The pendulum equation: from periodic to almost periodic forcings

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Dedicated to Patrick Habets and Jean Mawhin

1 Introduction

Consider the differential equation

$$\ddot{x} + a \sin x = p(t) \tag{1}$$

where $a > 0$ is a parameter and $p : \mathbb{R} \to \mathbb{R}$ is an almost periodic function. Almost periodicity will be understood in the classical sense defined by Bohr [7]. The existence of almost periodic solutions has been already discussed in several papers by Blot [5], Mawhin [19, 20] and by Belley and Saadi Drissi [2]. Also the papers by Fink [10] and by Fournier, Szulkin and Willem [12] contain results applicable to (1). In all these works there is some restriction on the size of the forcing. This size is measured with respect to different norms of p, always with the intention of locating the solution on an interval where the sine function is decreasing, say $]\frac{\pi}{2}, \frac{3\pi}{2}[$. The possible novelty of the present paper is that it searches for results valid for forcings of arbitrary size. There are many other papers on the forced pendulum equation but they deal with the periodic case. See [20] for a recent survey. A nice feature of the periodically forced pendulum is that most of the methods of Nonlinear Analysis can be applied and lead to interesting conclusions. In this sense the

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equation (1) becomes a good illustration for Nonlinear Mathematics. The almost periodic case is attractive just for opposite reasons. It seems that the standard techniques[1] are not applicable and that new phenomena appear. This is my main motivation for the present study but future applications in other fields cannot be discarded. Recently I read an interesting paper by Futakata and Iwasaki on animal locomotion. In [13] a neuronal circuit, the so-called Central Pattern Generator, is coupled with a forced pendulum modelling rhythmic body movements. Letting some parameters to go to zero one arrives at equation (1). The circuit considered in [13] is of Van der Pol type and so the torque $p(t)$ is periodic. The consideration of electric circuits with more degrees of freedom would lead easily to almost periodic torques.

The rest of the paper is organized in four sections. After some preliminaries on almost periodic functions we will review an old result on the periodic pendulum due to Hamel. See [14] and the section 3.3 of [20]. Hamel's theorem guarantees the existence of a periodic solution when p is periodic and has zero mean value. This result is proven using the variational method and it is not known if it can be extended to the almost periodic world. A discussion on the action functional acting on almost periodic functions will suggest the impossibility of extending Hamel's proof. Using different techniques we will prove that Hamel's result has at least a generic extension in the class of limit periodic functions. This is a subclass of the almost periodic functions that seems to be more treatable, the reason being that any function in this class can be approximated by periodic functions. The possibility of finding extensions of Hamel's result to other classes of almost periodic functions cannot be excluded and we refer to the work by Blot [4]. The quasi-periodic case seems particularly intriguing.

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2 Preliminaries on almost periodic functions

There are several equivalent definitions of an almost periodic function. Among them is Bohr's definition based on the quasi-periods and Bochner's characterization in terms of normal families of translates. For us it will be conve-

[1] Variational methods, continuation and degree theory, upper and lower solutions

nient to take a point of view based on Functional Analysis. All the results listed below were already proved in the memoir by Bohr [7].

Let us start with the Banach space

$$BC = \{\text{bounded and continuous functions } p : \mathbb{R} \to \mathbb{R}\},$$

with norm

$$||p||_\infty = \sup_{t \in \mathbb{R}} |p(t)|.$$

For each $T > 0$ we consider the linear subspace

$$Per_T = \{\text{continuous and } T - \text{periodic functions}\}.$$

The class of periodic functions

$$Per = \bigcup_{T>0} Per_T$$

has not a linear structure. We define AP as the algebraic and topological closure of Per in BC. That is,

$$Per \subset AP \subset BC$$

and AP is the smallest Banach space satifying this chain of inclusions. Two examples of functions lying in AP are

$$p_1(t) = \sin t + \sin \sqrt{2} t \quad \text{and} \quad p_2(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\frac{t}{n}).$$

The function p_1 belongs to the linear closure of Per. This closure is understood in an algebraic sense. The function p_2 belongs to the topological closure.

Functions in AP have an average or mean value. Given $p \in AP$ the mean value is defined as

$$M\{p\} = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_a^{a+\tau} p(t)dt,$$

uniformly in $a \in \mathbb{R}$. This limit exists and $M\{\cdot\}$ can be seen as a linear functional in the dual space $(AP)^\star$.

Given a periodic function, the primitive is also periodic whenever the average vanishes. This equivalence is no longer valid in almost periodic functions. Indeed, given $p \in AP$, the primitive

$$P(t) = \int_0^t p(s)ds$$

belongs to AP if and only if P is bounded. This requires in particular that $M\{p\} = 0$ but there are AP-functions with zero mean value and unbounded primitive. This fact seems to be at the origin of most differences between the periodic and the almost periodic worlds.

3 Remarks on Hamel's Theorem

Among other results it is proved in [14] that if $p(t) = \beta \sin t$ the equation (1) has a 2π-periodic solution. The proof extends to any periodic forcing with zero mean value. This result was rediscovered independently by Willem and Dancer. More details can be found in [20].

Theorem 1 (Hamel) *Assume that $p \in Per_T$ and $M\{p\} = 0$. Then the equation (1) has a T-periodic solution.*

There are several ways of proving this result and all of them are based on variational methods. See [14, 20, 24]. The best known proof considers the action functional

$$\mathcal{A}_T[x] = \int_0^T \{\frac{1}{2}\dot{x}(t)^2 + a\cos x(t) + p(t)x(t)\}dt,$$

acting on functions on the Sobolev space of T-periodic functions $H^1 = H^1(\mathbb{R}/T\mathbb{Z})$, and shows that it reaches a minimum. The minimizer is a periodic solution. This program is achieved following the direct method in the Calculus of Variations. First one proves that $\mathcal{A}_T$ is bounded below. This is easy after the integration by parts

$$\int_0^T p(t)x(t)dt = -\int_0^T P(t)\dot{x}(t)dt,$$

where we have used that the primitive $P(t)$ is also periodic. Now the action can be expressed as $\mathcal{A}_T[x] = I_1 + I_2$ with

$$I_1 = \int_0^T \{\frac{1}{2}\dot{x}(t)^2 - P(t)\dot{x}(t)\}dt \text{ and } I_2 = \int_0^T a\cos x(t)dt.$$

The quadratic function $y \mapsto \frac{1}{2}y^2 - P(t)y$ has the minimum value $-\frac{1}{2}P(t)^2$ and so $I_1 \geq -\frac{1}{2}||P||_\infty^2 T$. The second integral satisfies $I_2 \geq -aT$.

Once we know that $\mathcal{A}_T$ is bounded below it is possible to extract a minimizing sequence that is bounded in H^1. This uses that the functional is coercive on the hyperplane $M\{x\} = 0$ and the periodicity of the potential.

From this minimizing sequence one extracts a subsequence converging to the minimizer. The convergence is uniform in $\mathbb{R}$ and weak in H^1. This is possible since the inclusion $H^1 \subset C(\mathbb{R}/T\mathbb{Z})$ is compact and H^1 is a Hilbert space. Having finished the sketch of the proof it may be interesting to notice that this method always leads to a periodic solution that is unstable in the Lyapunov sense. This is a consequence of the results in [22] or [25].

We can now formulate the main question of the paper:

Assume that $p \in AP$ and $M\{p\} = 0$, is there a solution of (1) in AP?

The same question can be posed for the more restrictive class of forcings in AP with bounded primitive. As far as I know these questions are open but my impression is that they will have a negative answer. There are some examples in first order equations that seem to support this point of view. It is possible to construct a negative function p in AP such that the equation

$$\dot{x} = p(t) + \cos 2x$$

has a pathological behavior in the strip $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$. This means that there are no AP solutions in this region and at the same time there exist two bounded solutions

$$-\frac{\pi}{4} \leq x_1(t) < x_2(t) \leq \frac{\pi}{4}, \quad t \in \mathbb{R}$$

satisfying

$$\liminf_{t\to\pm\infty}(x_2(t) - x_1(t)) = 0 \quad \text{and} \quad \limsup_{t\to\pm\infty}(x_2(t) - x_1(t)) > 0.$$

This phenomenon of non-separated bounded solutions cannot occur in first order periodic equations. For more information and dynamical insight the reader is referred to the paper by Johnson [16]. This is perhaps a good starting point to produce an example refuting an almost periodic version of Hamel's theorem. Going back to our second order equation (1) it is interesting to observe that the variational proof previously discussed cannot be extended to the almost periodic case. The next section will be devoted to explain this.

4 The variational approach in AP

The action functional was defined on the Sobolev space $H^1 = H^1_T$ and the critical points will not change if we multiply it by a positive number. More precisely we define

$$\mathcal{A}[x] = \frac{1}{T}\int_0^T \{\frac{1}{2}\dot{x}(t)^2 + a\cos x(t) + p(t)x(t)\}dt, \quad x \in H^1_T.$$

This averaged functional has the advantage of unifying the value of the action for different periods. Indeed if $x \in H^1_T$ then the value of the modified action with respect to the periods $T, 2T, 3T, \dots$ is the same. Also we can interpret this action as the mean value of the Lagrangian and this point of view leads to a definition in the almost periodic setting.

Let AP^1 be the space of functions $x : \mathbb{R} \to \mathbb{R}$ of class C^1 such that x and $\dot{x}$ belong to AP. Given $p \in AP$ we define

$$\mathcal{A}[x] = M\{\frac{1}{2}\dot{x}^2 + a\cos x + px\}, \quad x \in AP^1.$$

The basic properties of AP functions imply that this quantity is well defined and extend the action from $Per^1 = Per \cap C^1(\mathbb{R})$ to AP^1. Incidentally we notice that $AP^1 \subsetneqq AP \cap C^1(\mathbb{R})$.

A similar functional but with cubic potential was introduced by Moser in [21] with the purpose of assigning a Hamiltonian structure to the KdV equation. Blot considered the functional associated to a general Lagrangian and studied in [3] the basic aspects of the so-called Calculus of Variations in mean.

The class AP^1 becomes a Banach space with the norm $||x||_\infty + ||\dot{x}||_\infty$ but it will be convenient to introduce new topologies in AP and AP^1. First we define the inner product

$$\langle p, q\rangle = M\{pq\} = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^T p(t)q(t)dt, \quad \text{if } p, q \in AP.$$

In this way AP becomes a pre-Hilbert space that will be denoted by $\mathbb{H}$. Analogously $\mathbb{V}$ will be the space AP^1 with the inner product

$$\langle p, q\rangle_{\mathbb{V}} = \langle p, q\rangle + \langle \dot{p}, \dot{q}\rangle.$$

The spaces $\mathbb{H}$ and $\mathbb{V}$ could be completed but the resulting spaces would be somehow unusual. Quoting the book on Functional Analysis by Riesz and Sz.-Nagy [23]: *we could complete* $\mathbb{H}$ *by adding certain ideal elements, but since these elements do not have an obvious representation as functions we prefer to use the incomplete space.* Later we will find additional reasons to remain in $\mathbb{H}$.

The rest of the section is devoted to the study of the functional

$$\mathcal{A} : \mathbb{V} \to \mathbb{R}, \quad \mathcal{A}[x] = M\{\frac{1}{2}\dot{x}^2 - U \circ x + px\}$$

where $U : \mathbb{R} \to \mathbb{R}$ is a C^1 function having a Lipschitz-continuous derivative. This includes the case $U(x) = -\cos x$.

Lemma 2 *The functional $\mathcal{A}$ is of class C^1 (in the Fréchet sense) with derivative*

$$\mathcal{A}'[x]\delta = \langle \dot{x}, \dot{\delta} \rangle - \langle U' \circ x, \delta \rangle + \langle p, \delta \rangle, \text{ if } \delta \in \mathbb{V}.$$

Proof. We consider the simpler functional

$$\mathcal{B} : \mathbb{V} \to \mathbb{R}, \quad \mathcal{B}[x] = M\{U \circ x\}.$$

The remaining terms appearing in the definition of $\mathcal{A}$ are discussed easily. First we prove the Gateaux differentiability of $\mathcal{B}$. For each $\delta \in \mathbb{V}$,

$$|\frac{1}{\epsilon}(\mathcal{B}[x + \epsilon\delta] - \mathcal{B}[x]) - \langle U' \circ x, \delta \rangle| \le |M\{\frac{U \circ (x + \epsilon\delta) - U \circ x}{\epsilon} - (U' \circ x)\delta\}|$$

$$\le ||\frac{U \circ (x + \epsilon\delta) - U \circ x}{\epsilon} - (U' \circ x)\delta||_\infty \le \epsilon[U']_{Lip}||\delta||_\infty^2,$$

where we have used the mean value theorem to arrive at the last inequality. The best Lipschitz constant of U' has been denoted by $[U']_{Lip}$.

The differential

$$\mathcal{B}'[x]\delta = \langle U' \circ x, \delta \rangle, \quad \delta \in \mathbb{V}$$

defines a map $x \in \mathbb{V} \mapsto \mathcal{B}'[x] \in \mathbb{V}^\star$. We will prove that this map is Lipschitz-continuous and this will complete the proof of the lemma. With this purpose we first observe that, given $x, y \in \mathbb{V}$,

$$||U' \circ x - U' \circ y||_{\mathbb{H}}^2 = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |U'(x(t)) - U'(y(t))|^2 dt \le$$

$$[U']_{Lip}^2 ||x - y||_{\mathbb{H}}^2 \le [U']_{Lip}^2 ||x - y||_{\mathbb{V}}^2.$$

Given $\delta \in \mathbb{V}$, $||\delta||_{\mathbb{V}} \le 1$, we apply Cauchy-Schwarz inequality and the estimate above to obtain,

$$|(\mathcal{B}'[x] - \mathcal{B}'[y])\delta| = |\langle U' \circ x - U' \circ y, \delta \rangle| \le ||U' \circ x - U' \circ y||_{\mathbb{H}} ||\delta||_{\mathbb{H}} \le$$

$$[U']_{Lip} ||x - y||_{\mathbb{V}}.$$

This proves that $[U']_{Lip}$ is also a Lipschitz constant for $\mathcal{B}'$.

Next we present a result that shows the consistency of the functional $\mathcal{A}$ with the almost periodic problem.

Proposition 3 *Assume that $x \in \mathbb{V}$ is a critical point of $\mathcal{A}$. Then $x(t)$ is of class C^2 and solves*

$$\ddot{x} + V'(x) = p(t). \tag{2}$$

Conversely, if $x(t)$ is a solution of (2) in AP then it also belongs to $\mathbb{V}$ and $\mathcal{A}'[x] = 0$.

This result is essentially a particular case of Theorem 1 in [3] but we will present a different proof based on the following result, that is inspired by the theory of distributions.

Lemma 4 *Assume that $u, v \in \mathbb{H}$ and*

$$\langle u, \dot{\phi} \rangle = -\langle v, \phi \rangle \quad for\ each\ \phi \in \mathbb{V}.$$

Then $u \in \mathbb{V}$ and $\dot{u} = v$.

We postpone the proof to the end of the Section.

Proof of Proposition 3. Assume first that $x \in \mathbb{V}$ and $\mathcal{A}'[x] = 0$. We apply the previous Lemma with $u = \dot{x}$ and $v = -U' \circ x + p$ and deduce that x is C^2 and satisfies (2). Conversely, if x is a solution of (2) in AP we will prove that also $\dot{x}$ and $\ddot{x}$ belong to AP. For the second derivative this is obvious from the equation. To prove that $\dot{x}$ is in AP it is sufficient to observe that it is the derivative of an almost periodic function and it is uniformly continuous ($||\ddot{x}||_\infty < \infty$). Now we know that $x \in \mathbb{V}$ and the conclusion $\mathcal{A}'[x] = 0$ follows from the identity $\langle \dot{x}, \dot{\delta} \rangle = -\langle \ddot{x}, \delta \rangle$.

The identity taken from AP^1 onto $\mathbb{V}$ is a bounded linear operator and so the previous results also hold if one takes AP^1 as the domain of $\mathcal{A}$. Thus, we can choose for the domain of the functional a pre-Hilbert space or a non-reflexive Banach space. These are not optimal settings from the point of view of the direct method of the Calculus of Variations and it seems natural to consider the completion of $\mathbb{V}$. This is the approach taken by Blot in [6]. The price to pay is that the consistency with the almost periodic problem is lost. We illustrate this phenomenon with a concrete example.

Example. Let us start with the function

$$b(t) = \sum_{n=1}^{\infty} \frac{1}{n} \sin^2(\frac{t}{n}).$$

This is a series of positive terms dominated by $\sum t^2 n^{-3}$. We also observe that the series obtained by successive differentiation are uniformly convergent in the whole line. From here it is easy to conclude that b is in $C^\infty(\mathbb{R})$

and the derivatives $\dot{b}, \ddot{b}, \ldots$ belong to AP. The function b is not almost periodic since it is unbounded. Indeed,

$$b(t_N) \geq \sum_{k=0}^{N} \frac{1}{2k+1} \to \infty \quad \text{if } t_N = \frac{\pi}{2} \cdot 1 \cdot 3 \cdot 5 \cdots (2N+1).$$

Consider now the action functional with $U \equiv 0$ and $p = \ddot{b}$. It can be expressed as

$$\mathcal{A}[x] = \frac{1}{2}||\dot{x}||^2_{\mathbb{H}} - \langle \dot{b}, \dot{x} \rangle$$

and so $\mathcal{A}$ is bounded below. More precisely, $\inf_{\mathbb{V}} \mathcal{A} = -\frac{1}{2}||\dot{b}||^2_{\mathbb{H}}$. A straightforward computation shows that $\mathcal{A}[x_n] \to \inf_{\mathbb{V}} \mathcal{A}$, with

$$x_N(t) = -\sum_{n=1}^{N} \frac{1}{2n} \cos(\frac{2t}{n}).$$

Moreover $\{x_N\}$ is a Cauchy sequence in $\mathbb{V}$. The limit in the completion of $\mathbb{V}$ should be a minimizer but the Euler-Lagrange equation associated to $\mathcal{A}$ has no almost periodic solutions. By construction this equation is $\ddot{x} = p(t)$ and the solutions are $x(t) = c_1 + c_2 t + b(t)$. This example is in contrast with the periodic situation where C^1-minimizers coincide with H^1-minimizers.

Next we show a second difference between the periodic and almost periodic situation. To this end we introduce the Banach space

$$\widetilde{AP} = \{p \in AP : \; M\{p\} = 0\}$$

and employ the notation $\mathcal{A}_p$ to emphasize the dependence of the action with respect to the forcing. We will prove that $\inf_{\mathbb{V}} \mathcal{A}_p = -\infty$ for a typical $p(t)$ in $\widetilde{AP}$. This is inspired by the results of Johnson in [15].

Proposition 5 *Assume that the potential U is bounded. Then the set*

$$\mathbf{B} = \{p \in \widetilde{AP} : \; \mathcal{A}_p \text{ is bounded below}\}$$

is of first category in $\widetilde{AP}$.

Proof. Consider the class of functions $p \in \widetilde{AP}$ satisfying

$$\sum |\frac{p_n}{\Lambda_n}|^2 < \infty,$$

where $p(t) \sim \sum p_n e^{i\Lambda_n t}$ is the Fourier expansion of p. Since $p(t)$ is real valued we are assuming that $\Lambda_{-n} = -\Lambda_n$, $p_{-n} = \overline{p}_n$ and $p_0 = 0$. This class is a proper linear subspace of $\widetilde{AP}$. Indeed it is of first category because the complement can be expressed as the countable intersection of the sets

$$\mathcal{G}_N = \{p \in \widetilde{AP} : \sum |\frac{p_n}{\Lambda_n}|^2 > N\},$$

that are open and dense. To justify that $\mathcal{G}_N$ is open it is sufficient to observe that the Fourier coefficients are continuous with respect to the uniform norm. To prove the density we select a function $q \in \widetilde{AP}$ with $\sum |\frac{q_n}{\Lambda_n}|^2 = \infty$, say $q(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\frac{t}{n^4})$. Given any $p \in \widetilde{AP} \setminus \mathcal{G}_N$ we observe that $p + \epsilon q \in \mathcal{G}_N$ if $\epsilon \neq 0$. Now it is sufficient to prove that **B** is contained in the class described above. A computation shows that

$$\mathcal{A}_p[x_N] \leq -\frac{1}{2} \sum_{0<|n|\leq N} |\frac{p_n}{\Lambda_n}|^2 + ||U||_\infty$$

with

$$x_N(t) = - \sum_{0<|n|\leq N} \frac{p_n}{\Lambda_n^2} e^{i\Lambda_n t}.$$

Proof of Lemma 4. Let us recall a construction employed by Bohr to prove his Fundamental Theorem in [7]. Given a countable set of real numbers, say $\{\Lambda_n : \ n \in \mathbb{Z}\}$ with $\Lambda_0 = 0$, one associates to it an appropriate double sequence of numbers $\{k_n^{(q)}\}_{n\in\mathbb{Z}, q\in\mathbb{N}}$ satisfying

$$0 \leq k_n^{(q)} \leq 1, \quad \lim_{q\to\infty} k_n^{(q)} = 1 \ \text{ for fixed } n.$$

$$\text{Given } q, \quad k_n^{(q)} = 0 \ \text{ for } |n| \text{ large enough.}$$

This sequence enjoys a universal property: for any $f \in AP$ with Fourier series

$$f(t) \sim \sum f_n e^{i\Lambda_n t},$$

it holds that

$$\lim_{q\to\infty} ||s_q(f) - f||_\infty = 0.$$

Here $s_q(f)$ stands for the trigonometric polynomial

$$s_q(f)(t) = \sum_n k_n^{(q)} f_n e^{i\Lambda_n t}.$$

Now we are ready to prove the Lemma. We choose for $\{\Lambda_n\}_{n\in\mathbb{Z}}$ a set containing all the exponents of the function $v(t)$. Notice that 0 is not an exponent of v, as can be checked by testing the function $\phi(t) \equiv 1$. With the help of the test functions $\phi(t) = e^{i\lambda t}$ we observe that the exponents of $u(t)$ are contained in $\{\Lambda_n\}_{n\neq 0} \cup \{0\}$ and

$$v(t) \sim \sum v_n e^{i\Lambda_n t}, \qquad u(t) \sim u_0 + \sum \frac{v_n}{i\Lambda_n} e^{i\Lambda_n t}.$$

Next we observe that $s_q(u)$ is a primitive of $s_q(v)$. At the same time, $s_q(u) \to u$ and $s_q(v) \to v$, uniformly in $\mathbb{R}$. The classical result on the passage to the limit under differentiation implies that u is C^1 and $\dot{u} = v$. In particular u belongs to $\mathbb{V}$.

5 Limit periodic forcings

Throughout this Section T is a positive number and $LPer_T$ is the class of functions obtained as uniform limits of periodic functions with periods of the type $T, 2T, \dots, NT, \dots$ From the point of view of Functional Analysis $LPer_T$ can be described as the closure in AP of a linear subspace, namely

$$LPer_T = \overline{\bigcup_{N\geq 1} Per_{NT}}.$$

This construction shows that $LPer_T$ is a Banach space with respect to the L^∞-norm. Several limit periodic functions have already appeared in the paper, in particular in the Example of last Section. The function $\dot{b}(t)$ belongs to $LPer_{2\pi}$. This is a function with zero mean value and unbounded primitive. More properties of limit periodic functions can be found in [1].

We will work in the Banach space

$$X = \{p \in LPer_T : \ M\{p\} = 0\}.$$

This is a closed hyperplane in $LPer_T$.

Theorem 6 *Given $a > 0$ there exists an open and dense set $\mathcal{G} \subset X$ such that the pendulum equation*

$$\ddot{x} + a \sin x = p(t)$$

has a solution in $LPer_T$ for each $p \in \mathcal{G}$.

Blot obtained a result that can be considered as the first Theorem of Hamel's type for almost periodic forcings. In [4] he proved that, given any $p \in AP$ with $M\{p\} = 0$, there exists $q \in AP$ with $||q - p||_B$ arbitrarily small and such that $\ddot{x} + a \sin x = q(t)$ has an almost periodic solution. Compared to the previous Theorem, Blot's result has the advantage of applying to general functions in AP and not only to those in $LPer_T$. However the norm employed in this density result is rather weak, namely

$$||p||_B = \sup\{\langle p, \phi \rangle : \phi \in \mathbb{V}, ||\phi||_{\mathbb{V}} \leq 1\}.$$

A nice feature of Theorem 6 is that the nonlinearity plays a role. Actually the conclusion is false for the linear equation obtained when $a = 0$. For that case it is enough to observe that the class

$$\{p \in X : \ddot{x} = p(t) \text{ has a solution in } LPer_T\}$$

is a proper linear subspace of first category in X. This can be justified using Theorem 4.4 in [15] or with Fourier analysis, as in the proof of Proposition 5. Notice that now all frequencies are commensurable with $\frac{2\pi}{T}$; that is, $\frac{2\pi}{T}/\Lambda_n$ is a rational number.

Proof of Theorem 6. First we recall some terminology on periodic equations. Assuming that $x(t)$ is a τ-periodic solution of

$$\ddot{x} + a \sin x = q(t), \quad q \in Per_\tau,$$

the linearized equation is

$$\ddot{y} + a \cos x(t) y = 0 \tag{3}$$

and the Floquet multipliers associated to this equation are μ_1 and μ_2. The solution $x(t)$ is called *degenerate* with respect to period τ if (3) has τ-periodic solutions different from $y \equiv 0$. This is equivalent to $\mu_1 = \mu_2 = 1$. When the Floquet multipliers are not in $\mathbb{S}^1$, $|\mu_i| \neq 1$, $i = 1, 2$, we say that $x(t)$ is *hyperbolic*.

The following sequence of Banach spaces will play an important role. For each integer $n \geq 1$ define

$$X_n = \{p \in Per_{nT} : M\{p\} = 0\},$$

so that

$$X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X.$$

Notice that $\bigcup X_n$ is dense in X.
The proof will be structured in three steps.

Step 1. *For each $n \geq 1$ there exists a set $\mathcal{G}_n$ open and dense in X_n such that if $q \in \mathcal{G}_n$ then the equation $\ddot{x} + a \sin x = q(t)$ has a hyperbolic periodic solution of period nT.*
The proof follows an argument already employed by Bosetto, Serra and Terracini in [8]. If $q \in X_n$ the periodic action functional

$$\mathcal{A}_{nT}[x] = \int_0^{nT} \{\frac{1}{2}\dot{x}(t)^2 + a\cos x(t) + q(t)x(t)\}dt, \quad x \in H^1_{nT}$$

has a minimizer, say x_n. By a well known argument in Calculus of Variations x_n is either degenerate or hyperbolic as a periodic solution (see Chapter 17 in [9] and Chapter 4 in [17]). The results in [18] say that degenerate solutions are unusual. More precisely, it follows from Theorem 3 of that paper that there exists a set $\mathcal{G}_n$ open and dense in X_n such that if $p \in \mathcal{G}_n$ then the minimizer x_n is non-degenerate and therefore hyperbolic.
Step 2. *Given $n \geq 1$ and $q \in \mathcal{G}_n$ there exists $\mathcal{U}_q$ open neighborhood of p in X such that $\ddot{x} + a \sin x = p(t)$ has a solution in $LPer_T$ if $p \in \mathcal{U}_q$.*
In the theory of almost periodic equations it is well known that an almost periodic solution is persistent if the linearized equation has an exponential dichotomy (see [11]). Moreover the module of frequencies of the perturbed solution is contained in the module of the equation. From step 1 we know that if $q \in \mathcal{G}_n$ then there is a hyperbolic periodic solution $x_n(t)$. In the periodic context this is equivalent to saying that the linearized equation has an exponential dichotomy. When $p \in LPer_T$ is such that $||p - q||_\infty$ is small, the equation $\ddot{x} + a \sin x = p(t)$ can be interpreted as a perturbation of $\ddot{x} + a \sin x = q(t)$. Therefore there exists an AP solution $x(t)$ of the perturbed equation that is close to $x_n(t)$ and such that its module of frequencies is contained in $\mathrm{mod}(p) \subset \frac{2\pi}{T}\mathbb{Q}$. This property of the module characterizes functions in $LPer_T$ (see for instance Proposition 2.7 in [1]).
Step 3. $\mathcal{G} = \bigcup_{n\geq 1} \bigcup_{q \in \mathcal{G}_n} \mathcal{U}_q$.
By construction there exists a solution in $LPer_T$ for each $p \in \mathcal{G}$. Moreover $\mathcal{G}$ is open since it is defined as a union of open sets. To prove the density of $\mathcal{G}$ in X we observe that each $\mathcal{G}_n$ is dense in X_n and $\bigcup X_n$ is dense in X.

The variational framework developed in Section 4 can be adapted to the limit periodic setting. We consider the space

$$LPer^1_T = \{x \in C^1(\mathbb{R}) : \ x, \dot{x} \in LPer_T\} = AP^1 \cap LPer_T$$

and, for each $p \in X$, the functional

$$\mathbf{A}_p : LPer^1_T \to \mathbb{R}, \quad \mathbf{A}_p[x] = M\{\frac{1}{2}\dot{x}^2 + a \cos x + px\}.$$

The proof of Proposition 3 can be adapted to conclude that the critical points of $\mathbf{A}_p$ are the solutions of (1) lying in $LPer_T$. Also Proposition 5 has a limit periodic version saying that

$$\mathbf{B}_L = \{p \in X : \ \mathbf{A}_p \text{ is bounded below}\}$$

is of first category. It is interesting to notice that $\mathbf{B}_L$ contains $\bigcup X_n$ and so is dense in X. To justify this take p in some periodic space X_n. There exists a periodic minimizer of $\mathbf{A}_p$ among functions of period nT, say $x_\star(t)$. A classical result in Calculus of Variations (again see [9]) implies that $x_\star$ is also a minimizer among the periodic functions with period a multiple of nT; that is, in the Sobolev spaces $H^1_{\nu nT}$, $\nu = 2, 3, \ldots$ This implies that

$$\mathbf{A}_p[x] \geq \mathbf{A}_p[x_\star] \quad \text{for each } x \in \bigcup_{\nu=1}^{\infty} Per^1_{\nu T}.$$

It is easy to prove that $\bigcup_{\nu=1}^{\infty} Per^1_{\nu T}$ is dense in $LPer^1_T$ and so the continuity of $\mathbf{A}_p$ implies that $x_\star$ is also a minimizer in $LPer^1_T$. Finally we can apply the previous Theorem to deduce that $\mathcal{G} \cap (X \setminus \mathbf{B}_L)$ is residual in X. This means that for a generic forcing p of X the functional $\mathbf{A}_p$ has critical points and $\inf \mathbf{A}_p = -\infty$. At the same time there is a dense set of forcings ($p \in \mathbf{B}_L$) for which $\mathbf{A}_p$ reaches its minimum.

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