hat{h}(\theta, r(\theta,\theta+c_1)) - \hat{h}(\theta,c_1)| + |\hat{h}(\theta,c_1) - c_1^2/2| \\ &\leq \Big|\int_{c_1}^{r(\theta,\theta+c_1)} \partial_r \hat{h}(\theta, r)\, dr\Big| + M \leq \frac{44}{43}(c_1 + 2m)m + M. \end{aligned}$$

According to (4.4),

$$V(\tilde{\theta},\tilde{\theta}_1) = V(\theta,\theta+c_1) + \int_\gamma dV = V(\theta,\theta+c_1) + \int_\gamma F\, d\theta + (G - F)\, d\theta_1$$

for the path γ connecting first $(\tilde{\theta},\tilde{\theta}_1)$ along the vertical line $\{\theta = \tilde{\theta}\}$ to the point $(\tilde{\theta}, \tilde{\theta} + c_1)$, and thereafter connecting $(\tilde{\theta}, \tilde{\theta} + c_1)$ to $(\theta, \theta + c_1)$ along the straight line $\{\theta_1 - \theta = c_1\}$. In particular, γ has length $|\gamma| \leq |\tilde{\theta}_1 - \tilde{\theta} - c_1| + T \leq \frac{1}{2}(b_1 - a_1) + T$. Thus the preceding estimates imply

$$|V(\tilde{\theta},\tilde{\theta}_1)| \leq \frac{44}{43}(c_1 + 2m)m + M + 3\Big(\frac{1}{2}(b_1 - a_1) + T\Big)m,$$

as claimed. □

Finally we consider an application of Theorem 4.3.

Theorem 4.5 *Let $\phi \in C^2(\mathbb{R})$ be such that ϕ and ϕ' are bounded. Then the map*

$$\theta_1 = \theta + r, \quad r_1 = r + \phi'(\theta + r),$$

has infinitely many complete orbits enjoying the properties described in Theorem 4.3.

In particular, this applies to the case of the maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin\omega_1(\theta + r) + \sin\omega_2(\theta + r))$$

for all $\lambda, \omega_1, \omega_2 > 0$.

Proof: Put $\hat{h}(\theta, r) = \frac{1}{2}r^2 + \phi(\theta + r)$, $F = 0$, and $G(\theta, r) = \phi'(\theta + r)$. Then $r_1 \partial_\theta \theta_1 - r = (r + \phi'(\theta, r)) - r = \phi'(\theta + r) = \partial_\theta \hat{h}$ and $r_1 \partial_r \theta_1 = r + \phi'(\theta + r) = \partial_r \hat{h}$ shows that $r_1 d\theta_1 - r d\theta = d\hat{h}$ holds. Since $|\hat{h}(\theta, r) - \frac{1}{2}r^2| = |\phi(\theta, r)| \leq \|\phi\|_\infty$, f is c-exact symplectic with $c(r) = \frac{1}{2}r^2$. Concerning (4.1), we only have to note that $\|G\|_\infty = \|\phi'\|_\infty < \infty$ by assumption. Hence the first claim follows from Theorem 4.3. For the special case, $\phi(\theta) = -\lambda(\omega_1^{-1}\cos\omega_1\theta + \omega_2^{-1}\cos\omega_2\theta)$ satisfies $\|\phi\|_\infty < \infty$ and $\|\phi'\|_\infty < \infty$. □

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