

# A ONE SIDE SUPERLINEAR AMBROSETTI-PRODI PROBLEM FOR THE DIRICHLET $p$ -LAPLACIAN

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ABSTRACT. We study the solvability of the quasilinear elliptic problem of parameter  $s$

$$-\Delta_p u \stackrel{\text{def}}{=} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = g(x, u) + s\varphi(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$ ,  $\varphi \geq 0$ ,  $g(\cdot; u)/|u|^{p-2}u$  lies for  $u < 0$  below, say, the first eigenvalue of the  $p$ -laplacian  $-\Delta_p$  and  $g$  grows for  $u > 0$  less than the lower Sobolev critical exponent  $p_*$ . We combine topological methods via upper and lower solutions and blow-up techniques to get a-priori bounds to prove a result of Ambrosetti-Prodi type : there exists  $s_* \leq s^*$  such that the problem possesses 2 solutions if  $s > s^*$ , it has at least one solution if  $s < s^*$ , and at least two solutions if  $s < s_*$ . We prove also that  $s_* = s^*$  in some cases.

## 1. INTRODUCTION

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$ . We are interested in the solvability of the following quasilinear boundary value problem

$$(P_s) \quad \begin{cases} -\Delta_p u = g(x, u) + s\varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Delta_p u$  for  $1 < p < +\infty$  is the usual  $p$ -Laplace operator,  $\varphi \in L^\infty(\Omega)$  with  $\varphi \succ 0$ , i.e.,  $\varphi$  is strictly positive on any compact set of  $\Omega$ ,  $s$  is a real parameter and the function  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function which grows in  $u$  below the *critical* Sobolev exponent  $p^* := \frac{Np}{N-p}$  if  $p < N$ ;  $p^* := +\infty$  if  $p \geq N$ , i.e.,

$$(G) \quad \exists C, D \in \mathbb{R}_+, 1 \leq q < p^* \text{ s.t. } |g(x, u)| \leq C|u|^{q-1} + D, \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Problem  $(P_s)$  is usually said to be of the *Ambrosetti-Prodi type* if the nonlinearity term  $g$  crosses the first eigenvalue  $\lambda_1$  of the Dirichlet  $p$ -Laplacian in  $\Omega$ , that is, when

$$\limsup_{t \rightarrow -\infty} \frac{g(x, t)}{|t|^{p-2}t} < \lambda_1 < \liminf_{t \rightarrow +\infty} \frac{g(x, t)}{|t|^{p-2}t}, \quad (1.1)$$

or the appropriate reversed inequalities. The expected classical Ambrosetti-Prodi result under these hypothesis will assure the existence of  $s^* \in \mathbb{R}$  such that “ $(P_s)$  has no solutions when  $s > s^*$ , at least one solution if  $s = s^*$  and at least two solution when  $s < s^*$ .”

Ambrosetti-Prodi type problems have been extensively treated in the semilinear case  $p = 2$ . In the case  $p \neq 2$  it has been recently studied by [2] and [7] assuming among other hypothesis that both of the limits in (1.1) are finite. Many of the proof of these results are based on the use of topological degree theory that combine upper and lower solutions techniques and a priori bounds. We will show in this paper that, in order to obtain a pair of upper and lower solutions of problem  $(P_s)$  for a large range of negative  $s$ , it is enough for instance to assume the following growth condition of  $g$  at  $-\infty$ :

$$\begin{aligned} (H1) \quad & \exists b \in L^\infty(\Omega), \lambda_1(b) > 1, \text{ and } C_1 > 0 \text{ s.t. } g(x, u) \geq b(x)|u|^{p-2}u - C_1, \\ (H2) \quad & \exists B \in L^\infty(\Omega), C_2 > 0 \text{ s.t. } g(x, u) \leq B(x)|u|^{p-2}u + C_2, \end{aligned}$$

for all  $u \leq 0$ , a.e.  $x \in \Omega$ . Here  $\lambda_1(m)$  denotes the principal eigenvalue of the Dirichlet problem for the  $p$ -Laplace operator with respect to the weight  $m$ . See next section for a precise definition of  $\lambda_1(m)$ . Let us observe that (H1) implies

$$\limsup_{t \rightarrow -\infty} \frac{g(x, t)}{|t|^{p-2}t} \leq b(x), \text{ a.e. } x \in \Omega,$$

and that condition  $\lambda_1(b) > 1$  is equivalent when  $b(x)$  is a constant, say  $b$ , to  $b < \lambda_1$ . Moreover,  $\lambda_1(b) > 1$  when  $b(x) < \lambda_1$  a.e.  $x \in \Omega$ , but one can easily find examples where condition  $\lambda_1(b) > 1$  is satisfied without having  $b(x) < \lambda_1$  a.e.  $\Omega$ .

In order to apply degree theory and to prove nonexistence results is essential to obtain a priori bounds for the solutions of  $(P_s)$  with  $s$  varying in an unbounded interval. To find a priori bounds when  $g$  has “superlinear” growth at  $+\infty$  is always a difficult task. Superlinear cases have been considered by [13] and [3] among others. In [13] the author proves the existence of positive solutions in the case when  $\varphi \equiv 1$  and the nonlinearity  $g(x, u)$  (which may depend also on  $\nabla u$  and satisfies some structure conditions) grows on  $u$  less than  $p_*$ , where  $p_* := \frac{N(p-1)}{N-p}$  if  $p < N$ ;  $p_* := +\infty$  if  $p \geq N$ , is the lower critical Sobolev exponent. However the structure hypothesis on  $g$  assumed by [13] do not imply that  $g$  crosses the first eigenvalue of the  $p$ -laplacian as it is assumed here. The limit growth  $p_*$  on  $u$  appears when one applies blow-up methods to have the desired a priori bounds of the solutions. Indeed it assures that the limiting Liouville type of problem either in  $\mathbb{R}^N$  or  $\mathbb{R}_+^N$  has no positive solutions. We will also use blow-up methods in this work to obtain a priori bounds, so we will suppose the following growth at  $+\infty$ :

$$\begin{aligned} (H3) \quad & \exists a > 0, C_3 > 0 \text{ and } p < q < p_* \text{ s.t. } g(x, u) \geq a|u|^{q-2}u - C_3, \\ (H4) \quad & \exists A > 0, C_4 > 0 \text{ and } p < q < p_* \text{ s.t. } g(x, u) \leq A|u|^{q-2}u + C_4, \end{aligned}$$

for all  $u \geq 0$ , a.e.  $x \in \Omega$ . Notice that as a consequence of (H3),

$$\liminf_{t \rightarrow +\infty} \frac{g(x, t)}{|t|^{p-2}t} = +\infty, \text{ a.e. } x \in \Omega,$$

so (1.1) holds and we are dealing with an Ambrosetti-Prodi type problem. Of course (H1), (H2), (H3) and (H4) imply (G).

Our main result is the following:

**Theorem 1.1.** *Assume (H1), (H2), (H3) and (H4) and let  $\varphi \in L^\infty(\Omega)$  with  $\varphi \succ 0$ . Then there exist  $s_* \leq s^* \in \mathbb{R}$  such that*

- (1)  $(P_s)$  has no solution if  $s > s^*$ ,
- (2)  $(P_s)$  has at least one solution if  $s < s^*$ ,
- (3)  $(P_s)$  has at least two solutions if  $s < s_*$ .

Moreover if we suppose that  $g$  is continuous, we prove in Theorem 5.1 that  $(P_{s^*})$  has at least one solution and that  $s_* = s^*$  when moreover  $\varphi$  is strictly positive on  $\bar{\Omega}$ .

This paper is organized as follows. In the next section we recall some results on the eigenvalue problems with weights and some particular nonhomogeneous problems related to them. We also recall there some regularity results for the solutions of quasilinear elliptic problems. In section 3 we prove the existence of lower and upper solutions for  $(P_s)$  and in section 4 we give a result on a-priori bounds. We complete the proofs of Theorem 1.1 and Theorem 5.1 using degree arguments in section 5.

## 2. PRELIMINARY RESULTS

Let us first recall some results concerning the eigenvalue problem

$$-\Delta_p u = \lambda m |u|^{p-2} u \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \quad (2.1)$$

From now on, given a measurable function  $a$  in  $\Omega$ , we will denote  $a^+ := \max\{a, 0\}$ ,  $a^- := a^+ - a$ . The following result can be found in [12] and [4] :

**Proposition 2.1.** *For any  $m \in L^\infty(\Omega)$  with  $m^+ \neq 0$  problem (2.1) possesses an unique principal eigenvalue  $\lambda_1(m)$  characterized by*

$$\lambda_1(m) \stackrel{\text{def}}{=} \inf \left\{ \|u\|^p \stackrel{\text{def}}{=} \int_\Omega |\nabla u|^p dx; u \in W_0^{1,p}(\Omega), \int_\Omega m |u|^p dx = 1 \right\}. \quad (2.2)$$

Moreover  $0 < \lambda_1(m)$  is simple, isolated in the spectrum. The map  $\lambda_1 : L^\infty(\Omega) \rightarrow \mathbb{R}^+$  is continuous and strictly decreasing in the sense that if  $m_1 \leq m_2$  and  $|\{m_1 \neq m_2\}| > 0$  then  $\lambda_1(m_1) > \lambda_1(m_2)$ .

By principal eigenvalue we mean that it possesses a positive eigenfunction. In the sequel we will denote  $\varphi_m$  the positive eigenfunction associated to  $\lambda_1(m)$  with  $\int_\Omega m(x) \varphi_m(x)^p dx = 1$ .

**Remark 2.1.** *If we assume (G) the regularity results of [8], [16] and [9] imply that the solutions of  $(P_s)$  belong to  $L^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$  for some  $\alpha \in ]0, 1[$ . Moreover, a careful reading of the proof of Theorem 7.1 Chapter IV of [8] shows that*

$$\|u^\pm\|_\infty \leq C(p, q, N, \|a\|_\infty, \|b\|_\infty, \|A\|_\infty, \|B\|_\infty, \|u^\pm\|_{p^*}).$$

The following result is a combination of Lemma 3.3 of [3] and Lemma 2.3. of [2].

**Proposition 2.2.** *For all  $f \in L^\infty(\Omega)$  there exists an unique solution  $u \in W_0^{1,p}(\Omega)$  of*

$$-\Delta_p u = f \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

The solution  $u$  belongs to  $C_0^{1,\alpha}(\overline{\Omega})$  for some  $\alpha = \alpha(p, N) \in ]0, 1[$  and there exists  $c > 0$  such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq c(\|f\|_\infty^{1/(p-1)} + 1). \quad (2.3)$$

Moreover the map  $\mathcal{K} : L^\infty(\Omega) \mapsto C_0^{1,\beta}(\overline{\Omega})$  defined as  $\mathcal{K}(f) = u$  is continuous and compact for any  $0 < \beta < \alpha$ .

We will also need the following result. When  $m > 0$  it can be found in [6].  $\nu$  stands here for the outer normal derivative.

**Proposition 2.3.** *Let  $m \in L^\infty(\Omega)$  be such that  $\lambda_1(m) > 1$ . Then for any  $f \in L^\infty(\Omega)$ ,  $f \geq 0$  there exists a unique solution  $u$  of*

$$-\Delta_p u = m |u|^{p-2} u + f \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \quad (2.4)$$

Moreover  $u \in C_0^{1,\alpha}(\overline{\Omega})$  for some  $\alpha \in ]0, 1[$  and, if  $f \not\equiv 0$ , then  $u > 0$  in  $\Omega$  and  $\frac{\partial u}{\partial \nu} < 0$  on  $\partial\Omega$ .

*Proof.* The existence of a positive solution of (2.4) follows by minimization of the functional  $J(u) := \frac{1}{p} \int_\Omega (|\nabla u|^p - m |u|^p) dx - \int_\Omega f u dx$  over  $W_0^{1,p}(\Omega)$ . Moreover any solution of (2.4) is nonnegative because, after multiplying by  $u^-$  and integrating the equation, we have

$$\int_\Omega |\nabla u^-|^p dx = \int_\Omega m (u^-)^p dx - \int_\Omega f u^- \leq \int_\Omega m (u^-)^p dx.$$

Since  $\lambda_1(m) > 1$  we get that  $u^- \equiv 0$ . By the Strong Maximum Principle of [17] we conclude that  $0 < u$  in  $\Omega$  and  $\frac{\partial u}{\partial \nu} < 0$  in  $\partial\Omega$ . Let us now prove that the solution is unique. To do so

let us prove that there is a minimal solution of problem (2.4). We define the following map  $T : L_+^\infty(\Omega) \mapsto C_0^{1,\beta}(\bar{\Omega})_+$  defined as  $T(z)$  = unique solution  $v$  of the problem

$$-\Delta_p v + m^- |v|^{p-1} v = z \quad \text{in } \Omega; \quad v = 0 \text{ on } \partial\Omega. \quad (2.5)$$

Notice that the positivity of  $T(z)$  and the uniqueness follows from the fact that problem (2.5) satisfies the weak comparison principle. Define then the sequence  $v_0 = T(f)$ ,  $v_n = T(m^+ v_{n-1}^{p-1} + f)$ . Then we have  $0 \leq v_n \leq v_{n+1}$  and, for any solution  $u$  of (2.4), it holds  $v_n \leq u$ . Thus the sequence  $v_n$  is bounded in  $L^\infty(\Omega)$  and, by the regularity results of [8], [16] and [9], the sequence  $v_n$  is uniformly bounded in  $C^{1,\alpha}(\bar{\Omega})$  for some  $\alpha \in ]0, 1[$ . Passing to a subsequence if necessary we get that  $v_n$  converges in  $C^{1,\beta}(\bar{\Omega})$  for any  $0 < \beta < \alpha$  to some  $v$  which will satisfy (2.4) and  $0 \leq v \leq u$ . Finally let us show that  $v = u$  to conclude. We can use for instance Picone's identity of [1] to get

$$0 \geq \int_{\Omega} f u \left(1 - \left(\frac{u}{v}\right)^{p-1}\right) dx = \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \left(\frac{u^p}{v^{p-1}}\right) dx \geq 0$$

hence  $u = v$  and the proof is complete.  $\square$

### 3. EXISTENCE OF UPPER AND LOWER SOLUTIONS

Let us recall the definition of upper and lower solutions.

**Definition 3.1.** Let  $f(x, s)$  be a Carathéodory function on  $\Omega \times \mathbb{R}$  with the property that for any  $s_0 > 0$ , there exists a constant  $M > 0$  such that  $|f(x, t)| \leq M$  a.e.  $x \in \Omega$  and all  $t \in [-s_0, s_0]$ . A function  $\alpha \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$  is called a (weak) lower solution of the problem

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

if  $\alpha \leq 0$  on  $\partial\Omega$  and

$$\int_{\Omega} |\nabla \alpha|^{p-2} \nabla \alpha \nabla \psi dx \leq \int_{\Omega} f(x, \alpha) \psi dx$$

for all  $\psi \in C_c^\infty(\Omega)$ ,  $\psi \geq 0$ . An upper solution is defined by reversing the inequality signs.

We will also use the following notations :

**Definition 3.2.** Let  $u, v$  two measurable functions in  $\Omega$ . We will say that  $u \prec v$  if  $\forall K \subset \Omega$  compact  $\exists \epsilon > 0$  such that  $u + \epsilon < v$  a.e. in  $K$ .

Notice that if  $u$  and  $v$  are two continuous functions, say that  $u \prec v$  is equivalent to say that  $u < v$  in  $\Omega$ .

**Definition 3.3.** Let  $u, v \in C^1(\bar{\Omega})$ . We will say that  $u \ll v$  if  $u < v$  in  $\Omega$  and when  $u = v$  on  $\partial\Omega$ , then  $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$  on  $\partial\Omega$ .

In order to obtain upper and lower solutions of the problem  $(P_s)$  we introduce the following two auxiliary problems:

$$(P_s^u) \quad \begin{cases} -\Delta_p u = -B(x)(u^-)^{p-1} + s\varphi(x) + C_2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P_s^l) \quad \begin{cases} -\Delta_p u = -b(x)(u^-)^{p-1} + s\varphi(x) - C_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hypothesis (H2) ensures that any non positive upper solution of  $(P_s^u)$  is an upper solution of  $(P_s)$  and, by (H1), any non positive lower solution of  $(P_s^l)$  is also a lower solution of  $(P_s)$ . The following proposition holds :

**Proposition 3.1.** *Assume (H2). There exists  $\underline{s} \in \mathbb{R}^-$  such that, for all  $s \leq \underline{s}$  there exists  $\bar{u} \in C_0^{1,\alpha}(\bar{\Omega})$  upper solution of  $(P_s^u)$  with  $\bar{u} \ll 0$ . Moreover there exists  $k \in ]0, 1[$  such that  $k\bar{u}$  is an upper solution of  $(P_s^u)$ .*

*Proof.* Let  $\Omega_0$  be an open subdomain of  $\Omega$  such that  $\bar{\Omega}_0 \subset \Omega$  and fix  $M > \|B\|_\infty + 2C_2$ . Let  $u_n$  be the solution of the the following problem

$$-\Delta_p u = h_n \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega,$$

where  $h_n(x) = -n^{p-1}$  if  $x \in \Omega_0$  and  $h_n(x) = M$  if  $x \in \Omega \setminus \Omega_0$ . Since trivially  $\frac{h_n}{n^{p-1}} \rightarrow -\chi_{\Omega_0}$  in  $L^\infty(\Omega)$  it follows by Proposition 2.2 that  $\frac{u_n}{n} \rightarrow v$  in  $C^{1,\beta}(\bar{\Omega})$  for any  $0 < \beta < \alpha$ , where  $v$  is the unique solution of

$$-\Delta_p u = -\chi_{\Omega_0} \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

Notice that  $v \ll 0$  by the Strong Maximum Principle of [17]. Then we can choose  $n_0$  large enough such that  $u_{n_0} \leq 0$  a.e. in  $\Omega$  and  $u_{n_0} < -1$  in  $\Omega_0$ . Let us check that  $\bar{u} := u_{n_0}$  is an upper solution of  $(P_s^u)$ , provided that

$$s < \underline{s} := \frac{-n_0^{p-1} - \|B\|_\infty \|u_{n_0}\|_\infty^{p-1} - C_2}{m} < 0,$$

where  $m := \inf\{\varphi(x) : x \in u_{n_0}^{-1}(-\infty, -1]\} > 0$ . We distinguish to cases : (a)  $u_{n_0}(x) \geq -1$ . Then  $x \notin \Omega_0$  and hence, by the choice of  $M$  we have for any  $s < 0$ ,

$$h_{n_0}(x) = M > -B(x)u_{n_0}^-(x)^{p-1} + s\varphi(x) + C_2;$$

(b)  $u_{n_0}(x) < -1$ . Then, by the choice of  $M$  and  $\underline{s}$ , we have

$$h_{n_0}(x) \geq -n_0^{p-1} > -B(x)u_{n_0}^-(x)^{p-1} + s\varphi(x) + C_2.$$

In any case,

$$\int_\Omega |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \psi \, dx \leq \int_\Omega (-B(x)(\bar{u}^-)^{p-1} + s\varphi(x) + C_2) \psi \, dx$$

for all  $\psi \in C_c^\infty(\Omega)$ ,  $\psi \geq 0$  and the first part of the proposition follows. To prove the last statement let us denote  $v = ku_{n_0}$  and notice that  $-\Delta_p v = k^{p-1}h_{n_0}$ . We have, when  $u_{n_0}(x) \geq -1$ ,

$$k^{p-1}M > -k^{p-1}B(x)u_{n_0}^-(x)^{p-1} + k^{p-1}2C_2 > -B(x)v^-(x)^{p-1} + s\varphi(x) + C_2$$

if  $k^{p-1}2 > 1$ , whereas when  $u_{n_0}(x) < -1$ ,

$$k^{p-1}h_{n_0} \geq -k^{p-1}n_0^{p-1} \geq -n_0^{p-1} > -B(x)v^-(x)^{p-1} + s\varphi(x) + C_2$$

since  $s \leq \underline{s}$  and  $\|v\|_\infty \leq \|u_{n_0}\|_\infty$ .  $\square$

**Proposition 3.2.** *Assume (H1). For all  $\omega \in C_0^1(\bar{\Omega})$  and for all  $s \in \mathbb{R}$  there exists  $\underline{u} \in C_0^{1,\alpha}(\bar{\Omega})$  a lower solution of  $(P_s^l)$  with  $\underline{u} \ll \omega$  and  $\underline{u} \ll 0$ . Moreover  $K\underline{u}$  is a lower solution of  $(P_s^l)$  for all  $K > 1$ .*

*Proof.* Let  $c > 0$  be such that  $-c - C_1 + s\varphi < 0$  and consider the unique solution  $u_c$  of

$$-\Delta_p u = b(x)|u|^{p-2}u - c - C_1 + s\varphi \quad \text{in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

By (H1) we infer from Proposition 2.3 that  $u_c \ll 0$  and therefore  $u_c$  is a lower solution of  $(P_s^l)$ . We claim that for  $c$  large enough  $u_c \ll \omega$ . Indeed, take  $c_n \rightarrow +\infty$  and denote  $v_n = \frac{u_{c_n}}{c_n^{1/p-1}}$ . Then

$$-\Delta_p v_n = b(x)|v_n|^{p-2}v_n - 1 - \frac{C_1 - s\varphi}{c_n} \quad \text{in } \Omega; \quad v_n = 0 \text{ on } \partial\Omega.$$

Since  $-1 - \frac{C_1 - s\varphi}{c_n} \rightarrow -1$  in  $L^\infty(\Omega)$ , we get that the sequence  $v_n$  tends in  $C_0^{1,\beta}(\overline{\Omega})$  to the solution  $v$  of

$$-\Delta_p v = b(x)|v|^{p-2}v - 1 \quad \text{in } \Omega; \quad v = 0 \text{ on } \partial\Omega.$$

By Proposition 2.3  $v \ll 0$  so, for  $n$  large,  $u_{c_n} \ll \omega$ . Finally we observe for all  $K > 1$

$$-\Delta_p(Ku) = b(x)|Ku|^{p-2}Ku - K^{p-1}(c + C_1 - s\varphi) \leq b(x)|Ku|^{p-2}Ku - c - C_1 + s\varphi,$$

since  $c + C_1 - s\varphi > 0$ . Thus  $Ku$  is a lower solution of  $(P_s^!)$ .  $\square$

#### 4. A PRIORI BOUNDS

Let us consider the following problem

$$(P_f) \quad \begin{cases} -\Delta_p u = g(x, u) + f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

for any  $f \in L^\infty(\Omega)$ . The following result states that the negative part of the solutions of (4.1) is bounded in terms of the negative part of  $f$ . Since we are going to use the  $L^\infty$ -bound of a solution in terms of his  $W_0^{1,p}$  norm we need to assume in this section that  $g$  satisfies (G).

**Lemma 4.1.** *Suppose that  $g$  satisfies hypothesis (H1) and (G). For any  $K > 0$  there exists  $M > 0$  such that for all solution  $u$  of  $(P_f)$  with  $\|f^-\|_\infty \leq K$  we have  $\|u^-\|_\infty \leq M$ .*

*Proof.* By the results of [8] (see also Remark 2.1) it is enough to find an estimate of  $\|u^-\|$  in  $W_0^{1,p}(\Omega)$ . Assume by contradiction that there exists a sequence  $f_n \in L^\infty(\Omega)$ ,  $\|f_n^-\|_\infty \leq K$  and a sequence  $u_n$  solution of  $(P_{f_n})$  such that  $\|u_n^-\| \rightarrow +\infty$ . Let us denote  $v_n := \frac{u_n}{\|u_n^-\|}$ . Then it holds

$$-\Delta_p v_n = \frac{g(x, u_n)}{\|u_n^-\|^{p-1}} + \frac{f_n}{\|u_n^-\|^{p-1}} \quad \text{in } \Omega, \quad v_n = 0 \text{ on } \partial\Omega.$$

By multiplying the previous equation by  $-v_n^-$  and using (H1) we have

$$1 = \int_\Omega |\nabla v_n^-|^p dx \leq \int_\Omega b(x)|v_n^-|^p dx + C_1 \int_\Omega \frac{v_n^-}{\|u_n^-\|^{p-1}} dx + \int_\Omega \frac{f_n^- v_n^-}{\|u_n^-\|^{p-1}} dx.$$

Up to a subsequence, there exists  $v_0 \in W_0^{1,p}(\Omega)$  such that  $v_n^- \rightharpoonup v_0$  in  $W_0^{1,p}(\Omega)$  and strongly in  $L^p(\Omega)$ . Going to infinity we have

$$\int_\Omega |\nabla v_0|^p dx \leq \liminf_{n \rightarrow \infty} \int_\Omega |\nabla v_n^-|^p dx = \int_\Omega b(x)|v_0|^p dx$$

and

$$1 \leq \int_\Omega b(x)|v_0|^p dx.$$

But since  $\lambda_1(b) > 1$  we conclude from the first inequality that  $v_0 = 0$ , a contradiction with the second one.  $\square$

**Lemma 4.2.** *Assume that  $g$  satisfies hypothesis (G), (H1) and (H3). Given  $s_0 \in \mathbb{R}$  there exists  $R > 0$  and  $M > 0$  such that for all  $s > s_0$  and for all solution  $u$  of  $(P_s)$  it holds*

$$\frac{s}{(\|u\|_\infty + M)^{1-p}} \leq R. \quad (4.2)$$

*Proof.* We apply Lemma 4.1 with  $f = s\varphi$ . Since  $\|(s\varphi_1)^-\|_\infty \leq s_0^- \|\varphi_1\|_\infty$  for all  $s \geq s_0$ , there exists  $M > 1$  such that  $\|u^-\|_\infty < M - 1$  for all  $u$  solution of  $(P_s)$  with  $s \geq s_0$ . Applying Picone's inequality to  $u + M$  and  $\varphi_1$  we obtain

$$0 \leq \int_\Omega |\nabla \varphi_1|^p dx - \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \left( \frac{\varphi_1^p}{(u + M)^{p-1}} \right) dx =$$

$$= \lambda_1 - \int_{\Omega} \{g(x, u) + s\varphi(x)\} \frac{\varphi_1^p(x)}{(u(x) + M)^{p-1}} dx.$$

By using that  $\varphi, \varphi_1$  are nonnegative and that  $u(x) + M > 1$  for all  $x \in \Omega$  we find, on the one hand,

$$\frac{sb}{(\|u\|_{\infty} + M)^{p-1}} \leq \lambda_1 - \int_{\Omega} g(x, u) \frac{\varphi_1^p(x)}{(u(x) + M)^{p-1}} dx. \quad (4.3)$$

where  $b := \int_{\Omega} \varphi(x) \varphi_1^p(x) dx$ . On the other hand, using that  $g$  satisfies (H1) and (H3) we have

$$\inf_{x \in \Omega, l > -M+1} \frac{g(x, l) \varphi_1^p(x)}{(l + M)^{p-1}} > -\infty. \quad (4.4)$$

From (4.3) and (4.4) we easily get (4.2).  $\square$

Finally we prove a result on a-priori bounds for the solutions of  $(P_s)$  using a blow-up argument. Notice that here we allow only the nonlinearity  $g$  to growth at  $+\infty$  as  $u^{q-1}$  with  $p < q < p_*$ . We have followed here some ideas of [13]. For the Liouville problem in a half space we use the recent results of [10].

**Proposition 4.3.** *Assume (H1), (H2), (H3) and (H4). Then given  $s_0 \in \mathbb{R}$  there exists  $R > 0$  such that, for all  $s \geq s_0$  and for all solution  $u$  of  $(P_s)$ , we have  $\|u\|_{\infty} \leq R$ .*

*Proof.* Assume by contradiction that there exist a sequence  $s_n \geq s_0$  and a solution  $u_n$  of  $(P_{s_n})$  such that  $\|u_n\|_{\infty} \rightarrow \infty$ . We know by Lemma 4.1 that  $\|u_n^-\|_{\infty}$  is bounded. Let us put  $\gamma_n := \|u_n\|_{\infty} = u_n(x_n)$  for some  $x_n \in \Omega$  and  $\delta_n := \text{dist}(x_n, \partial\Omega)$ . In what follows we will denote by  $C, D, \dots$  generic constant *independent of  $n$* . We first prove that there exist a constant  $C > 0$  such that

$$\delta_n \gamma_n^{\frac{q-p}{p}} > C. \quad (4.5)$$

Define  $w_n(y) = \gamma_n^{-1} u_n(\gamma_n^{\frac{p-q}{p}} y + x_n)$  for  $y \in \Omega_n := \gamma_n^{\frac{q-p}{p}} (\Omega - x_n)$ . Then  $w_n(0) = 1$ ,  $\|w_n\|_{\infty} = 1$ ,  $\nabla w_n = \gamma_n^{-\frac{q}{p}} \nabla u_n$ ,  $\Delta_p w_n = \gamma_n^{1-q} \Delta_p u_n$  and

$$\begin{cases} -\Delta_p w_n = \theta_n(y, w_n) & \text{in } \Omega_n, \\ w_n = 0 & \text{on } \partial\Omega_n \end{cases} \quad (4.6)$$

where  $\theta_n(y, w) = \gamma_n^{1-q} |g(x, \gamma_n w)| + \gamma_n^{1-q} s_n \varphi$ . Using (H1) to (H4) we have for a.e  $y \in \Omega_n$  and for all  $w \in \mathbb{R}$  that  $\gamma_n^{1-q} |g(x, \gamma_n w)| \leq D_1 |w|^{q-1} + D_2$  for some  $D_1, D_2$  independent of  $n$ . For the second term of  $\theta_n$  we use the estimate (4.2) and the fact that  $\gamma_n \rightarrow +\infty$  to get

$$\gamma_n^{1-q} s_n \leq C \gamma_n^{p-q} \rightarrow 0.$$

Thus we have for all  $w \in \mathbb{R}$ ,  $\theta_n(\cdot, w) \leq D_1 |w|^{q-1} + D_3$  for some  $D_3$  independent of  $n$ . Therefore by the regularity results already quoted we infer that  $\|\nabla w_n\|_{\infty} \leq C$  independent of  $n$ . If we now choose  $z_n \in \partial\Omega$  such that  $\text{dist}(x_n, z_n) = \delta_n$ , we have

$$1 = w_n(0) - w_n(\gamma_n^{\frac{q-p}{p}} (z_n - x_n)) \leq \|\nabla w_n\|_{\infty} \gamma_n^{\frac{q-p}{p}} \delta_n$$

and (4.5) follows. We then consider two cases :

**Case 1.**  $\gamma_n^{\frac{q-p}{p}} \delta_n \rightarrow +\infty$ . By a diagonal argument we can prove that there exists  $w \in W^{1,p}(\mathbb{R}^N)$  such that for all  $R > 0$ , up to a subsequence,  $w_n \rightharpoonup w$ , strongly in  $C(B(0, R))$ . By (H3), and having in mind that  $\|w_n^-\|_{\infty} \rightarrow 0$  (because of  $\|u_n^-\|_{\infty}$  is bounded), we conclude that  $w$  is a positive solution of

$$-\Delta_p w \geq a w^{q-1} \quad \text{in } \mathbb{R}^N.$$

By the results of [11] it must be  $w \equiv 0$ , a contradiction with  $w(0) = 1$ .

**Case 2.**  $\gamma_n^{\frac{q-p}{p}} \delta_n \leq C$  for some  $C > 0$ . Notice that in particular  $\delta_n \rightarrow 0$ . Again by a

diagonal argument there exists  $w \in W^{1,p}(\mathbb{R}_+^N)$  such that for all  $R > 0$ , up to a subsequence,  $w_n \rightharpoonup w$ , strongly in  $C(B(0,R) \cap \mathbb{R}_+^N)$ . By (H3) and (H4)  $w$  is a positive solution of

$$Aw^{q-1} \geq -\Delta_p w \geq aw^{q-1} \quad \text{in } \mathbb{R}_+^N.$$

By the results of [10] we now conclude that  $w \equiv 0$  in contradiction again with  $w(0) = 1$ .  $\square$

**Remark 4.1.** *The results of [10] are stated for  $1 < p < N$  but the proof can be easily extended to all  $p > 1$ . Indeed Lemma 4.1 of [15] is valid for  $p \geq N$  (with no restriction on  $\gamma$ ) and the weak Harnack inequality as well (see remark on page 154 of [14]).*

**Remark 4.2.** *Hypothesis (H4) is only needed to assure that the limiting Liouville type of problem in the half-space has no positive solutions. Indeed, in  $\mathbb{R}^N$  it is known that there is no positive solution  $u$  satisfying  $-\Delta_p u \leq u^{q-1}$  with  $p < q < p_*$  (c.f. [11]) whereas in the half space  $\mathbb{R}_+^N$  the same conclusion holds for positive solutions of  $Du^{q-1} \leq -\Delta_p u \leq u^{q-1}$  with  $p < q < p_*$ . The validity of this conclusion for positive solutions of the inequality  $-\Delta_p u \leq u^{q-1}$  with  $p < q < p_*$  in  $\mathbb{R}_+^n$  is to our knowledge an open question.*

**Corollary 4.4.** *Under the hypothesis of Proposition 4.3, there exists  $\bar{s} \in \mathbb{R}$  such that  $(P_s)$  has no solution if  $s > \bar{s}$ .*

*Proof.* Fix any  $s_0 \in \mathbb{R}$ . By Proposition 4.3, there exist  $D > 0$  such that for all  $s \geq s_0$  and for all solution  $u$  of  $(P_s)$  it holds  $\|u\|_\infty \leq D$ . Then by using (4.2) we get that the range of  $s > s_0$  for which  $(P_s)$  has a solution is a bounded. Thus  $(P_s)$  have no solutions when  $s$  is large enough.  $\square$

## 5. PROOF OF THEOREM 1.1.

(1) The nonexistence has already been proved in Corollary 4.4. Define then

$$s^* := \inf\{\bar{s} \in \mathbb{R}; (P_s) \text{ has no solution for all } s \geq \bar{s}\}.$$

(2) Let us show that  $(P_s)$  has at least one solution when  $s < s^*$ . By definition, given  $s < s^*$  there exists  $s_1 > s$  such that  $(P_{s_1})$  has a solution  $u_1$ . Using that  $u_1$  is an upper solution of  $(P_s)$  we get by Proposition 3.2 that there exists  $\underline{u} \ll u_1$  a lower solution of  $(P_s)$ . Then (see Theorem 8.1 of [5])  $(P_s)$  has a solution  $u$  with  $\underline{u} \leq u \leq u_1$ .

(3) Let us show that  $(P_s)$  has at least two solutions for all  $s \leq \underline{s}$ , where  $\underline{s}$  has been found in Proposition 3.1. Let us denote  $X := \{u \in C^1(\bar{\Omega}); u = 0 \text{ on } \partial\Omega\}$ . Given  $s \in \mathbb{R}$ , we define  $\mathcal{K}_s : X \rightarrow X$  by  $\mathcal{K}_s v = u$  if and only if  $u$  is the unique solution of

$$\begin{cases} -\Delta_p u = g(x, v) + s\varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $g(\cdot, v) \in L^\infty(\Omega)$  for any  $v \in X$  (as a consequence of (G) and the regularity results), it follows from Proposition 2.2 that the map  $\mathcal{K}_s$  is well defined and compact. Moreover,  $u$  is a solution of  $(P_s)$  if and only if  $\mathcal{K}_s u = u$ , that is, if and only if  $(I - \mathcal{K}_s)u = 0$ . By the properties of the Leray-Schauder Degree and Proposition 4.3,  $\deg(I - \mathcal{K}_s, X, 0) = 0$ . Besides let us remark that we can use the upper solution  $\bar{u} < 0$  found in Proposition 3.1 and the lower solution  $\underline{u}$  found in Proposition 3.2 to get a solution  $u$  of  $(P_s)$  in between. Moreover we have that, for some  $0 < k < 1 < K$ , the functions  $\bar{v} := k\bar{u}$  and  $\underline{v} := K\underline{u}$  are respectively upper and lower solution of  $(P_s)$ . Notice that  $\bar{v} \gg \bar{u}$  and  $\underline{v} \ll \underline{u}$  so our solution  $u$  satisfies  $\underline{v} \ll u \ll \bar{v}$  and  $u \leq 0$ . Let us denote  $\mathcal{C} := \{w \in X : \underline{v} \ll w \ll \bar{v}\}$ . It is known (see, for instance, [5]) that  $\deg(I - \mathcal{K}_s, \mathcal{C}, 0) = 1$ . Then, by the scision property of the Leray-Schauder degree,  $(P_s)$  has a second solution. It is enough to define

$$s_* := \sup\{s \in \mathbb{R}; (P_{s'}) \text{ has at least two solutions for any } s' \leq s\},$$

in order to complete the proof.  $\square$



Theorem 1.1 can be improve when the nonlinearity  $g$  is continuous. In fact we have

**Theorem 5.1.** *Under the hypothesis of Theorem 1.1, if  $g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, then  $(P_{s^*})$  has at least one solution. If moreover there exists  $\mu > 0$  such that  $\varphi(x) \geq \mu$ , for all  $x \in \Omega$ , then  $s_* = s^*$ .*

*Proof.* We will first prove that  $(P_{s^*})$  has a solution. Let  $\{s_n\} \in \mathbb{R}$  be a sequence such that  $s^* > s_n \rightarrow s^*$  and let  $u_n$  be a solution of  $(P_{s_n})$ . By Proposition 4.3, the sequence  $\{u_n\}$  is bounded in  $W_0^{1,p}(\Omega)$ . Up to a subsequence, there exists  $u_0 \in W_0^{1,p}(\Omega)$  such that  $u_n \rightarrow u_0$  in  $W^{1,p}(\Omega)$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$ . Then,

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u_0) dx = \int_{\Omega} (g(x, u_n) + s_n \varphi)(u_n - u_0) dx \rightarrow 0,$$

so

$$\int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0) (\nabla u_n - \nabla u_0) dx \rightarrow 0,$$

and consequently  $u_n \rightarrow u_0$  in  $W^{1,p}(\Omega)$ . Hence, for any fixed  $\omega \in W_0^{1,p}(\Omega)$ , we have after taking limits in the equality

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \omega dx = \int_{\Omega} (g(x, u_n) + s_n \varphi) \omega dx,$$

and using the Lebesgue Dominate Convergence Theorem in the right side, we obtain

$$\int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla \omega dx = \int_{\Omega} (g(x, u_0) + s^* \varphi) \omega dx.$$

Thus  $u_0$  is a solution of  $(P_{s^*})$ .

Assume finally that there exists  $\mu > 0$  such that  $\varphi(x) \geq \mu$ , for all  $x \in \Omega$ . Let  $s < s^*$ , take  $s_1 < s < s_2 < s^*$  and let  $u_i$  be a solution of  $(P_{s_i})$ ,  $i = 1, 2$ . Thus  $u_1$  and  $u_2$  are lower and upper solutions of  $(P_s)$ . Arguing as in the proof of step (3) above, we only need to find another couple of lower and upper solutions  $\underline{v}, \bar{v}$  such that  $\underline{v} \ll u$  and  $u \ll \bar{v}$  to prove the result. We will show how to find the lower solutions; the proof for the upper solutions follows in a similar way. Take  $0 < \varepsilon < \frac{s-s_1}{2} \mu$ . Since  $g$  and  $u_1$  are continuous functions, there exists  $0 < \delta < 1$  such that

$$|g(x, u) - g(x, v)| \leq \varepsilon, \quad \forall x \in \bar{\Omega}, \quad u, v \in [-\|u_1\|_{\infty} - 1, \|u_1\|_{\infty} + 1], \quad |u - v| < \delta.$$

Define  $\underline{v} = u_1 - \delta/2$ . We have

$$-\Delta_p \underline{v} = -\Delta_p u_1 = g(x, \underline{v} + \delta/2) + s_1 \varphi \leq g(x, \underline{v}) + \varepsilon + s_1 \varphi < g(x, \underline{v}) + s \varphi,$$

since  $\varepsilon < \frac{s-s_1}{2} \mu \leq \frac{s-s_1}{2} \varphi(x)$  in  $\Omega$ , so  $\varepsilon + s_1 \varphi < s \varphi$ . Moreover  $\underline{v} = -\delta/2, x \in \partial\Omega$ . Thus  $\underline{v}$  is a lower solution of  $(P_s)$  and  $\underline{v} \ll u$  because  $\underline{v}(x) < u_1(x) \leq u(x)$  in  $\Omega$  and  $\underline{v} = -\delta/2 < 0 = u(x)$  in  $\partial\Omega$ . □

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