Complete orbits for twist maps on the plane: extensions and applications

Markus Kunze[1] & Rafael Ortega[2]

[1] Universität Duisburg-Essen, Fachbereich Mathematik, D - 45117 Essen, Germany

[2] Departamento de Matemática Aplicada, Universidad de Granada, E-18071 Granada, Spain

Key words: twist map, Aubry-Mather orbit, rotation number

Dedicated to Professor R. Johnson on the occasion of his 60th birthday

Abstract

Twist maps $(\theta_1, r_1) = f(\theta, r)$ on the plane are considered which do not exhibit any kind of periodicity in their dependence on θ. Some general results are obtained which typically yield the existence of infinitely many complete and bounded orbits. Examples that can be treated with this theory include oscillators of the type $\ddot{x} + V'(x) = p(t)$ under appropriate hypotheses, the bouncing ball system, and the standard map.

1 Introduction

In this paper we continue our earlier work [6] on non-periodic twist maps $(\theta_1, r_1) = f(\theta, r)$ on the plane by providing further general results and applications. As soon as no hypothesis like periodicity, quasi-periodicity, or almost periodicity is imposed on the dependence of f on θ, the highly developed machineries of KAM theory and Aubry-Mather theory are no longer useful. In [6] we were able to devise a theory that allows the construction of some special orbits for maps of this type when the associated generating function grows quadratically. This means that

$$h(\theta, \theta_1) \sim (\theta_1 - \theta)^2$$

where $h = h(\theta, \theta_1)$ is a function satisfying

$$r = \partial_\theta h(\theta, \theta_1), \quad r_1 = -\partial_{\theta_1} h(\theta, \theta_1).$$

In Section 2 we further illustrate our results by proving the existence of running solutions of equations of the type

$$\ddot{x} + V'(x) = p(t)$$

for periodic forcings p of zero average and C^2-bounded potentials V. The so-called running solutions are solutions $x = x(t)$ having a bounded and uniformly positive derivative, i.e., they satisfy

$$0 < \inf_{t\in\mathbb{R}} \dot{x}(t) \leq \sup_{t\in\mathbb{R}} \dot{x}(t) < \infty.$$

In particular x must be unbounded and a new criterion for unbounded motions can be derived. As an illustration, it can be proved that bounded and unbounded solutions coexist for the equation

$$\ddot{x} + \frac{x}{1+x^4} = \lambda \sin t. \tag{1.1}$$

A delicate point concerning the applicability of the theory of twist maps to these differential equations is that the generating function is not explicitly known. The required estimates on h are obtained using the connection between generating functions and Lagrangians as described e.g. in [9].

The general theory is extended in Section 3.1 to include generating functions with superlinear growth,

$$h(\theta, \theta_1) \sim (\theta_1 - \theta)^\kappa$$

for some $\kappa > 1$. The example of a ball falling down under the influence of gravity and bouncing back from a moving wall is considered Section 3.2; in this case it turns out that $\kappa = 3$. This is a model related to the so-called Fermi-Ulam acceleration and has been studied by several authors, in particular in the case where the wall moves periodically; see [1, 3] and the references therein. We can also treat non-periodic walls, as long as they have bounded velocity and move in a compact region. Our theory then leads to the existence of infinitely many bounded motions of the ball.

Section 4 deals with a somewhat different but closely related issue. For the standard map

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda \sin(\theta + r),$$

Aubry-Mather theory implies that given $\alpha > 0$ there exists a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ that is monotone ($\theta_n < \theta_{n+1}$), bounded in r ($\sup_{n\in\mathbb{N}} |r_n| < \infty$), and has a rotation number

$$\alpha = \lim_{|n|\to\infty} \frac{\theta_n}{n}.$$

Our theory applies to maps

$$f_\varphi : \quad \theta_1 = \theta + r, \quad r_1 = r + \varphi(\theta + r),$$

for suitable non-periodic functions φ. In general the solutions that we construct are monotone and bounded, but they will not have a well-defined rotation number; examples for this are easily found when φ is positive. Nevertheless we can show that in a typical situation there will be a 'large' set of ψ's such that each f_ψ allows for what we call an AM-orbit, i.e., one that is monotone, bounded, and admits a rotation number. This is achieved by adapting to our discrete setting the theory of the rotation number for non-autonomous linear differential equations; see [5].

Acknowledgement: The second author thanks Alberto Abbondandolo for a conversation on the existence of orbits with rotation number for maps on the annulus.

2 Quadratic generating functions

In this section we prove the existence of particular solutions to the equation $\ddot{x} + V'(x) = p(t)$ for a bounded and periodic forcing p of zero average, however imposing no periodicity hypothesis on the potential V.

Definition 2.1 *A running solution of $\ddot{x} + V'(x) = p(t)$ is a solution satisfying*

$$0 < \delta \leq \dot{x}(t) \leq \Delta < \infty \quad \textit{for} \quad t \in \mathbb{R}.$$

Notice that then the mean value theorem implies that

$$\delta \leq x(t+1) - x(t) \leq \Delta \quad \text{for} \quad t \in \mathbb{R},$$

and from the generalized L'Hospital rule,

$$\delta \leq \liminf_{|t| \to \infty} \frac{x(t)}{t} \leq \limsup_{|t| \to \infty} \frac{x(t)}{t} \leq \Delta.$$

It turns out that for potentials V satisfying certain bounds, the problem of the existence of running solutions can be recast in terms of a quadratic generating map, so that the results from [6] apply.

Theorem 2.2 *Let $V \in C^2(\mathbb{R})$ be such that $\|V\|_{L^\infty(\mathbb{R})} + \|V'\|_{L^\infty(\mathbb{R})} + \|V''\|_{L^\infty(\mathbb{R})} < \infty$. Suppose that $p \in L^\infty(\mathbb{R})$ is T-periodic and such that $\int_0^T p(t)\, dt = 0$. Then there exists a sequence (x_j) of running solutions of*

$$\ddot{x} + V'(x) = p(t) \tag{2.1}$$

such that $\inf_{t \in \mathbb{R}} \dot{x}_j(t) \to \infty$ as $j \to \infty$.

Remark 2.3 As mentioned in the introduction, running solutions are unbounded, and hence Theorem 2.2 leads to a new result concerning the existence of unbounded solutions. For the example (1.1) we can take $V(x) = (1/2)\arctan(x^2)$ and $p(t) = \lambda \sin t$. Then our theorem says that there are many running solutions. On the other hand, the force $-V'(x)$ is attractive and thus there is a periodic solution by [7]. It follows that for (1.1) bounded and unbounded motions do coexist. It is an interesting problem to clarify the same question for

$$\ddot{x} + \frac{x}{1+x^2} = \lambda \sin t,$$

since Theorem 2.2 does not apply in this case.

Before going on to the proof, we present two further examples. Somehow they illustrate extreme cases of the range of applicability of the theorem.

Example 2.4 (Periodic and quasiperiodic potentials) First we assume that V and p are periodic with period $T = 1$ and furthermore $p \in C^2(\mathbb{R})$. In this case, more precise information can be obtained from Moser's version of the Aubry-Mather theorem; see [9]. For every $\alpha \in \mathbb{R}$ there exists a solution satisfying

$$\lim_{|t| \to \infty} \frac{x(t)}{t} = \alpha$$

and

$$|x(t+1) - x(t) - \alpha| \leq 1 \quad \text{for} \quad t \in \mathbb{R}.$$

Since $\|\ddot{x}\|_{L^\infty(\mathbb{R})} \leq \|V'\|_{L^\infty(\mathbb{R})} + \|p\|_{L^\infty(\mathbb{R})}$ is finite, it is easy to deduce that x is a running solution for large α. Assuming some additional smoothness for V, one can say that all solutions have a bounded derivative [8, 10, 13], and all solutions with large initial velocity are running solutions. Analogous conclusions were obtained in [2] for V quasiperiodic and satisfying appropriate diophantine conditions. Our result is applicable to any trigonometric sum of the type

$$V(x) = \sum_{n=1}^{N} a_n \sin(\omega_n t + \varphi_n),$$

without imposing any condition on the ω_n's.

Example 2.5 (arctan-**potentials)** For $V(x) = \arctan x$ it is possible to obtain more refined information by rather elementary methods. Let $P(t)$ be a T-periodic solution of $\ddot{P} = p(t)$. The change of variables $x = y + P(t)$ leads us to the equation

$$\ddot{y} + \frac{1}{1 + (y + P(t))^2} = 0.$$

Then $\ddot{y}(t) < 0$ for every solution, so $\dot{y}(t)$ is strictly decreasing. Let us denote by $\dot{y}(\pm\infty)$ the limits, be they finite or infinite. In any case, $\dot{y}(-\infty) > \dot{y}(+\infty)$, and thus

$$\lim_{t\to\infty} \frac{x(t)}{t} = \dot{y}(+\infty) < \dot{y}(-\infty) = \lim_{t\to-\infty} \frac{x(t)}{t}$$

for any solution x of the original equation. This is in contrast to the case of periodic potentials and shows the impossibility of defining rotation numbers for general potentials. Let us now prove that in this case nevertheless all solutions have a bounded derivative. If, for instance, $\dot{y}(+\infty) = -\infty$ would hold, then $|y(t)| \geq t$ for large t. Hence the function $\frac{1}{1+(y(t)+P(t))^2}$ is integrable in $[0, \infty[$. Thus

$$\dot{y}(+\infty) = \dot{y}(0) - \int_0^\infty \frac{dt}{1 + (y(t) + P(t))^2} > -\infty,$$

a contradiction. Our result also applies to more complicated potentials like $V(x) = \arctan x + \cos x$, for which the preceding direct argument is no longer useful.

Now we turn to the proof of Theorem 2.2. As an auxiliary step, we state a quantitative version of the Riemann-Lebesgue lemma. The classical case of this lemma corresponds to $\psi(x) = \sin(2\pi x)$ and $z = 0$.

Lemma 2.6 *Let $\psi \in C(\mathbb{R})$ be such that its primitive $\Psi(x) = \int_0^x \psi(\xi)\, d\xi$ is bounded, i.e., $\Psi \in L^\infty(\mathbb{R})$. Suppose that $z \in C^2([0,1])$ satisfies $\|\dot{z}\|_{L^\infty([0,1])}, \|\ddot{z}\|_{L^\infty([0,1])} \leq C_1$ and let $\alpha \in C^1([0,1])$ be such that $\alpha(0) = \alpha(1) = 0$ and $\|\alpha\|_{L^\infty([0,1])}, \|\dot{\alpha}\|_{L^\infty([0,1])} \leq C_2$. Then*

$$\sup_{\theta\in\mathbb{R}} \left| \int_0^1 \alpha(t)\psi(\theta + rt + z(t))\, dt \right| \leq \frac{4C_2}{r} \|\Psi\|_{L^\infty(\mathbb{R})} \quad \textit{for} \quad r \geq 2C_1.$$

Proof: Let I denote the integral that we want to estimate. From $r + \dot{z}(t) \geq r - C_1 \geq r/2 > 0$ we deduce that it is possible to integrate by parts. It follows that

$$\begin{aligned} |I| &= \Big| \int_0^1 \frac{\alpha(t)}{r+\dot{z}(t)} \frac{d}{dt}[\Psi(\theta + rt + z(t))]\, dt \Big| \\ &= \Big| \int_0^1 \Big(\frac{\dot{\alpha}(t)}{r+\dot{z}(t)} - \frac{\alpha(t)\ddot{z}(t)}{(r+\dot{z}(t))^2} \Big) \Psi(\theta + rt + z(t))\, dt \Big| \\ &\leq \Big(\frac{2C_2}{r} + \frac{4C_1C_2}{r^2} \Big) \|\Psi\|_{L^\infty(\mathbb{R})} \leq \frac{4C_2}{r} \|\Psi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

completing the proof. □

To simplify the notation we henceforth take $T = 1$ in Theorem 2.2. For every $\theta, r \in \mathbb{R}$ let $x(t) = x(t;\theta, r)$ denote the unique global solution to (2.1) with initial data $x(0) = \theta$ and $\dot{x}(0) = r$. Since V' and p are bounded, this solution does exist, and it is differentiable w.r. to θ and r. Moreover, let

$$\theta_1 = \theta_1(\theta, r) = x(1;\theta, r) \quad \text{and} \quad r_1 = r_1(\theta, r) = \dot{x}(1;\theta, r)$$

as well as

$$y(t) = y(t;\theta, r) = x(t;\theta, r) - \theta - (\theta_1 - \theta)t \quad \text{and} \quad F(\theta, r) = \theta_1 - \theta - r.$$

In the following lemma bounds are obtained on y and F.

Lemma 2.7 *Let $C_1 = \|V'\|_{L^\infty(\mathbb{R})} + \|p\|_{L^\infty(\mathbb{R})}$. Then*

$$\|y\|_{L^\infty([0,1])} \leq C_1, \quad \|\dot{y}\|_{L^\infty([0,1])} \leq C_1, \quad \textit{and} \quad \|F\|_{L^\infty(\mathbb{R}^2)} \leq C_1.$$

In addition,

$$|F_r(\theta, r)| \leq \frac{C_2}{r} \quad \textit{for} \quad \theta \in \mathbb{R} \quad \textit{and} \quad r \geq 2C_1,$$

where $C_2 = 8\|V'\|_{L^\infty(\mathbb{R})}(1 + \|V''\|_{L^\infty(\mathbb{R})}) \exp(\|V''\|_{L^\infty(\mathbb{R})})$.

Proof: By definition y solves the Dirichlet problem $\ddot{y} + V'(x) = p(t)$, $y(0) = 0$, $y(1) = 0$. Let

$$G(t, s) = \begin{cases} s(1-t) & : \quad s \leq t \\ t(1-s) & : \quad s \geq t \end{cases}, \quad 0 \leq t, s \leq 1, \tag{2.2}$$

denote Green's function for $[0, 1]$. Then

$$y(t) = \int_0^1 G(t, s)[V'(x(s)) - p(s)]\, ds$$

yields the estimate $\|y\|_{L^\infty([0,1])} \leq (1/8)(\|V'\|_{L^\infty(\mathbb{R})} + \|p\|_{L^\infty(\mathbb{R})}) \leq C_1$, and moreover

$$\dot{y}(t) = \int_0^1 \partial_t G(t, s)[V'(x(s)) - p(s)]\, ds,$$

so that $\|\dot{y}\|_{L^\infty([0,1])} \leq (1/2)(\|V'\|_{L^\infty(\mathbb{R})} + \|p\|_{L^\infty(\mathbb{R})}) \leq C_1$.

Concerning F, note first that

$$F(\theta,r)=\int_0^1 (1-t)\,\ddot{x}(t)\,dt=\int_0^1 (1-t)\,[p(t)-V'(x(t))]\,dt. \tag{2.3}$$

Therefore $\|F\|_{L^\infty(\mathbb{R}^2)} \le (1/2)(\|V'\|_{L^\infty(\mathbb{R})}+\|p\|_{L^\infty(\mathbb{R})}) \le C_1$. In addition, (2.3) leads to

$$F_r(\theta,r)=-\int_0^1 (1-t)\,V''(x(t))x_r(t)\,dt=\int_0^1 \alpha(t)\,V''(x(t))\,dt, \tag{2.4}$$

where $x_r=\partial_r x$ and $\alpha(t)=\alpha(t;\theta,r)=-(1-t)x_r(t;\theta,r)$. We are going to apply Lemma 2.6 to estimate this integral. To begin with, note that $\alpha(0)=\alpha(1)=0$. According to (2.1), $\xi=x_r$ solves $(\frac{d^2}{dt^2}+V''(x))\xi=0$, $\xi(0)=0$, $\dot{\xi}(0)=1$. It follows that

$$\xi(t)=t-\int_0^t (t-s)V''(x(s))\,\xi(s)\,ds.$$

Hence by Gronwall's inequality, $|\xi(t)|\le \exp(\|V''\|_{L^\infty(\mathbb{R})})$ for $t\in[0,1]$. Also

$$\dot{\xi}(t)=1-\int_0^t V''(x(s))\,\xi(s)\,ds,$$

and accordingly $|\dot{\xi}(t)|\le 1+\|V''\|_{L^\infty(\mathbb{R})}\exp(\|V''\|_{L^\infty(\mathbb{R})})$ for $t\in[0,1]$. To summarize,

$$|x_r(t)|\le \exp(\|V''\|_{L^\infty(\mathbb{R})})\le \tilde{C}_2 \quad\text{and}\quad |\dot{x}_r(t)|\le 1+\|V''\|_{L^\infty(\mathbb{R})}\exp(\|V''\|_{L^\infty(\mathbb{R})})\le \tilde{C}_2$$

for $t\in[0,1]$, where $\tilde{C}_2=(1+\|V''\|_{L^\infty(\mathbb{R})})\exp(\|V''\|_{L^\infty(\mathbb{R})})$. In particular,

$$\|\alpha\|_{L^\infty([0,1])},\,\|\dot{\alpha}\|_{L^\infty([0,1])}\le 2\tilde{C}_2. \tag{2.5}$$

Next denote $z(t)=z(t;\theta,r)=x(t;\theta,r)-\theta-rt$. Then z solves $\ddot{z}+V'(x)=p(t)$, $z(0)=0$, $\dot{z}(0)=0$. Thus

$$\dot{z}(t)=\int_0^t [p(s)-V'(x(s))]\,ds$$

yields $\|\dot{z}\|_{L^\infty([0,1])}\le \|V'\|_{L^\infty(\mathbb{R})}+\|p\|_{L^\infty(\mathbb{R})}\le C_1$. From $\ddot{z}+V'(x)=p(t)$ also $\|\ddot{z}\|_{L^\infty([0,1])}\le C_1$ is obtained. Recalling (2.4) and (2.5), Lemma 2.6 gives

$$|F_r(\theta,r)|\le \frac{8\tilde{C}_2}{r}\,\|V'\|_{L^\infty(\mathbb{R})} \quad\text{for}\quad r\ge 2C_1,$$

as was to be shown. □

The proof of Theorem 2.2 relies on an application of [6, Thm. 2.1], which we recall first. See also Theorem 3.1 below for a more general result.

Theorem 2.8 *Let $\Delta>\delta>0$. Suppose that $h:\Omega=\{(\theta,\theta')\in\mathbb{R}^2:\delta\le\theta'-\theta\le\Delta\}\to\mathbb{R}$ is C^1 and such that*

$$\underline{\alpha}(\theta'-\theta)^2\le h(\theta,\theta')\le\overline{\alpha}(\theta'-\theta)^2,\quad (\theta,\theta')\in\Omega,$$

for some constants $\overline{\alpha} \geq \underline{\alpha} > 0$ *so that* $\overline{\alpha} < \frac{3}{2}\underline{\alpha}$. *Then there is a constant* $\sigma_{**} \geq 1$ *(depending only on* $\overline{\alpha}/\underline{\alpha} \in [1, \frac{3}{2}[$*) with the following property. If*

$$\sigma_{**}\delta < \sigma_{**}^{-1}\Delta,$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ *such that* $|\theta_0^*| \leq \Delta$, $\delta \leq \theta_{n+1}^* - \theta_n^* \leq \Delta$ *for* $n \in \mathbb{Z}$*, and*

$$\partial_2 h(\theta_{n-1}^*, \theta_n^*) + \partial_1 h(\theta_n^*, \theta_{n+1}^*) = 0, \quad n \in \mathbb{Z}.$$

Moreover,

$$\delta \leq \liminf_{n\to\infty} \frac{\theta_n^*}{n} \leq \limsup_{n\to\infty} \frac{\theta_n^*}{n} \leq \Delta, \quad \delta \leq \liminf_{n\to-\infty} \frac{\theta_n^*}{n} \leq \limsup_{n\to-\infty} \frac{\theta_n^*}{n} \leq \Delta.$$

Proof of Theorem 2.2 : Denote

$$C_3 = 3(C_1 + C_2) + 1,$$

where $C_1 > 0$ and $C_2 > 0$ are defined in Lemma 2.7. Let $\theta, \theta_1 \in \mathbb{R}$ be fixed such that $\theta_1 - \theta \geq C_3$. Then

$$\theta_1 = \theta + r + F(\theta, r)$$

has a unique solution $r = r(\theta, \theta_1) \in [2(C_1 + C_2), \infty[$, since $\varphi(r) = r + F(\theta, r)$ has $\varphi(2(C_1 + C_2)) \leq 2(C_1 + C_2) + \|F\|_{L^\infty(\mathbb{R}^2)} \leq 3C_1 + 2C_2 < C_3 \leq \theta_1 - \theta$ and $\varphi'(r) = 1 + F_r(\theta, r) \geq 1 - C_2/r \geq 1/2$ for $r \geq 2(C_1 + C_2)$ by Lemma 2.7. Let

$$X(t) = X(t; \theta, \theta_1) = x(t; \theta, r(\theta, \theta_1)), \quad \theta_1 - \theta \geq C_3.$$

Then X solves $\ddot{X} + V'(X) = p(t)$, and moreover $X(0) = \theta$ as well as

$$X(1) = x(1; \theta, r(\theta, \theta_1)) = \theta + r(\theta, \theta_1) + F(\theta, r(\theta, \theta_1)) = \theta_1$$

by the definitions of F and $r(\theta, \theta_1)$. Let $Y(t) = Y(t; \theta, \theta_1) = X(t; \theta, \theta_1) - \theta - (\theta_1 - \theta)t$. Then Y is the solution to $\ddot{Y} + V'(X) = p(t)$, $Y(0) = 0$, $Y(1) = 0$. As in Lemma 2.7 for y, it follows that $\|Y\|_{L^\infty([0,1])} \leq C_1$ and $\|\dot{Y}\|_{L^\infty([0,1])} \leq C_1$. Next we introduce the restricted action functional

$$h(\theta, \theta_1) = \int_0^1 L(t, X(t; \theta, \theta_1), \dot{X}(t; \theta, \theta_1))\, dt, \quad \theta_1 - \theta \geq C_3,$$

where $L(t, X, \dot{X}) = \frac{1}{2}\dot{X}^2 - V(X) + p(t)X$ is the Lagrange function. It is well known that h is the generating function of the Poincaré map $P : (\theta, r) = (x(0), \dot{x}(0)) \mapsto (x(1), \dot{x}(1)) = (\theta_1, r_1)$; see [9, p. 84]. In particular,

$$-\partial_\theta h(\theta, \theta_1) = r(\theta, \theta_1), \quad \partial_{\theta_1} h(\theta, \theta_1) = \dot{x}(1; \theta, r(\theta, \theta_1)), \tag{2.6}$$

holds, as could also be checked directly. Expressing X in terms of Y,

$$\begin{aligned} h(\theta, \theta_1) &= \int_0^1 \Big(\frac{1}{2}\,[\dot{Y}(t) + (\theta_1 - \theta)]^2 - V(X(t)) + p(t)[Y(t) + \theta + (\theta_1 - \theta)t]\Big)\, dt \\ &= \frac{1}{2}\,(\theta_1 - \theta)^2 + (\theta_1 - \theta)\Big(\int_0^1 tp(t)\, dt\Big) + \int_0^1 \Big(\frac{1}{2}\,\dot{Y}(t)^2 - V(X(t)) + p(t)Y(t)\Big)\, dt. \end{aligned}$$

Observe that for this expansion it has been used that p has zero average. Denoting $R(\theta,\theta_1) = h(\theta,\theta_1) - (\theta_1-\theta)^2/2$ the remainder term, it follows from the previous observations that

$$|R(\theta,\theta_1)| \le C_4(\theta_1-\theta) + C_5, \quad \theta_1-\theta \ge C_3,$$

for

$$C_4 = \Big|\int_0^1 tp(t)\,dt\Big| \quad \text{and} \quad C_5 = \frac{1}{2}C_1^2 + \|V\|_{L^\infty(\mathbb{R})} + C_1\|p\|_{L^\infty(\mathbb{R})}.$$

Since $C_3 \ge 1$, we obtain $|R(\theta,\theta_1)| \le C_6(\theta_1-\theta)$ for $\theta_1-\theta \ge C_3$, where $C_6 = C_4 + C_5$.

In order to apply Theorem 2.8 to h, take $\underline{\alpha} = 7/16$, $\overline{\alpha} = 9/16$, and select the constant $\sigma_{**} \ge 1$ accordingly. Next fix sequences (δ_j) and (Δ_j) such that $\delta_j < \Delta_j < \delta_{j+1}$ and also $\delta_j > C_3 + 16C_6$ and $\Delta_j > \sigma_{**}^2\delta_j$ are satisfied for $j \in \mathbb{N}$. Then Theorem 2.8 can be used for h on every $\Omega_j = \{(\theta,\theta_1) \in \mathbb{R}^2 : \delta_j \le \theta_1-\theta \le \Delta_j\}$, since $|R(\theta,\theta_1)| \le C_6(\theta_1-\theta) \le (1/16)(\theta_1-\theta)^2$ yields $(7/16)(\theta_1-\theta)^2 \le h(\theta,\theta_1) \le (9/16)(\theta_1-\theta)^2$ on Ω_j. Thus by Theorem 2.8, for every $j \in \mathbb{N}$ there is a sequence $(\theta_n^{(j)})_{n\in\mathbb{Z}}$ such that $|\theta_0^{(j)}| \le \Delta_j$,

$$\delta_j \le \theta_{n+1}^{(j)} - \theta_n^{(j)} \le \Delta_j, \quad \text{and} \quad \partial_2 h(\theta_{n-1}^{(j)},\theta_n^{(j)}) + \partial_1 h(\theta_n^{(j)},\theta_{n+1}^{(j)}) = 0$$

holds for $n \in \mathbb{Z}$. Let $r_n^{(j)} = -\partial_\theta h(\theta_n^{(j)},\theta_{n+1}^{(j)}) = r(\theta_n^{(j)},\theta_{n+1}^{(j)})$ for $n \in \mathbb{Z}$ and $j \in \mathbb{N}$, cf. (2.6). Consider the solutions

$$x_j(t) = x(t;\theta_0^{(j)},r_0^{(j)}).$$

Then

$$\theta_n^{(j)} = x_j(n) \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad j \in \mathbb{N}. \tag{2.7}$$

For this, the definition of $r = r(\theta,\theta_1)$ implies that $x_j(1) = \theta_1^{(j)}$ and

$$\dot{x}_j(1) = \partial_2 h(\theta_0^{(j)},\theta_1^{(j)}) = -\partial_1 h(\theta_1^{(j)},\theta_2^{(j)}) = r_1^{(j)},$$

because h is the generating function of the Poincaré map. The periodicity in time then allows us to conclude that

$$(x_j(n),\dot{x}_j(n)) = P^n(\theta_0^{(j)},r_0^{(j)}) = (\theta_n^{(j)},r_n^{(j)}), \quad n \in \mathbb{Z},$$

proving (2.7). Since $\delta_j \le x_j(n+1) - x_j(n) \le \Delta_j$, there is $\zeta \in [n,n+1]$ such that $\delta_j \le \dot{x}_j(\zeta) \le \Delta_j$. Observing $\|\ddot{x}_j\|_{L^\infty(\mathbb{R})} \le \|V'\|_{L^\infty(\mathbb{R})} + \|p\|_{L^\infty(\mathbb{R})} \le C_1$ by (2.1), the estimates

$$\delta_j - C_1 \le \dot{x}_j(\zeta) - |\dot{x}_j(t) - \dot{x}_j(\zeta)| \le \dot{x}_j(t) \le \dot{x}_j(\zeta) + |\dot{x}_j(t) - \dot{x}_j(\zeta)| \le \Delta_j + C_1$$

are obtained for $t \in [n,n+1]$. Thus every x_j is a running solution to (2.1) and $\inf_{t\in\mathbb{R}} \dot{x}_j(t) \ge \delta_j - C_1 \to \infty$ as $j \to \infty$. □

3 More general generating functions

In view of further applications it is too restrictive to be fixed to generating functions that grow quadratically, as it was the case in Section 2. Thus we first broaden this class in Section 3.1 to include generating functions of the type $h(\theta,\theta') \sim (\theta'-\theta)^\kappa$ for some $\kappa > 1$. As an application we consider in the subsequent Section 3.2 the example of a bouncing ball, where the existence of infinitely many motions of bounded velocity is obtained.

3.1 An abstract result

We are going to prove the following generalization of [6, Thm. 2.1] (cited in Theorem 2.8 above), where we had $\kappa = 2$ corresponding to the case of quadratically growing generating functions.

Theorem 3.1 *Fix $\kappa > 1$ and let $\Delta > \delta > 0$. Suppose that the mapping $h : \Omega = \{(\theta, \theta') \in \mathbb{R}^2 : \delta \leq \theta' - \theta \leq \Delta\} \to \mathbb{R}$ is C^1 and such that*

$$\underline{\alpha}(\theta' - \theta)^\kappa \leq h(\theta, \theta') \leq \overline{\alpha}(\theta' - \theta)^\kappa, \quad (\theta, \theta') \in \Omega, \tag{3.1}$$

for some constants $\overline{\alpha} \geq \underline{\alpha} > 0$ so that $\overline{\alpha} < (\frac{1}{2} + \frac{1}{2^k})^{-1}\,\underline{\alpha}$. Then there is a constant $\sigma_{**} \geq 1$ (depending only on $\overline{\alpha}/\underline{\alpha}$) with the following property. If

$$\sigma_{**}\delta < \sigma_{**}^{-1}\Delta,$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ such that $|\theta_0^*| \leq \Delta$, $\delta \leq \theta_{n+1}^* - \theta_n^* \leq \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}^*, \theta_n^*) + \partial_1 h(\theta_n^*, \theta_{n+1}^*) = 0, \quad n \in \mathbb{Z}.$$

Moreover,

$$\delta \leq \liminf_{n\to\infty} \frac{\theta_n^*}{n} \leq \limsup_{n\to\infty} \frac{\theta_n^*}{n} \leq \Delta, \quad \delta \leq \liminf_{n\to-\infty} \frac{\theta_n^*}{n} \leq \limsup_{n\to-\infty} \frac{\theta_n^*}{n} \leq \Delta.$$

The proof of Theorem 3.1 is along the lines of the proof to [6, Thm. 2.1]. Therefore we keep the presentation short, but indicate at which places changes are needed. For fixed $A > 0$, $N \in \mathbb{N}$, and $\Delta > \delta > 0$, define

$$\Sigma^{(N)} = \Big\{\Theta = (\theta_n)_{-N\leq n\leq N} : \theta_{\pm N} = \pm A,\ \delta \leq \theta_{n+1} - \theta_n \leq \Delta \text{ for } n = -N, \dots, N-1\Big\}.$$

Since later $A = A_N$ will be chosen to depend on N, the dependence of $\Sigma^{(N)}$ on A is suppressed in our notation. The following observation is [6, Lemma 2.3].

Lemma 3.2 *If $\delta \leq \frac{A}{N} \leq \Delta$, then $\Sigma^{(N)} \neq \emptyset$, and $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is compact.*

Let

$$S(\Theta) = \sum_{n=-N}^{N-1} h(\theta_n, \theta_{n+1}), \quad \Theta = (\theta_n)_{-N\leq n\leq N} \in \Sigma^{(N)}.$$

Since $S : \Sigma^{(N)} \to \mathbb{R}$ is continuous, there exists a minimizer, i.e.,

$$S(\Theta^{(N)}) = \min_{\Theta\in\Sigma^{(N)}} S(\Theta)$$

for a suitable $\Theta^{(N)} = (\theta_n^{(N)})_{-N\leq n\leq N} \in \Sigma^{(N)}$, which henceforth we consider to be fixed. The bounds obtained in the next lemma are of central importance to the proof.

Lemma 3.3 *Suppose that $\overline{\alpha} < (\frac{1}{2} + \frac{1}{2^k})^{-1}\,\underline{\alpha}$. There exists a constant $\sigma_* = \sigma_*(\overline{\alpha}/\underline{\alpha}) \geq 1$ such that for all $N \in \mathbb{N}$,*

$$\sigma_*^{-1}(\theta_n^{(N)} - \theta_{n-1}^{(N)}) \leq \theta_{n+1}^{(N)} - \theta_n^{(N)} \leq \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}), \quad -N+1 \leq n \leq N-1.$$

Proof: To derive the upper bound, write $\theta_n^{(N)} - \theta_{n-1}^{(N)} = L$ and $\theta_{n+1}^{(N)} - \theta_n^{(N)} = \sigma L$ for $L, \sigma > 0$. Using the method from [6, Lemma 2.4], it follows that

$$h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \le h(\theta_{n-1}^{(N)}, s) + h(s, \theta_{n+1}^{(N)})$$

for $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_{n-1}^{(N)})$. Thus by (3.1),

$$\begin{aligned}\underline{\alpha}(1+\sigma^\kappa)L^\kappa &= \underline{\alpha}(\theta_n^{(N)} - \theta_{n-1}^{(N)})^\kappa + \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^\kappa \le \overline{\alpha}(s - \theta_{n-1}^{(N)})^\kappa + \overline{\alpha}(\theta_{n+1}^{(N)} - s)^\kappa \\ &= \frac{1}{2^{\kappa-1}}\,\overline{\alpha}(\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)})^\kappa = \frac{1}{2^{\kappa-1}}\,\overline{\alpha}(1+\sigma)^\kappa L^\kappa.\end{aligned}$$

In other words, $\varphi(\sigma) \le q = \overline{\alpha}/\underline{\alpha}$ for the strictly increasing function $\varphi(\sigma) = \frac{2^{\kappa-1}(1+\sigma^\kappa)}{(1+\sigma)^\kappa} : [1,\infty[\to [1, 2^{k-1}[$. Since $1 \le q < (\frac{1}{2} + \frac{1}{2^k})^{-1} < 2^{k-1}$ by assumption, there is a unique $\sigma_* \in [1,\infty[$ so that $\varphi(\sigma_*) = q$. It follows that $\sigma \le \sigma_*$, which yields the upper bound.

For the lower bound, it is again sufficient to make use of the upper bound for the function

$$h_0(\theta, \theta') = h(-\theta', -\theta), \quad (\theta, \theta') \in \Omega;$$

see [6, Lemma 2.4]. □

For $N \in \mathbb{N}$, put

$$\delta^{(N)} = \min_{-N \le n \le N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \quad \text{and} \quad \Delta^{(N)} = \max_{-N \le n \le N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}).$$

Then $\delta \le \delta^{(N)} \le \Delta^{(N)} \le \Delta$, due to $\Theta^{(N)} = (\theta_n^{(N)})_{-N \le n \le N} \in \Sigma^{(N)}$. Next we need to generalize [6, Lemma 2.5].

Lemma 3.4 *Suppose that* $\overline{\alpha} < (\frac{1}{2} + \frac{1}{2^k})^{-1}\,\underline{\alpha}$. *There exists a constant* $\sigma_{**} = \sigma_{**}(\overline{\alpha}/\underline{\alpha}) \ge 1$ *such that for all* $N \in \mathbb{N}$,

$$\Delta^{(N)} \le \sigma_{**}\delta^{(N)}.$$

Proof: Put

$$\sigma_{**} = 4(2^{k-1} - 1)^{-1/k}\sigma_*.$$

Since $\sigma_{**} \ge 2\sigma_* \ge 2$, we still may assume without loss of generality that $\delta^{(N)} \le \frac{1}{2\sigma_*}\Delta$ and $\Delta^{(N)} \ge 2\delta$ holds. Select $-N \le m, n \le N-1$ such that

$$\delta^{(N)} = \theta_{m+1}^{(N)} - \theta_m^{(N)} \quad \text{and} \quad \Delta^{(N)} = \theta_{n+1}^{(N)} - \theta_n^{(N)}.$$

It is no restriction to suppose that $m \le n$. If $m = n$, then the assertion of the lemma is a consequence of $\sigma_{**} \ge 1$. If $m + 1 = n$, then it suffices to use Lemma 3.3, because $\sigma_{**} \ge \sigma_*$. Hence we may moreover suppose that $m + 2 \le n$. Following the argument from the proof to [6, Lemma 2.5], we thus get

$$\begin{aligned}&h(\theta_m^{(N)}, \theta_{m+1}^{(N)}) + h(\theta_{m+1}^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\le h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}),\end{aligned}$$

where $s = \frac{1}{2}(\theta_{n+1}^{(N)} + \theta_n^{(N)})$. Since $h \geq 0$, (3.1) yields

$$\begin{aligned}
\underline{\alpha}(\Delta^{(N)})^\kappa &= \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^\kappa \leq h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\
&\leq h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}) \\
&\leq \overline{\alpha}(\theta_{m+2}^{(N)} - \theta_m^{(N)})^\kappa + 2\overline{\alpha}\gamma^\kappa \\
&\leq 2^\kappa\, \overline{\alpha}\, \sigma_*^\kappa (\delta^{(N)})^\kappa + \frac{1}{2^{\kappa-1}}\, \overline{\alpha}\, (\Delta^{(N)})^\kappa,
\end{aligned}$$

denoting $\gamma = s - \theta_n^{(N)} = \theta_{n+1}^{(N)} - s = \frac{1}{2}(\theta_{n+1}^{(N)} - \theta_n^{(N)}) = \frac{1}{2}\Delta^{(N)}$. Observe that

$$\theta_{m+2}^{(N)} - \theta_m^{(N)} = (\theta_{m+2}^{(N)} - \theta_{m+1}^{(N)}) + (\theta_{m+1}^{(N)} - \theta_m^{(N)}) \leq (\sigma_* + 1)(\theta_{m+1}^{(N)} - \theta_m^{(N)}) \leq 2\sigma_*\delta^{(N)}$$

by Lemma 3.3. Therefore $\overline{\alpha} < (\frac{1}{2} + \frac{1}{2^k})^{-1}\, \underline{\alpha}$ yields

$$\Big(\frac{1}{2} - \frac{1}{2^k}\Big)\, (\Delta^{(N)})^\kappa \leq \Big(\underline{\alpha}/\overline{\alpha} - \frac{1}{2^{\kappa-1}}\Big)\, (\Delta^{(N)})^\kappa \leq 2^\kappa\, \sigma_*^\kappa (\delta^{(N)})^\kappa,$$

and hence the claim. □

Having at hand Lemma 3.4, the proof of the following result is the same as in [6, Cor. 2.6].

Corollary 3.5 *Suppose that the assumptions of Lemma 3.4 are satisfied. If*

$$\sigma_{**}\delta < \frac{A}{N} < \sigma_{**}^{-1}\Delta,$$

then for all $N \in \mathbb{N}$ *and* $-N \leq n \leq N-1$,

$$\delta < \delta^{(N)} \leq \theta_{n+1}^{(N)} - \theta_n^{(N)} \leq \Delta^{(N)} < \Delta.$$

After deriving the needed auxiliary results, the actual proof of Theorem 3.1 can now be copied verbatim from [6], taking once more $A = A_N = \frac{1}{2}(\sigma_{**}^{-1}\Delta + \sigma_{**}\delta)N$ in the definition of $\Sigma^{(N)}$. □

Remark 3.6 (a) The relevant constants σ_* and $\sigma_{**} = 4(2^{k-1}-1)^{-1/k}\sigma_*$ depend only on $q = \overline{\alpha}/\underline{\alpha} \in [1, (\frac{1}{2} + \frac{1}{2^k})^{-1}[\subset [1, 2^{k-1}[$. Here $\sigma_* \in [1, \infty[$ is the unique solution to $\frac{2^{\kappa-1}(1+\sigma_*^\kappa)}{(1+\sigma_*)^\kappa} = q$.
(b) For $\kappa = 2$, the hypothesis $q < \frac{3}{2}$ from [6] seems to be less restrictive than $q < \frac{4}{3}$, as obtained in Theorem 3.1. However, it is clear from the proof of Lemma 3.4 that a sharper estimate is also possible. In applications it is always sufficient to have a result which covers the values of q sufficiently close to 1.

3.2 A model of a bouncing ball

As an example for the application of Theorem 3.1 for $\kappa = 3$, we consider a ball at the vertical position $x(t) \geq w(t)$ which is in free fall (i.e. its motion is governed by the equation $\ddot{x} = -g$) until it hits a horizontal plate that is located at $w(t)$. Denoting $y = x - w \geq 0$, at an instant τ of impact the change of velocity $\dot{y}(\tau^+) = -\dot{y}(\tau^-)$ is assumed to be elastic. We thus analyze the dynamics of the equation with an obstacle

$$\ddot{y}(t) = -(g + \ddot{w}(t)), \quad y \geq 0 \text{ everywhere}, \quad y(\tau) = 0 \;\Rightarrow\; \dot{y}(\tau^+) = -\dot{y}(\tau^-). \tag{3.2}$$

Henceforth we suppose that the given function $w \in C^2(\mathbb{R})$ satisfies $\|w\|_{L^\infty(\mathbb{R})} + \|\dot{w}\|_{L^\infty(\mathbb{R})} < \infty$. By a solution of (3.2) we understand a function $y \in C(\mathbb{R})$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of impact times such that

(i) $\inf_{n\in\mathbb{Z}}(t_{n+1} - t_n) > 0$,

(ii) $y(t_n) = 0$ for $n \in \mathbb{Z}$ and $y(t) > 0$ for $t \in]t_n, t_{n+1}[$, and

(iii) the function y is of class C^2 on every interval $[t_n, t_{n+1}]$ and satisfies the linear differential equation on this interval. Moreover, $\dot{y}(t_n^+) = -\dot{y}(t_n^-)$.

A solution will be called bounded, if furthermore

(iv) $y, \dot{y} \in L^\infty(\mathbb{R})$, and

(v) $\sup_{n\in\mathbb{Z}}(t_{n+1} - t_n) < \infty$

are satisfied.

To get a clue on how to construct bounded solutions, let us start with the Dirichlet problem

$$\ddot{y}(t) = -(g + \ddot{w}(t)), \quad y(t_0) = y(t_1) = 0.$$

It is uniquely solved by the function

$$y(t; t_0, t_1) = \frac{g}{2}(t_1 - t)(t - t_0) + \frac{w(t_1) - w(t_0)}{t_1 - t_0}(t - t_0) + w(t_0) - w(t). \tag{3.3}$$

Lemma 3.7 *Suppose that*

$$t_1 - t_0 > \frac{8}{g}\|\dot{w}\|_{L^\infty(\mathbb{R})} =: \delta_*. \tag{3.4}$$

Then $y(t) > 0$ *for* $t \in]t_0, t_1[$.

Proof: If, in general, $f \in C^1([t_0, t_1])$ is such that $f(t_0) = f(t_1) = 0$, then

$$|f(t)| \leq \min\{t - t_0, t_1 - t\}\,\|\dot{f}\|_{L^\infty([t_0,t_1])}, \quad t \in [t_0, t_1].$$

This estimate is applied to $f(t) = \frac{w(t_1)-w(t_0)}{t_1-t_0}(t-t_0) + w(t_0) - w(t)$, where $\|\dot{f}\|_{L^\infty([t_0,t_1])} \leq 2\|\dot{w}\|_{L^\infty(\mathbb{R})}$. If $t \in [t_0, (t_0 + t_1)/2]$, thus

$$\begin{aligned} y(t) &\geq \frac{g}{2}(t_1 - t)(t - t_0) - |f(t)| \geq (t - t_0)\Big[\frac{g}{2}(t_1 - t) - 2\|\dot{w}\|_{L^\infty(\mathbb{R})}\Big] \\ &\geq (t - t_0)\Big[\frac{g}{4}(t_1 - t_0) - 2\|\dot{w}\|_{L^\infty(\mathbb{R})}\Big]. \end{aligned}$$

The same estimate is obtained for $t \in [(t_0+t_1)/2, t_1]$, which yields the claim. $\square$

Hence a first attempt to obtain solutions of the obstacle problem (3.2) would be to select an arbitrary sequence $(t_n)_{n\in\mathbb{N}}$ such that $\inf_{n\in\mathbb{Z}}(t_{n+1}-t_n) > \delta_*$ and juxtapose the corresponding solutions $y(t;t_n,t_{n+1})$ of the Dirichlet problems. However, this would not lead to the desired solution, because we cannot guarantee that the condition of elastic bouncing $\dot{y}(t_n^+) = -\dot{y}(t_n^-)$ will be satisfied. To overcome this difficulty, we consider the generating function $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(t_0,t_1) = \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t))\, dt,$$

where

$$L(t,y,\dot{y}) = \frac{1}{2}\,\dot{y}^2 - (g + \ddot{w}(t))y$$

is the Lagrangian associated to the differential equation and $y(t)$ stands for $y(t;t_0,t_1)$ from (3.3). Next we observe that h is of class C^1 on $\{(t_0,t_1) : t_0 < t_1\}$. Using integration by parts and the Euler-Lagrange equation for y on $]t_0,t_1[$, the partial derivative $\partial_{t_0}h$ is found to be

$$\begin{aligned}
\partial_{t_0}h(t_0,t_1) &= -L(t_0,y(t_0),\dot{y}(t_0^+)) + \int_{t_0}^{t_1} \Big\{(\partial_y L)(\partial_{t_0}y) + (\partial_{\dot{y}}L)(\partial_{t_0}\dot{y})\Big\}\, dt \\
&= -L(t_0,y(t_0),\dot{y}(t_0^+)) + (\partial_{\dot{y}}L)\,(\partial_{t_0}y)\Big|_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} \Big\{(\partial_y L) - \frac{d}{dt}(\partial_{\dot{y}}L)\Big\}\,(\partial_{t_0}y)\, dt \\
&= -\frac{1}{2}\,\dot{y}(t_0^+)^2 + \dot{y}(t_1^-)\,\partial_{t_0}y(t_1) - \dot{y}(t_0^+)\,\partial_{t_0}y(t_0).
\end{aligned}$$

Differentiating the relations $y(t_1;t_0,t_1) = 0$ and $y(t_0;t_0,t_1) = 0$, it follows that $\partial_{t_0}y(t_1) = 0$ and $\dot{y}(t_0^+) + \partial_{t_0}y(t_0) = 0$. Therefore

$$\partial_{t_0}h(t_0,t_1) = \frac{1}{2}\,\dot{y}(t_0^+)^2$$

is obtained, and in the same way it is shown that

$$\partial_{t_1}h(t_0,t_1) = -\frac{1}{2}\,\dot{y}(t_1^-)^2.$$

Assume now that we could find a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\inf_{n\in\mathbb{Z}}(t_{n+1}-t_n) > \delta_*$ and

$$\partial_{t_1}h(t_{n-1},t_n) + \partial_{t_0}h(t_n,t_{n+1}) = 0, \quad n \in \mathbb{Z}. \tag{3.5}$$

Then we can construct a solution y to the obstacle problem (3.2) by defining

$$y(t) = y(t;t_n,t_{n+1}), \quad t \in [t_n,t_{n+1}],$$

since then the previous observations imply that

$$\dot{y}(t_n^+)^2 = \dot{y}(t_n^-)^2,$$

which in our context is equivalent to the condition of elastic bouncing.

Assume moreover that we know in addition that $\sup_{n\in\mathbb{Z}}(t_{n+1}-t_n) \le \Delta < \infty$. Then

$$0 \le y(t) \le \frac{g}{8}\,\Delta^2 + 4\,\|w\|_{L^\infty(\mathbb{R})}, \quad t \in \mathbb{R},$$

is a consequence of (3.3). Supposing that $\dot{w} \in L^\infty(\mathbb{R})$, we obtain a similar estimate for the derivative,

$$|\dot{y}(t)| \leq \frac{g}{2}\,\Delta + 2\,\|\dot{w}\|_{L^\infty(\mathbb{R})}, \quad t \in \mathbb{R};$$

it $t = t_n$, then $\dot{y}(t)$ refers to the one-sided derivatives $\dot{y}(t_n^\pm)$. The above discussions leads to a method for proving the existence of a bounded solution to the obstacle problem (3.2). All we have to do is to find solutions $(t_n)_{n\in\mathbb{N}}$ of (3.5) satisfying

$$\delta_* < t_{n+1} - t_n \leq \Delta \quad \text{for each} \quad n \in \mathbb{Z}.$$

This is a question that can be treated using Theorem 3.1. First we compute the dominant terms of the expansion of h in powers of $(t_1 - t_0)$.

Lemma 3.8 *We have*

$$h(t_0, t_1) = -\frac{g^2}{24}(t_1 - t_0)^3 + R(t_0, t_1),$$

where

$$|R(t_0, t_1)| \leq C_*(t_1 - t_0), \quad t_1 > t_0,$$

for $C_* = 2g\|w\|_{L^\infty(\mathbb{R})} + \|\dot{w}\|^2_{L^\infty(\mathbb{R})}$.

Proof: First we note that integrating $\int_{t_0}^{t_1} dt$ the relation $\frac{d}{dt}(y\dot{y}) = y\ddot{y} + \dot{y}^2$ implies that

$$h(t_0, t_1) = -\frac{1}{2}\int_{t_0}^{t_1} (g + \ddot{w})y\,dt.$$

Then we substitute the explicitly known y from (3.3) in $-(g/2)\int_{t_0}^{t_1} y\,dt$ and integrate by parts twice in $-(1/2)\int_{t_0}^{t_1} \ddot{w}y\,dt$, thereafter replacing $\ddot{y}$ by $-(g + \ddot{w})$. After a lengthy but straightforward calculation it is found that

$$\begin{aligned} h(t_0, t_1) &= -\frac{g^2}{24}(t_1 - t_0)^3 - \frac{g}{2}(w(t_1) + w(t_0))(t_1 - t_0) + \frac{(w(t_1) - w(t_0))^2}{2(t_1 - t_0)} \\ &\quad + g\int_{t_0}^{t_1} w(t)\,dt - \frac{1}{2}\int_{t_0}^{t_1} \dot{w}(t)^2\,dt. \end{aligned}$$

This yields the claimed bound on R. □

Now we can formulate the main result of this section.

Theorem 3.9 *Suppose that* $w \in C^2(\mathbb{R})$ *is such that* $\|w\|_{L^\infty(\mathbb{R})} + \|\dot{w}\|_{L^\infty(\mathbb{R})} < \infty$. *Then there are infinitely many bounded solutions* y_j *of the obstacle problem (3.2) such that the associated sequences* $(t_n^{(j)})_{n\in\mathbb{N}}$ *of impact times satisfy*

$$\inf_{n\in\mathbb{Z}}(t_{n+1}^{(j)} - t_n^{(j)}) \to \infty \quad as \quad j \to \infty.$$

Proof: We are going to apply Theorem 3.1 for $\kappa = 3$ to $\tilde{h}(t_0, t_1) = -h(t_0, t_1)$. Let $\underline{\alpha} = g^2/30$ and $\overline{\alpha} = g^2/20$. Then $\overline{\alpha} < (8/5)\underline{\alpha}$. Next select the constant $\sigma_{**} \geq 1$ according to Theorem 3.1 and put

$$\delta_{**} = \delta_* + \frac{1}{g}\sqrt{120\, C_*},$$

where δ_* and C_* are defined in (3.4) and Lemma 3.8, respectively.

Fix sequences (δ_j) and (Δ_j) such that $\delta_{**} < \delta_j < \Delta_j < \delta_{j+1}$ and $\Delta_j > \sigma_{**}^2 \delta_j$ are satisfied for $j \in \mathbb{N}$. We claim that Theorem 3.1 can be used for $\tilde{h} : \Omega_j \to \mathbb{R}$ and every $j \in \mathbb{N}$, where $\Omega_j = \{(t_0, t_1) \in \mathbb{R}^2 : \delta_j \leq t_1 - t_0 \leq \Delta_j\}$. In fact, if $t_1 - t_0 \geq \delta_j > \delta_{**}$, then

$$|R(t_0, t_1)| \leq C_*(t_1 - t_0) \leq \frac{g^2}{120}(t_1 - t_0)^3$$

by Lemma 3.8 and the definition of δ_{**}. Hence

$$\underline{\alpha}(t_1 - t_0)^3 = \frac{g^2}{30}\,(t_1 - t_0)^3 \leq \frac{g^2}{24}(t_1 - t_0)^3 - R(t_0, t_1) \leq \frac{g^2}{20}\,(t_1 - t_0)^3 = \overline{\alpha}(t_1 - t_0)^3$$

yields (3.1) for $\tilde{h}$. Therefore by Theorem 3.1 for every $j \in \mathbb{N}$ there is a sequence $(t_n^{(j)})_{n \in \mathbb{Z}}$ such that in particular

$$\delta_j \leq t_{n+1}^{(j)} - t_n^{(j)} \leq \Delta_j, \quad j \in \mathbb{N}, \quad n \in \mathbb{Z},$$

and

$$\partial_2 h(t_{n-1}^{(j)}, t_n^{(j)}) + \partial_1 h(t_n^{(j)}, t_{n+1}^{(j)}) = 0, \quad n \in \mathbb{Z},$$

hold, where $\partial_1 h = \partial_{t_0} h$ and $\partial_2 h = \partial_{t_1} h$. By the remarks preceding the theorem, this is sufficient to conclude the proof. □

4 The remains of the Aubry-Mather theory

The complete and bounded orbits that are constructed in Theorems 2.8 or 3.1 will in general not admit a rotation number. In this section we will investigate this point in greater detail for the generalized standard maps

$$f_\varphi(\theta, r) = (\theta_1, r_1), \quad \text{where} \quad \theta_1 = \theta + r \quad \text{and} \quad r_1 = r + \varphi(\theta + r), \tag{4.6}$$

for a given function $\varphi \in \mathrm{BUC}(\mathbb{R})$, the space of bounded and uniformly continuous maps from $\mathbb{R}$ to $\mathbb{R}$. This is a homeomorphism of $\mathbb{R}^2$ with the inverse

$$f_\varphi^{-1}(\theta_1, r_1) = (\theta, r), \quad \text{where} \quad \theta = \theta_1 - r_1 + \varphi(\theta_1) \quad \text{and} \quad r = r_1 - \varphi(\theta_1).$$

Note that if φ is a trigonometric polynomial, then f_φ becomes the standard map. The periodicity of φ then allows one to interpret θ as a periodic angle lying in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and f_φ induces a homeomorphism of the cylinder $\mathbb{T} \times \mathbb{R}$.

In [6, Thm. 4.5] we have proved the following

Theorem 4.1 *Let $\phi \in C^2(\mathbb{R})$ be such that ϕ and $\varphi = \phi'$ are bounded. Then f_φ has infinitely many complete orbits $(\theta_n^j, r_n^j)_{n \in \mathbb{Z}}$ such that $R_j \leq r_n^j \leq R_{j+1}$ for $n \in \mathbb{Z}$, where $R_j < R_{j+1} \to \infty$ as $j \to \infty$. Moreover,*

$$R_j \leq \liminf_{|n| \to \infty} \frac{\theta_n^j}{n} \leq \limsup_{|n| \to \infty} \frac{\theta_n^j}{n} \leq R_{j+1}.$$

We introduce a useful definition that will help to formalize our results.

Definition 4.2 *An* AM-*orbit of f_φ is a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ such that*

(a) $(\theta_n)_{n\in\mathbb{Z}}$ is strictly increasing (monotonicity),

(b) $(r_n)_{n\in\mathbb{Z}}$ is bounded, and

(c) the limit $\lim\limits_{|n|\to\infty} \dfrac{\theta_n}{n}$ does exist (rotation number).

Unlike in the case of Aubry-Mather sets in the theory of twist maps on the cylinder [9, Thm. 3.3.3], the rotation numbers in (c) will in general not exist. This is made evident by the next example which is similar to Example 2.5.

Example 4.3 Let $\phi(\theta) = \arctan\theta$ and $\varphi(\theta) = \frac{1}{1+\theta^2}$. Take a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ of f_φ such that (a) and (b) from Definition 4.2 are satisfied. Since φ is positive and $r_{n+1} = r_n + \varphi(\theta_n + r_n)$, the finite limits $r_\pm = \lim_{n\to\pm\infty} r_n$ exist and $r_+ > r_-$. Thus

$$\lim_{n\to\infty} \frac{\theta_n}{n} = \lim_{n\to\infty} \left(\frac{\theta_0}{n} + \frac{1}{n}\sum_{j=0}^{n-1} r_j\right) = r_+$$

and similarly $\lim_{n\to-\infty} \frac{\theta_n}{n} = r_-$. Hence $\lim_{|n|\to\infty} \frac{\theta_n}{n}$ does not exist.

To obtain positive results concerning the existence of AM-orbits for f_φ we will use the Bebutov flow which is usually employed to associate a dynamical system to a non-autonomous differential equation; see [12]. We will see below that this concept can also be useful in the study of twist maps with non-periodic angles.

First we recall some definitions. The space $C(\mathbb{R})$ of continuous functions is endowed with a distance inducing the topology of uniform convergence on compact sets. Given $\theta \in \mathbb{R}$ and $\varphi \in C(\mathbb{R})$, let

$$(T_\theta\varphi)(\tau) = \varphi(\theta + \tau), \quad \tau \in \mathbb{R},$$

denote the translate of φ by θ. The map

$$\mathcal{T} : \mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R}), \quad \mathcal{T}(\theta, \varphi) = T_\theta\varphi,$$

defines a continuous flow on $C(\mathbb{R})$ which is usually called the Bebutov flow. If $\varphi \in \mathrm{BUC}(\mathbb{R})$, then its orbit $\{T_\theta\varphi : \theta \in \mathbb{R}\} \subset C(\mathbb{R})$ is relatively compact. Thus its closure

$$\mathcal{H}_\varphi = \overline{\{T_\theta\varphi : \theta \in \mathbb{R}\}} \tag{4.7}$$

in $C(\mathbb{R})$ becomes a compact metric space, the so-called hull of φ. For instance, if $\varphi(\theta) = \frac{1}{1+\theta^2}$ is as in Example 4.3, then

$$\mathcal{H}_\varphi = \{T_\theta\varphi : \theta \in \mathbb{R}\} \cup \{0\} \tag{4.8}$$

which is homeomorphic to $\mathbb{S}^1$ by means of the map $T_\theta\varphi \mapsto \exp(2i\arctan\theta)$, $0 \mapsto \exp(\pm i\pi) = (-1, 0)$. The topology of $\mathcal{H}_\varphi$ can be quite complicated in general, but the advantage is that now the angle is immersed in a compact space. If φ is fixed or clear from the context then we will only write $\mathcal{H}$ instead of $\mathcal{H}_\varphi$.

It turns out that $\mathcal{H} \times \mathbb{R}$ can play a similar role as does $\mathbb{T} \times \mathbb{R}$ in the periodic case. For this consider $\mathcal{F} : \mathcal{H} \times \mathbb{R} \to \mathcal{H} \times \mathbb{R}$ defined by

$$\mathcal{F}(\psi, r) = (\psi_1, r_1), \quad \text{where} \quad \psi_1 = T_r \psi \quad \text{and} \quad r_1 = r + \psi(r);$$

note that this prescription makes $\mathcal{F}$ well-defined by (4.7). It is a homeomorphism with the inverse

$$\mathcal{F}^{-1}(\psi_1, r_1) = (\psi, r), \quad \text{where} \quad \psi = T_{-r_1+\psi_1(0)}\psi_1 \quad \text{and} \quad r = r_1 - \psi_1(0).$$

Strictly speaking, we should also write $\mathcal{F}_\varphi$ or $\mathcal{F}_\mathcal{H}$ instead of only $\mathcal{F}$. Then f_φ from (4.6) can be dynamically immersed inside $\mathcal{F}$. To be precise about the meaning of this statement, introduce $\iota : \mathbb{R} \times \mathbb{R} \to \mathcal{H} \times \mathbb{R}$ by

$$\iota(\theta, r) = (T_\theta \varphi, r).$$

Then

$$\mathcal{F} \circ \iota = \iota \circ f_\varphi$$

holds and thus $\mathcal{F}^n \circ \iota = \iota \circ f_\varphi^n$. In particular, if $(\theta_n, r_n)_{n\in\mathbb{Z}}$ is a complete orbit of f_φ, then $(\theta_n, r_n) = f_\varphi^n(\theta_0, r_0)$ and hence

$$\mathcal{F}^n(\iota(\theta_0, r_0)) = \iota(\theta_n, r_n), \quad n \in \mathbb{Z}.$$

This means that $(\psi_n, r_n) = \iota(\theta_n, r_n) = (T_{\theta_n}\varphi, r_n)$ is a complete orbit for $\mathcal{F}$.

Although f_φ from Example 4.3 has no AM-obits, there are obviously AM-orbits for f_ψ with $\psi = 0 \in \mathcal{H}_\varphi$; see (4.8). The next result extends this observation.

Theorem 4.4 *Let $\phi \in C^2(\mathbb{R})$ be bounded and such that $\varphi = \phi' \in \mathrm{BUC}(\mathbb{R})$. Then f_ψ admits an* AM-*orbit for some ψ in the hull $\mathcal{H}$ of φ.*

Proof: We adapt ideas taken from the theory of non-autonomous linear differential equations; see [5]. According to Theorem 4.1 there exists a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ of f_φ such that

$$0 < \underline{R} \le r_n \le \overline{R} < \infty, \quad n \in \mathbb{Z}. \tag{4.9}$$

According to the remarks preceding this theorem then $(\psi_n, r_n)_{n\in\mathbb{Z}} = (T_{\theta_n}\varphi, r_n)_{n\in\mathbb{Z}}$ is a complete orbit for $\mathcal{F}$ on $\mathcal{H} \times \mathbb{R}$. Thus it follows from (4.9) that

$$K = \overline{\{(T_{\theta_n}\varphi, r_n) : n \in \mathbb{Z}\}}$$

is a compact metric space and an $\mathcal{F}$-invariant subset of $\mathcal{H} \times [\underline{R}, \overline{R}]$. Hence by [11, Prop. 9.5] there exists at least one $\mathcal{F}$-ergodic probability measure μ on K. Since $\mathcal{F} : K \to K$ is homeomorphic, note that μ is also $\mathcal{F}^{-1}$-ergodic. Next consider the projection

$$\pi_2 : \ K \to \mathbb{R}, \quad \pi_2(\psi, r) = r.$$

Then π_2 is continuous and thus μ-integrable. Therefore it follows from the Birkhoff ergodic theorem [11, Thm. 10.2] that we can choose $(\psi^*, \rho^*) \in K$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \pi_2(\mathcal{F}^k(\psi^*, \rho^*)) = \int_K \pi_2 \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \pi_2(\mathcal{F}^{-k}(\psi^*, \rho^*)).$$

If we let $(\psi_k, \rho_k) = \mathcal{F}^k(\psi^*, \rho^*)$ for $k \in \mathbb{Z}$, then $(\psi_k, \rho_k)_{k\in\mathbb{Z}}$ is a complete orbit for $\mathcal{F}$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho_k = \int_K \pi_2 \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho_{-k} \tag{4.10}$$

is satisfied. Define

$$r_n = \rho_n \quad \text{and} \quad \theta_n = \begin{cases} \sum_{k=0}^{n-1} \rho_k & : \quad n \geq 1 \\ 0 & : \quad n = 0 \\ -\sum_{k=1}^{|n|} \rho_{-k} & : \quad n \leq -1 \end{cases}$$

for $n \in \mathbb{Z}$. Then it is straightforward to check that $(\theta_n, r_n)_{n\in\mathbb{Z}}$ is a complete orbit for f_{ψ^*}. It is also an AM-orbit, since (a) and (b) from Definition 4.2 are verified due to $\theta_{n+1} - \theta_n = r_n = \rho_n \in [\underline{R}, \overline{R}]$ and the rotation number $\lim_{|n|\to\infty} \theta_n/n = \int_K \pi_2 \, d\mu$ exists by (4.10). □

Following [4] we say that a function $\varphi \in \mathrm{BUC}(\mathbb{R})$ is minimal, if the Bebutov flow $\mathcal{T}$ on $\mathcal{H}_\varphi$ is minimal. Then the orbits $\{T_\theta\psi : \theta \in \mathbb{R}\} \subset \mathcal{H}_\varphi$ are dense for any $\psi \in \mathcal{H}_\varphi$. Almost periodic functions are examples of minimal functions.

Corollary 4.5 *Under the assumptions of Theorem 4.4 suppose that furthermore φ is minimal. Then there exists a dense subset $\mathcal{D} \subset \mathcal{H}_\varphi$ such that f_ψ admits an* AM*-orbit for every $\psi \in \mathcal{D}$.*

Proof: Let ψ_* be as constructed in Theorem 4.4 and denote $(\theta_n, r_n)_{n\in\mathbb{Z}}$ an AM-orbit of f_{ψ^*}. If $\theta \in \mathbb{R}$, then $(\theta_n - \theta, r_n)_{n\in\mathbb{Z}}$ is an AM-orbit of $f_{T_\theta\psi^*}$. Since φ is minimal we can thus take $\mathcal{D} = \{T_\theta\psi^* : \theta \in \mathbb{R}\}$ to be the orbit of ψ_*. □

To illustrate the previous result we include an example.

Corollary 4.6 *Consider the quasi-periodic standard map*

$$\theta_1 = \theta + r, \quad r_1 = r + 2\sin(\omega_1\theta + \phi_1) + 2\sin(\omega_2\theta + \phi_2), \tag{4.11}$$

where $\omega_1, \omega_2 > 0$ are fixed and not commensurable ($\omega_1/\omega_2 \notin \mathbb{Q}$); furthermore, $\phi_1, \phi_2 \in [0, 2\pi]$ are viewed as parameters. There exists a dense subset $D \subset [0, 2\pi] \times [0, 2\pi]$ such that if $(\phi_1, \phi_2) \in D$, then (4.11) has an AM*-orbit.*

Proof: To apply Corollary 4.5 take $\phi(\theta) = -(2/\omega_1)\cos(\omega_1\theta) - (2/\omega_2)\cos(\omega_2\theta)$ and $\varphi(\theta) = \phi'(\theta) = 2\sin(\omega_1\theta) + 2\sin(\omega_2\theta)$. Then φ is quasi-periodic, hence almost periodic, and thus minimal. Note that $\mathcal{H}_\varphi$ is homeomorphic to a torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$ via the map $\mathbb{T}^2 \ni (\overline{\phi_1}, \overline{\phi_2}) \mapsto \varphi^* \in \mathcal{H}_\varphi$, where $\varphi^*(\theta) = 2\sin(\omega_1\theta + \phi_1) + 2\sin(\omega_2\theta + \phi_2)$. Therefore the claim follows from Corollary 4.5. □

References

[1] BUTTAZZO G. & PERCIVALE D.: On the approximation of the elastic bounce problem on Riemannian manifolds, *J. Differential Equations* **47**, 227-245 (1983)

[2] CHIERCHIA L. & ZEHNDER E.: Asymptotic expansions of quasiperiodic solutions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **16**, 245-258 (1989)

[3] DOLGOPYAT D.: Bouncing balls in non-linear potentials, *Discrete Contin. Dyn. Syst.* **22**, 165-182 (2008)

[4] JOHNSON R.: Minimal functions with unbounded integral, *Israel J. Math.* **31**, 133-141 (1978)

[5] JOHNSON R.: Nonautonomous differential systems in two dimensions, Chapter 2 of *Handbook of Differential Equations*, Vol. 4, pp. 133-268. Elsevier, Amsterdam 2008

[6] KUNZE M. & ORTEGA R.: Complete orbits for twist maps on the plane, *Ergodic Theory Dynam. Systems* **28**, 1197-1213 (2008)

[7] LAZER A.C.: On Schauder's fixed point theorem and forced second-order nonlinear oscillations, *J. Math. Anal. Appl.* **21**, 421-425 (1968)

[8] LEVI M.: KAM theory for particles in periodic potentials, *Ergodic Theory Dynam. Systems* **10**, 777-785 (1990)

[9] MOSER J.: *Selected Chapters in the Calculus of Variations*, Birkhäuser, Basel-Boston 2003

[10] MOSER J.: Minimal foliations on a torus, in *Topics in calculus of variations, Montecatini Terme 1987*, LNM 1365, Springer, Berlin-New York 1989, pp. 62-99

[11] POLLICOTT M. & YURI M.: *Dynamical Systems and Ergodic Theory*, Cambridge University Press, Cambridge 1998

[12] SELL G.: *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold Mathematical Studies, No. 33. Van Nostrand Reinhold, London 1971

[13] YOU J.: Invariant tori and Lagrange stability of pendulum-type equations, *J. Differential Equations* **85**, 54-65 (1990)