Maximum principles around an eigenvalue with constant eigenfunctions

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Abstract

A class of linear operators $L+\lambda I$ between suitable function spaces is considered, when 0 is an eigenvalue of L with constant eigenfunctions. It is proved that $L+\lambda I$ satisfies a strong maximum principle when λ belongs to a suitable pointed left-neighborhood of 0, and satisfies a strong uniform antimaximum principle when λ belongs to a suitable pointed right-neighborhood of 0. Applications are given to various type of ordinary or partial differential operators with periodic or Neumann boundary conditions.

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1 Introduction

Consider the problem

$$\Delta u + \lambda u = f(x) \text{ in } D, \quad Bu = 0 \text{ in } \partial D, \tag{1}$$

where D is a smooth bounded domain in $\mathbb{R}^N$ and $Bu = 0$ represents either the Dirichlet or Neumann homogenous boundary conditions.

It is a standard consequence of the **maximum principle (MP)** that *if $\lambda < \lambda_1$, where λ_1 represents the principal eigenvalue of $-\Delta$ under the corresponding boundary conditions, and if f is a nonnegative function, then* *the solution u of (1) is nonpositive in $\overline{D}$.* Indeed, a stronger conclusion holds for (1), namely a **strong maximum principle (SMP)** : *if f is nonnegative and not identically zero, then u is negative in D.* We refer to the classical book [29] for more details and early references, to [4] for a general statement, and to the recent monograph [30] for more details on this important tool. Those references essentially deal with partial differential equations or systems of elliptic or of parabolic type.

In 1979, Clément and Pelletier [12] investigated the problem (1) in the situation where $\lambda > \lambda_1$ and proved the following **anti-maximum principle (AMP) :** *Given a nonnegative function f, there exists $\delta = \delta(f) > 0$ such that if $\lambda_1 < \lambda < \lambda_1 + \delta$, then any solution u of (1) is nonnegative in D.* Furthermore, a **strong anti-maximum principle (SAMP)** holds in this case : *if f is nonnegative and not identically zero, then u is positive in D.* They also showed in [12] that δ can be taken independent of f for the Neumann problem in dimension $N = 1$, in which case one speaks of a **uniform antimaximum principle (UAMP)**. Recent work with those (possibly strong) AMP and UAMP for linear elliptic operators with various boundary conditions include the papers of Hess [21] (AMP for elliptic problems with weight), de Figueiredo-Gossez [15] (connection to the Fučik spectrum), Birindelli [5] (irregular domains), Takač [35] (MP and AMP for abstract linear elliptic boundary value problems in strongly ordered space), Cabada-Lois [10] (UAMP for higher order ordinary differential operators with periodic boundary conditions), Sweers [34] (exact L^p space where f should be taken), Pinchover [28] (MP and AMP via perturbation theory of positive solutions), Alziary-Fleckinger-Takáč [1] (MP and AMP for Schrödinger equation in $\mathbb{R}^2$), Godoy-Gossez-Paczka [17, 18] (AMP and UAMP for Dirichlet, Neumann or Robin problems with weight), Clément-Sweers [13, 14] (UAMP for second or higher order elliptic operators with homogeneous boundary conditions), Stavrakakis-de Thélin [33] (AMP for elliptic equation on $\mathbb{R}^N$), Grunau-Sweers [20] (optimal conditions for AMP or UAMP for polyharmonic boundary value problems), Barteneva-Cabada-Ignatyev [3] and Reichel [31] (MP and AMP for second order ordinary differential operators with variable coefficients), Arcoya-Gámez [2] and Shi Junping [32] (proofs of AMP using bifurcation) and others.

In the case of Neumann problem in dimension one

$$u'' + \lambda u = f(x) \quad in \quad (0, \pi), \quad u'(0) = 0 = u'(\pi)$$

zero is the principal eigenvalue, and, if $f \geq 0$ on $(0, \pi)$, the SMP tells that $u < 0$ on $(0, \pi)$ when $\lambda < 0$, and the SUAMP tells that $u > 0$ on $(0, \pi)$ when $\lambda \in (0, 1/4]$. In other words, $\lambda u > 0$ when $\lambda \in (-\infty, 0) \cup (0, 1/4]$.

Less standard maximum principles have been obtained recently for the time-periodic or the time-bounded solutions of telegraph equations of the form

$$u_{tt} + cu_t - \Delta u + \lambda u = f(t, x) \tag{2}$$

with periodic spatial boundary conditions (see [27, 23, 24]). In the case of time-periodic solutions, a SMP holds for $\lambda \in (0, \lambda_+]$, for some finite $\lambda_+ > 0$ depending upon c. A natural question is the existence of an anti-maximum principle for such a problem.

The aim of this paper is to identify a class of abstract linear operators $L + \lambda I$ acting on some function spaces, $\lambda = 0$ being an eigenvalue of L with constant eigenfunctions, for which a MP holds when $\lambda \in [\lambda_-, 0)$, a SMP holds when $\lambda \in (\lambda_-, 0)$, a UAMP holds when $\lambda \in (0, \lambda_+]$, and a SUAMP holds when $\lambda \in (0, \lambda_+)$, for some $-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty$. A precise statement is given in Theorem 1 of Section 2. The proof is based upon a detailed study of the resolvent operator R_λ of L.

Various applications are given in Section 3, namely to linear differential operators of arbitrary order with constant coefficients and periodic boundary conditions, first order difference equations with periodic boundary conditions, a two-point boundary value problem of order four, polyharmonic operators with Neumann-type boundary conditions on a smooth bounded domain of $\mathbb{R}^N$, and some hyperbolic operators on a torus, generalizing the telegraph equation with periodic boundary conditions in space and time. This shows in particular that, in this case, a SUAMP holds in addition to the SMP proved in [27].

2 Abstract setting of the results

Let Ω be a compact metric space and let μ be a positive and bounded measure over Ω. The notion of measure is understood as in [16]. We shall work with the Banach spaces $\mathcal{C} = C(\Omega)$ with its standard norm $\|u\|_\infty = \max_{\omega\in\Omega} |u(\omega)|$ and $\mathcal{L} = L^1(\Omega, \mu)$ with its standard norm $\|f\|_1 = \int_\Omega |f| d\mu$. Given $f \in \mathcal{L}$ we employ the notations

$$\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu, \quad \widetilde{f} = f - \overline{f}, \quad \widetilde{\mathcal{L}} = \{f \in \mathcal{L} \mid \overline{f} = 0\}, \quad \widetilde{\mathcal{C}} = \mathcal{C} \cap \widetilde{\mathcal{L}}.$$

Let $L : \mathrm{Dom}(L) \subset \mathcal{C} \to \mathcal{L}$ be a linear operator satisfying

$$\mathrm{Ker}(L) = \{\text{constant functions}\}, \;\; \mathrm{Im}(L) = \widetilde{\mathcal{L}} \tag{3}$$

and such that equation

$$Lu = \widetilde{f}$$

has a unique solution $\widetilde{u} \in \widetilde{\mathcal{C}}$. Moreover assume that

$$\|\widetilde{u}\|_\infty \leq K\|\widetilde{f}\|_1 \tag{4}$$

where K only depends upon L. Thus L is a closed Fredholm operator of index zero.

Definition 1 *Given $\lambda \in \mathbb{R}\backslash\{0\}$, the operator $L + \lambda I$ satisfies a* maximum principle *if for each $f \in \mathcal{L}$, the equation*

$$Lu + \lambda u = f, \quad u \in \mathrm{Dom}(L) \tag{5}$$

has a unique solution and $\lambda u \geq 0$ for any $f \geq 0$. The maximum principle is said to be strong *if $\lambda u(x) > 0$ for any $x \in \Omega$ when $f \geq 0$ and $f(x) > 0$ in a subset of Ω with positive measure.*

Such a definition includes both maximum and anti-maximum principles. For example, if $Lu = u''$ with the Neumann boundary conditions on $\Omega = [0, 1]$, our definition corresponds to a maximum principle when $\lambda < 0$ and to an anti-maximum principle when $\lambda > 0$.

The main result of this paper is the following

Theorem 1 *Assume that conditions (3) and (4) hold. Then there exist numbers λ_- and λ_+, with*

$$-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty$$

such that $L+\lambda I$ has a maximum principle if and only if $\lambda \in [\lambda_-, 0)\cup(0, \lambda_+]$. Moreover if $\lambda \in (\lambda_-, 0) \cup (0, \lambda_+)$ the maximum principle is strong.

Before giving the proof of Theorem 1, we introduce some notations, and prove three lemmas. The inverse of $L + \lambda I$, whenever it exists, is called the *resolvent* of L, and is denoted by R_λ, so that $R_\lambda : \mathcal{L} \to \mathcal{C}$. The *partial resolvent* for $\lambda = 0$, denoted by $\widetilde{R}_0$, is the operator $\widetilde{R}_0 : \widetilde{\mathcal{L}} \to \widetilde{\mathcal{C}}$, defined by

$$\widetilde{u} = \widetilde{R}_0\widetilde{f} \quad \Longleftrightarrow \quad L\widetilde{u} = \widetilde{f}.$$

We know by assumption (4) that $\widetilde{R}_0$ is continuous.

Lemma 1 *There exist some* $\Lambda_1 > 0$ *such that, if* $\lambda \in [-\Lambda_1, \Lambda_1]\backslash\{0\}$, *then the resolvent* R_λ *of* L *exists. Moreover*

$$\|R_\lambda \widetilde{f}\|_\infty \leq \left(\frac{\|\widetilde{R}_0\|_{\widetilde{\mathcal{L}}\to\widetilde{\mathcal{C}}}}{1 - \Lambda_1 \|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}}} \right) \|\widetilde{f}\|_1 \tag{6}$$

if $\widetilde{f} \in \widetilde{\mathcal{L}}$, $0 < |\lambda| \leq \Lambda_1$.

Proof. Split equation (5) in the following way

$$\left\{ \begin{array}{c} L\widetilde{u} + \lambda\widetilde{u} = \widetilde{f}, \\ \lambda\overline{u} = \overline{f}. \end{array} \right. \tag{7}$$

Using the partial resolvent of L, the first equation in (7) can be written as

$$\widetilde{u} + \lambda \widetilde{R}_0 \widetilde{u} = \widetilde{R}_0 \widetilde{f}. \tag{8}$$

Let us observe that $\widetilde{R}_0$ is also a continuous map when restricted to $\widetilde{\mathcal{C}}$. If $|\lambda|\|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}} < 1$ then $I + \lambda\widetilde{R}_0$ is invertible from $\widetilde{\mathcal{C}}$ to $\widetilde{\mathcal{C}}$ and (8) is solved as

$$\widetilde{u} = (I + \lambda\widetilde{R}_0)^{-1}\widetilde{R}_0\widetilde{f}.$$

To finish this lemma, we take $0 < \Lambda_1 < \frac{1}{\|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}}}$ and obtain from (8), for $|\lambda| \leq \Lambda_1$,

$$\begin{aligned} \|\widetilde{u}\|_\infty - \Lambda_1\|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}}\|\widetilde{u}\|_\infty &\leq \|\widetilde{u}\|_\infty - |\lambda|\|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}}\|\widetilde{u}\|_\infty \\ &\leq \|(I + \lambda\widetilde{R}_0)\widetilde{u}\|_\infty = \|\widetilde{R}_0\widetilde{f}\|_\infty \\ &\leq \|\widetilde{R}_0\|_{\widetilde{\mathcal{L}}\to\widetilde{\mathcal{C}}}\|\widetilde{f}\|_1, \end{aligned}$$

and hence

$$\|\widetilde{u}\|_\infty \leq \left(\frac{\|\widetilde{R}_0\|_{\widetilde{\mathcal{L}}\to\widetilde{\mathcal{C}}}}{1 - \Lambda_1 \|\widetilde{R}_0\|_{\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}}}} \right) \|\widetilde{f}\|_1,$$

which gives (6). ∎

Lemma 2 *There exists* $\Lambda_2 \in (0, \Lambda_1)$ *such that, if* $0 < |\lambda| \leq \Lambda_2$, *then* $L + \lambda I$ *has a strong maximum principle.*

Proof. If we take $f \geq 0$, then $\overline{f} = \frac{1}{\mu(\Omega)}\|f\|_1$, so that, using the second equation in (7),

$$\lambda R_\lambda(\overline{f} + \widetilde{f}) = \overline{f} + \lambda R_\lambda(\widetilde{f}) = \frac{1}{\mu(\Omega)}\|f\|_1 + \lambda R_\lambda \widetilde{f} \geq \frac{1}{\mu(\Omega)}\|f\|_1 - |\lambda| \|R_\lambda \widetilde{f}\|_\infty$$

and using (6), if $0 < |\lambda| \leq \Lambda_1$,

$$\lambda u = \lambda R_\lambda(\overline{f} + \widetilde{f}) \geq \frac{1}{\mu(\Omega)}\|f\|_1 - |\lambda| \left(\frac{\|\widetilde{R}_0\|_{\widetilde{\mathcal{L}} \to \widetilde{\mathcal{C}}}}{1 - \Lambda_1 \|\widetilde{R}_0\|_{\widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}}} \right) \|\widetilde{f}\|_1.$$

Since $\|\widetilde{f}\|_1 \leq \|f\|_1 + \|\overline{f}\|_1 = 2\|f\|_1$, the result follows by taking

$$\Lambda_2 < \min \left\{ \Lambda_1, \frac{1 - \Lambda_1 \|\widetilde{R}_0\|_{\widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}}}{2\mu(\Omega) \|\widetilde{R}_0\|_{\widetilde{\mathcal{L}} \to \widetilde{\mathcal{C}}}} \right\}$$

. ∎

Lemma 3 *Assume that $L + \lambda_0 I$ has a maximum principle for some $\lambda_0 > 0$. Then, for any $\lambda \in (0, \lambda_0)$, $L + \lambda I$ has a strong maximum principle. Moreover R_λ is defined for any $\lambda \in (0, 2\lambda_0)$ and for any $f \in \mathcal{L}$ the function $\lambda \to R_\lambda f$ is analytic.*

Proof. A first observation is that the existence of a maximum principle implies the continuity of the resolvent from $\mathcal{C} \to \mathcal{C}$. Indeed, because of (5), $R_{\lambda_0} 1 = \frac{1}{\lambda_0}$ and if $f \in \mathcal{C}$, $-\|f\|_\infty \leq f \leq \|f\|_\infty$. So using the maximum principle,

$$-\frac{\|f\|_\infty}{\lambda_0} \leq R_{\lambda_0} f \leq \frac{\|f\|_\infty}{\lambda_0},$$

i.e.

$$\|R_{\lambda_0}\|_{\mathcal{C} \to \mathcal{C}} = \frac{1}{\lambda_0}. \tag{9}$$

Now, let us take $\lambda \in \mathbb{R}$ and write (5) as

$$Lu + \lambda_0 u = (\lambda_0 - \lambda) u + f,$$

or, equivalently,

$$u - (\lambda_0 - \lambda) R_{\lambda_0} u = R_{\lambda_0} f.$$

This equation is solvable if

$$\|(\lambda_0 - \lambda) R_{\lambda_0}\|_{\mathcal{C} \to \mathcal{C}} < 1,$$

which, using estimate (9), holds if

$$\left|\frac{\lambda_0 - \lambda}{\lambda_0}\right| < 1 \quad \Longleftrightarrow \quad 0 < \lambda < 2\lambda_0.$$

Thus equation (5) is solvable in $(0, 2\lambda_0)$.

If now $f \in \mathcal{L}$, then $R_{\lambda_0} f \in \mathcal{C}$ and

$$R_\lambda f = [I - (\lambda_0 - \lambda)R_{\lambda_0}]^{-1} R_{\lambda_0} f = \left(\sum_{n\in\mathbb{N}} (\lambda_0 - \lambda)^n R_{\lambda_0}^n\right) R_{\lambda_0} f. \tag{10}$$

This expression is analytic in λ, and nonnegative if $\lambda < \lambda_0$ and $f \geq 0$. This gives the maximum principle. To finish the lemma, it remains to show that the strong maximum principle holds for any $\lambda \in (0, \lambda_0)$. Take $f \geq 0$ with $\int_\Omega f \, d\mu > 0$. Then from (10), if $0 < \lambda_2 < \lambda_1 < \lambda_0$, one has $R_{\lambda_2} f \geq R_{\lambda_1} f$. Fix any $x \in \Omega$ and consider the function $\varphi(\lambda) = R_\lambda f(x), \;\; \lambda \in (0, \lambda_0]$. The previous remarks imply that this function in non-negative and monotone non-increasing. From Lemma 2 we know that $\varphi(\lambda) > 0$ if $\lambda \in (0, \Lambda_2]$. We invoke the analyticity of φ in $(0, \lambda_0]$ and conclude that $\varphi(\lambda) > 0$ everywhere in $(0, \lambda_0)$. ■

Proof of Theorem 1. It is enough to prove the result for $\lambda > 0$. For the case of negative λ it is enough to replace L by $-L$. From Lemmas 2 and 3 we know that

$$I_+ = \{\lambda > 0 \mid L + \lambda I \text{ has a maximum principle}\}$$

is a nonempty interval. Assuming that $\lambda_+ = \sup I_+$ is finite, it remains to prove that $\lambda_+ \in I_+$. Take $\lambda_n \nearrow \lambda_+$, and select n_0 large enough such that $\lambda_+ < 2\lambda_{n_0}$. Thus R_{λ_+} exists and we can apply the identity (10) with $\lambda_0 = \lambda_+$ to conclude that $u_n = R_{\lambda_n} f \to u = R_{\lambda_+} f$ uniformly in Ω. If $f \geq 0$ then $u_n \geq 0$ and so $u \geq 0$. ■

A natural question after Theorem 1 is whether there exist some links between the numbers $\lambda_\pm$ and the spectrum of $-L$. To adjust to the usual notations we consider the resolvent set $\mathcal{R}$ of the operator $-L$, that is

$$\mathcal{R} = \{\lambda \in \mathbb{C} \mid L + \lambda I \;\text{ is one-to-one, onto and } (L + \lambda I)^{-1} \text{ is continuous}\}.$$

As usual in spectral theory the symbol L also denotes the complex extension. Assuming that λ_+ is finite we observe from the proof of Lemma 3 that the open disk of center λ_+ and radius λ_+ is contained in $\mathcal{R}$. The same applies to the disk centered at λ_- and of radius $|\lambda_-|$ when this number is finite. In particular $\lambda_\pm$ belong to $\mathcal{R}$ and cannot be eigenvalues of $-L$. Define Σ_+ as the largest positive number such that the open disk of center Σ_+ and radius Σ_+ is contained in $\mathcal{R}$. In an analogous way one can define Σ_- and it is now obvious that

$$\Sigma_- \leq \lambda_- < 0 < \lambda_+ \leq \Sigma_+,$$

with the conventions $\Sigma_+ = +\infty$ if $\lambda_+ = +\infty$, $\Sigma_- = -\infty$ if $\lambda_- = -\infty$. The next example shows that these estimates are sharp.

Let Ω be the metrizable space composed by two points A and B. We consider the measure satisfying $\mu(\{A\}) = \alpha$ and $\mu(\{B\}) = \beta$, where $\alpha > 0$ and $\beta > 0$ are real parameters. In this case the spaces $\mathcal{C}$ and $\mathcal{L}$ can be identified to $\mathbb{R}^2$ with the corresponding norms

$$\|(x,y)\|_\infty = \max\{|x|,|y|\} \text{ and } \|(x,y)\|_{\alpha,\beta} = \alpha|x| + \beta|y|.$$

The operator $L : \mathrm{Dom}(L) = \mathbb{R}^2 \to \mathbb{R}^2$ is given by the matrix

$$\begin{pmatrix} \beta & -\beta \\ -\alpha & \alpha \end{pmatrix}.$$

An easy computation shows that $\mathrm{Ker}(L)$ is spanned by the vector $(1,1)^T$ while $\mathrm{Im}(L)$ is the line of equation $\alpha x + \beta y = 0$, which coincides with $\widetilde{\mathcal{L}}$. Also,

$$(L + \lambda I)^{-1} = \frac{1}{\lambda(\alpha + \beta + \lambda)} \begin{pmatrix} \alpha + \lambda & \beta \\ \alpha & \beta + \lambda \end{pmatrix}$$

for every $\lambda \in \mathcal{R} = \mathbb{C} \setminus \{0, -(\alpha + \beta)\}$.

The maximum principle holds whenever this matrix has non-negative coefficients. Hence $\lambda_+ = \infty$ and $\lambda_- = -\min\{\alpha, \beta\}$. Since $\Sigma_- = -(\frac{\alpha+\beta}{2})$ we obtain the expected inequality $\Sigma_- \leq \lambda_-$ which is strict except in the case $\alpha = \beta$.

3 Some examples

3.1 Periodic solutions of ordinary differential equations

We discuss the maximum principle for problems of the type

$$\begin{cases} u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1 u' + \lambda u = f(t) \\ u \quad 2\pi\text{-periodic}, \end{cases} \tag{11}$$

where $a_{n-1}, a_{n-2}, \ldots a_1$ are real coefficients and f is 2π-periodic.

The space Ω is the quotient group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and μ is the associated Haar measure. We normalize the measure so that $\mu(\mathbb{T}) = 2\pi$. This just means that for any continuous function $\phi \in C(\mathbb{T})$,

$$\int_{\mathbb{T}} \phi d\mu = \int_0^{2\pi} \phi(s) ds.$$

The space $\mathcal{L} = L^1(\mathbb{T})$ is the usual space of periodic functions that are integrable over the period. The operator L will be taken as

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1 u', \quad u \in \mathrm{Dom}(L) = W^{n,1}(\mathbb{T}),$$

where $W^{n,1}(\mathbb{T})$ is the Sobolev space composed by 2π-periodic functions which are of class C^{n-1} and such that $u^{(n-1)}$ is absolutely continuous. To guarantee that $\mathrm{Ker}(L)$ only contains constant functions we must impose a condition on the characteristic polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda,$$

namely,

$$p(ki) \neq 0 \ \text{ for each } k \in \mathbb{Z} \setminus \{0\}. \tag{12}$$

To prove that

$$\mathrm{Im}(L) = \left\{ f \in \mathcal{L} \;\middle|\; \int_0^{2\pi} f(t) dt = 0 \right\}$$

we employ the Fredholm alternative. According to this principle the space $\mathrm{Im}(L)$ is composed by those functions f which are orthogonal (in the L^2-sense) to the 2π-periodic solutions of $L^\star u = 0$, where $L^\star$ is the adjoint operator

$$L^\star(v) := (-1)^n v^{(n)} + (-1)^{n-1} a_{n-1} v^{(n-1)} + \ldots + (-1) a_1 v'.$$

The only 2π-periodic solutions of the adjoint equation are again the constants. Indeed, the characteristic polynomial associated to $L^\star$ is $p^\star(\lambda) = p(-\lambda)$ and so condition (12) holds for $p^\star$ and p simultaneously. When $\widetilde{W}^{n,1}(\mathbb{T}) = \{u \in W^{n,1}(\mathbb{T}) : \overline{u} = 0\}$ and $\widetilde{L}^1(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \overline{f} = 0\}$ are endowed with their natural norms, so that they become Banach spaces, the map

$$\widetilde{u} \in \widetilde{W}^{n,1}(\mathbb{T}) \mapsto \widetilde{u}^{(n)} + a_{n-1}\widetilde{u}^{(n-1)} + \cdots + a_1\widetilde{u}' = \widetilde{f} \in \widetilde{L}^1(\mathbb{T})$$

is a continuous isomorphism. Therefore the inverse is also continuous and so the estimate $\|\widetilde{u}\|_{W^{n,1}} \leq K_1 \|\widetilde{f}\|_1$ holds. We are in dimension one and so the space $\widetilde{W}^{n,1}(\mathbb{T})$ is immersed in $\widetilde{C}(\mathbb{T}) = \{u \in C(\mathbb{T}) : \overline{u} = 0\}$, implying that $\|\widetilde{u}\|_\infty \le K_2 \|\widetilde{u}\|_{W^{n,1}}$. These two estimates lead to the $L^1 - L^\infty$ condition required for the applicability of Theorem 1 and we obtain maximum principles for λ positive or negative.

For $n = 1$ or 2, condition (12) always holds and the numbers λ_+ and λ_- can be computed in several ways. They are $\lambda_\pm = \pm\infty$ if $n = 1$ and $\lambda_+ = \frac{1+a_1^2}{4}$, $\lambda_- = -\infty$ if $n = 2$. Since every constant is explicit in this case, we can discuss the relationships with the spectrum of $-L$ and the numbers $\Sigma_\pm$. For $n = 1$ this spectrum is the sequence $\lambda_k = ik$ with $k \in \mathbb{Z}$. It lies on the imaginary axis and so $\Sigma_\pm = \pm\infty$. For $n = 2$ the spectrum becomes $\lambda_k = k^2 - ia_1k$, $k \in \mathbb{Z}$. A computation shows that $\Sigma_- = -\infty$, $\Sigma_+ = \frac{1+a_1^2}{2} = 2\lambda_+$.

The determination of the exact value of $\lambda_\pm$ is more delicate for $n \ge 3$ and we refer to [7, 8, 9, 10, 25, 26]. See also [11] for Neumann boundary conditions and [3, 31] for second order operators with variable coefficients.

3.2 Periodic solutions of difference equations

Let $\Delta u_m := u_{m+1} - u_m$ denote the forward difference operator, and, given the positive integer n, let us consider the difference equation with periodic boundary conditions

$$\Delta u_m + \lambda u_m = f_m \quad (0 \le m \le n-1), \quad u_0 = u_n, \tag{13}$$

where $\lambda \in \mathbb{R}$ and $f_m \in \mathbb{R}$ $(0 \le m \le n-1)$. Problem (13) is equivalent to the problem

$$\Delta u_m + \lambda u_m = f_m \quad (0 \le m \le n-2), \quad u_0 - u_{n-1} + \lambda u_{m-1} = f_{n-1}.$$

If Ω is the metrizable space composed of n points $A_0, A_2, \ldots A_{n-1}$, and μ the measure on Ω satisfying $\mu(\{A_m\}) = 1$ $(0 \le m \le n-1)$, then $\mathcal{C}$ can be identified with $\mathbb{R}^n = \{u = (u_0, \ldots, u_{n-1})\}$ endowed with the norm $\|u\|_\infty = \max_{0 \le m \le n-1} |u_m|$, and $\mathcal{L}$ can be identified with $\mathbb{R}^n$ endowed with the norm $\|u\|_1 = \sum_{m=0}^{n-1} |u_m|$. It is easy to see that the operator $L : \mathrm{Dom}(L) = \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$L(u) = (\Delta u_0, \Delta u_1, \ldots, \Delta u_{n-2}, u_0 - u_{n-1})$$

is such that

$$\mathrm{Ker}(L) = \{(c, \ldots, c) : c \in \mathbb{R}\}, \quad \mathrm{Im}(L) = \left\{ y \in \mathbb{R}^n : \sum_{m=0}^{n-1} y_m = 0 \right\}$$

so that Assumption (3) holds, with $\overline{u} := \frac{1}{n}\sum_{m=0}^{n-1} u_m$, and Assumption (4) is a direct consequence of the finite dimension of $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{L}}$. Thus, all conditions of Theorem 1 hold for (13).

To obtain more information about the corresponding λ_- and λ_+, notice that problem (13) is equivalent to the linear algebraic system in $\mathbb{R}^n$

$$u_{m+1} + (\lambda - 1)u_m = f_m \quad (0 \le m \le n-2), \quad u_0 + (\lambda - 1)u_{n-1} = f_{n-1}. \tag{14}$$

Such a system is uniquely solvable if and only if $(1-\lambda)^n \neq 1$, and a direct step-by-step computation gives, for those λ the solution

$$u_m = \frac{1}{1-(1-\lambda)^n}\left[\sum_{k=0}^{m-1}(1-\lambda)^{m-k-1} f_k + \sum_{k=m}^{n-1}(1-\lambda)^{n+m-k-1} f_k\right] \quad (0 \le m \le n-1).$$

Consequently, if $f_m \ge 0$ for all $0 \le m \le n-1$, we will have $u_m \ge 0$ for all $0 \le m \le n-1$ if $1-(1-\lambda)^n$ and all powers of $(1-\lambda)$ have the same sign. The only possible case is when they are both positive, i.e. when $0 < \lambda < 1$. Similarly, we will have $u_m \le 0$ for all $0 \le m \le n-1$ if $1-(1-\lambda)^n$ and all powers of $(1-\lambda)$ have opposite signs. The only possible case is where $1-(1-\lambda)^n \le 0$ and all powers of $(1-\lambda)$ are positive, i.e. when $\lambda < 0$. Thus for problem (13)

$$-\infty = \lambda_- < 0 < \lambda_+ = 1.$$

So we necessarily have $\Lambda_- = -\infty$. Now, the (complex) spectrum of $-L$ is the set of $\lambda \in \mathbb{C}$ such that $(1-\lambda)^n = 1$, and hence is made of the n eigenvalues

$$\lambda_m = 1 - e^{\frac{2\pi m i}{n}} \quad (0 \le m \le n-1),$$

which are located on the circle of center 1 and of radius 1. This implies that $\Lambda_+ = 1$.

Notice the difference between the periodic problem for difference equations (13) and the corresponding periodic problem for an ordinary differential equation

$$u' + \lambda u = f(t), \quad u(0) = u(2\pi) \tag{15}$$

considered in Subsection 3.1. We had found in this case

$$-\infty = \Lambda_- = \lambda_-, \quad \lambda_+ = \Lambda_+ = +\infty.$$

The difference is related to the nature of the spectra of the two problem : a purely imaginary one for (15), and a spectrum on the circle $\partial B(1,1)$ for (13).

3.3 A less standard two-point boundary value problem

We now derive a maximum principle for the boundary value problem

$$\begin{cases} u'''' + \lambda u = f(t) \\ u'(0) = u''(0) = u'(\pi) = u''(\pi) = 0. \end{cases} \tag{16}$$

To this end we consider the space $\Omega = [0, \pi]$ with the measure $d\mu = v(t)dt$, where $v(t) := t(\pi - t)$. This choice of μ leads to the space of integrable functions

$$\mathcal{L} = \left\{ f : [0, T] \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \int_0^\pi |f(s)|v(s)ds < \infty \right\}$$

with the norm

$$\|f\|_1 = \int_0^\pi |f(t)|v(t)dt.$$

The operator $Lu = u''''$ is defined on the space $\mathrm{Dom}(L)$ composed by those functions $u \in C^3[0, \pi]$ such that

$$u'(0) = u''(0) = u'(\pi) = u''(\pi) = 0$$

and the fourth derivative u'''' belongs to $\mathcal{L}$. This derivative is understood in the sense of distributions. It is easy to check that

$$\mathrm{Ker}(L) = \{\text{constant functions}\}.$$

To compute the range we split the problem

$$Lu = f, \;\; u \in \mathrm{Dom}(L), \;\; f \in \mathcal{L}$$

in two second order problems. First we observe that $z = u''$ satisfies

$$\begin{cases} z'' = f(t) \\ z(0) = z(\pi) = 0. \end{cases}$$

This Dirichlet problem has the unique solution $u(t) = \int_0^\pi G(t, s)f(s)ds$ with

$$G(t, s) = \begin{cases} \frac{(s-\pi)t}{\pi} & 0 < t \leq s \\ \frac{(t-\pi)s}{\pi} & s < t < \pi \end{cases}$$

Notice that $|G(t, s)| \leq \frac{v(s)}{\pi}$ and so the integral makes sense if $f \in \mathcal{L}$.

Next we must solve the Neumann problem

$$\begin{cases} u'' = z(t) \\ u'(0) = u'(\pi) = 0. \end{cases}$$

This is solvable if and only if

$$0 = \int_0^\pi z(t)dt = \int_0^\pi f(s) \int_0^\pi G(t,s)dtds = -\frac{1}{2}\int_0^\pi f(s)v(s)ds.$$

From here we deduce that $\mathrm{Im}(L)$ is precisely $\widetilde{\mathcal{L}}$. When $f \in \widetilde{\mathcal{L}}$ the Neumann problem has a continuum of solutions but only one of them lies in $\widetilde{\mathcal{C}}$. This fact leads to the unique solvability of

$$L\widetilde{u} = \widetilde{f}, \quad \widetilde{u} \in \widetilde{\mathcal{C}}, \quad \widetilde{f} \in \widetilde{\mathcal{L}}.$$

The above discussions imply in particular that $\|z\|_\infty \leq \frac{1}{\pi}\|f\|_1$ and from here it is easy to arrive at the estimate $\|\widetilde{u}\|_\infty \leq K\|\widetilde{f}\|_1$. We are in the conditions of Theorem 1 and so we have obtained a maximum principle for (16).

3.4 The polyharmonic operator with Neumann boundary conditions

The classical maximum principles for the Laplace operator cannot be proved using our abstract setting. To illustrate the reasons for this obstruction we consider the Neumann problem

$$\begin{cases} \Delta u + \lambda u = f(x), & x \in D \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \end{cases} \tag{17}$$

where $D \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain of class C^2. From [29], the maximum principle holds for $\lambda < 0$ but, as shown in [15], it is not valid for any $\lambda > 0$ (no UAMP). That is, $\lambda_- = -\infty, \lambda_+ = 0$, a situation which cannot occur when Theorem 1 is applicable. Indeed, if we would try to apply our abstract setting we would be stopped after observing that there is no $L^1 - L^\infty$ regularity for the problem

$$\begin{cases} \Delta u = \tilde{f}(x), & x \in D \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{18}$$

Given $\tilde{f} \in L^1(D)$ with $\int_D \tilde{f} = 0$, there exists a unique solution with $\int_D \tilde{u} = 0$. In general this solution will not be continuous but it belongs to $L^q(D)$ with

$q < \frac{N}{N-2}$. This can be proved using the duality method as employed in the final remarks of Chapter IX in [6]. This method also leads to the estimate $||\tilde{u}||_q \leq K||\tilde{f}||_1$. By a solution of (18) we understand a function $u \in L^q(D)$ such that

$$\int_D u\Delta\phi = \int_D \tilde{f}\phi$$

for each $\phi \in C^\infty(\overline{D})$ with $\frac{\partial \phi}{\partial \eta} = 0$ on ∂D.

Assume now that k is an integer with

$$2k > N$$

and consider the problem

$$\begin{cases} \Delta^k u = \tilde{f}(x), & x \in D \\ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = ... = \frac{\partial}{\partial n}\Delta^{k-1}u = 0 & \text{on } \partial D \end{cases}$$

where $\Delta^k = \Delta \circ \ldots^{(k)} \circ \Delta$ is the polyharmonic operator. This problem is equivalent to the system

$$\begin{cases} \Delta v_i = v_{i+1}, & \text{in } D, \quad i = 0, 1, \ldots, k-1 \\ \frac{\partial v_i}{\partial n} = 0 & \text{on } \partial D, \quad i = 0, 1, \ldots, k-2 \\ v_0 = u, \quad v_k = \tilde{f}. & \end{cases}$$

Now it is easy to prove that if $\tilde{f} \in \tilde{L}^1(D)$ then there exists a unique solution $\tilde{u} \in \tilde{C}(\overline{D})$ with $||\tilde{u}||_\infty \leq K||\tilde{f}||_1$. To prove this we select q satisfying

$$\frac{N}{N-2} > q, \quad \frac{1}{q} - \frac{2(k-1)}{N} < 0.$$

The second inequality is imposed to guarantee that a certain Sobolev space is contained in $C(\overline{D})$. From the previous discussion for (18) we know that $v_{k-1} \in L^q(D)$ and, by standard elliptic regularity, $u \in W^{2(k-1),q}(D) \subset C(\overline{D})$. This discussion shows that Theorem 1 can be applied to the problem below when $k > 2N$,

$$\begin{cases} \Delta^k u + \lambda u = f(x), & x \in D \\ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = ... = \frac{\partial}{\partial n}\Delta^{k-1}u = 0 & \text{on } \partial D \end{cases}$$

In this case $\Omega = \overline{D}$, μ is the Lebesgue measure and $Lu = \Delta^k u$. The definition of the domain of L is induced by the formulation of the equation as a system and the duality method. Namely, $\text{Dom}(L)$ is composed by those functions $u \in C(\overline{D})$ such that there exist $v_0 = u, v_1, \ldots, v_k \in L^1(D)$ satisfying

$$\int_D v_i \Delta\phi = \int_D v_{i+1}\phi, \quad I = 0, 1, \ldots, k-1$$

for each $\phi \in C^\infty(\overline{D})$ with $\frac{\partial \phi}{\partial \eta} = 0$ on ∂D. Notice that $Lu = u_k$.

Maximum or anti-maximum principles for the polyharmonic operator and Dirichlet boundary conditions have been obtained in [13, 14, 19, 20, 22].

3.5 Some hyperbolic equations on a torus

The space Ω is here the two dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with the associated Haar measure μ. It is assumed that $\mu(\mathbb{T}^2) = (2\pi)^2$ and generic points on the torus are denoted by $(\bar{t}, \bar{x})$ with $\bar{t} = t + 2\pi\mathbb{Z}$, $\bar{x} = x + 2\pi\mathbb{Z}$.

We consider the differential operator

$$Lu = \partial_t^2 u + c\partial_t u + (-1)^\gamma \partial_x^{2\gamma} u$$

acting on doubly periodic functions $u : \mathbb{T}^2 \to \mathbb{R}$, $u = u(t, x)$, $c > 0$, and $\gamma = 1, 2, ...$ This operator is related to the model of telegraph transmission for $\gamma = 1$ and to the vibration of beams for $\gamma = 2$. The derivatives of u are understood as distributions on the torus so that

$$< Lu, \phi >= \int_{\mathbb{T}^2} u L^\star \phi$$

for each $\phi \in \mathcal{D}(\mathbb{T}^2) = C^\infty(\mathbb{T}^2)$, where $L^\star = \partial_t^2 - c\partial_t u + (-1)^\gamma \partial_x^{2\gamma}$ is the adjoint operator. We consider L as an operator from $\mathcal{C} = C(\mathbb{T}^2)$ to $\mathcal{L} = L^1(\mathbb{T}^2)$ with

$$\mathrm{Dom}(L) = \{u \in C(\mathbb{T}^2) \mid Lu \in L^1(\mathbb{T}^2)\}.$$

We claim that all the conditions of the abstract setting hold and so we obtain maximum principles for the doubly periodic solutions of

$$\partial_t^2 u + c\partial_t u + (-1)^\gamma \partial_x^{2\gamma} u + \lambda u = f(t, x) \text{ in } \mathcal{D}'(\mathbb{T}^2)$$

whenever $\lambda \in [\lambda_-, \lambda_+] \setminus \{0\}$. For $\gamma = 1$ this is the maximum principle found in [27]. The result seems to be new for $\gamma = 1$ and $\lambda \in [\lambda_-, 0)$ or for $\gamma \geq 2$.

To justify that L satisfies the conditions of the abstract setting we will assume that $\gamma \geq 2$. The case $\gamma = 1$ follows from Proposition 4.4 in [27]. First we state an auxiliary result for equation

$$\partial_t^2 u + c\partial_t u + (-1)^\gamma \partial_x^{2\gamma} u + \frac{c^2}{4} u = f(t, x) \text{ in } \mathcal{D}'(T^2) \tag{19}$$

Lemma 4 *For each $f \in L^1(\mathbb{T}^2)$ equation (19) has a unique solution u in $C(\mathbb{T}^2)$. Moreover the resolvent operator*

$$R : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2),\ f \to u$$

is compact if $\gamma \geq 2$.

Notice that in the abstract setting the map R corresponds to the resolvent $R_{\frac{c^2}{4}}$.

We postpone the proof of this lemma, and first discuss how to apply it. By selecting the test function $\phi(t,x) = e^{i(nt+mx)}$ it is easy to conclude that

$$\mathrm{Ker}(L) = \{\text{constant functions}\}$$

The choice $\phi \equiv 1$ lead to

$$\mathrm{Im}(L) \subset \widetilde{\mathcal{L}}.$$

It remains to prove that this inclusion is indeed an equality and that the generalized inverse is continuous as a map from $\mathcal{L}$ to $\mathcal{C}$. To this end we observe that the equation $Lu = f$ can be rewritten as $Lu + \frac{c^2}{4}u = f + \frac{c^2}{4}u$ and, after an application of Lemma 4 we obtain the equivalence

$$Lu = f \Longleftrightarrow \left(I - \frac{c^2}{4}R\right) u = Rf.$$

With the symbol R we also indicate the restriction of R to $\mathcal{C}$. This restriction becomes a compact endomorphism of $\mathcal{C}$. The previous discussion implies that

$$\mathrm{Ker}\left(I - \frac{c^2}{4}R\right) = \{\text{constant functions}\},\ \mathrm{Im}\left(I - \frac{c^2}{4}R\right) \subset \widetilde{\mathcal{C}}.$$

To obtain the last inclusion we observe that $R(\widetilde{\mathcal{L}}) \subset \widetilde{\mathcal{C}}$. Since R is compact we notice that $I - \frac{c^2}{4}R$ is a Fredholm operator of zero index. This implies that $\mathrm{Im}(I - \frac{c^2}{4}R) = \widetilde{C}$ and so $\mathrm{Im}(L) = \widetilde{\mathcal{C}}$. The continuity of the generalized inverse can be deduced as a consequence of the general theory of compact operators.

It is possible to obtain some upper estimates of the numbers $|\lambda_-|$ and λ_+. The spectrum of L is given by

$$\sigma(L) = \{n^2 - m^{2\gamma} + cin \mid (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

implying that $-\frac{1}{2} \leq \lambda_- < 0 < \lambda_+ \leq \frac{c^2+1}{2}$.

We conclude with the

Proof of Lemma 4. The uniqueness follows from Fourier analysis. Again one uses $e^{i(nt+mx)}$ as test functions. To prove the existence of a solution and the compactness of the resolvent we construct a fundamental solution of $L+\frac{c^2}{4}I$ on the torus. This is a function $U \in C(\mathbb{T}^2)$ satisfying

$$LU + \frac{c^2}{4}U = \delta \text{ in } \mathcal{D}'(\mathbb{T}^2).$$

Here $\delta \in D'(\mathbb{T}^2)$ is the Dirac measure concentrated at $(\overline{0}, \overline{0})$ that is

$$< \delta, \phi >= \phi(0,0) \text{ for each } \phi \in \mathcal{D}(\mathbb{T}^2).$$

Once the function U has been constructed the solution of $\left(L + \frac{c^2}{4}I\right) u = f$ can be expressed as a convolution on $\mathbb{T}^2$, namely

$$u(t,x) = (U * f)(t,x) = \int_{\mathbb{T}^2} U(t-\tau, x-\xi) f(\tau,\xi) d\tau d\xi$$

The compactness of $\mathcal{R}f = U * f$ is now a consequence of Ascoli-Arzelá's theorem.

After this discussion it remains to construct U. It is defined as the uniformly convergent series $U = \sum_{n=0}^{\infty} U_n$, where $U_n \in C(\mathbb{T}^2) \cap C^2([0,\pi]\times\mathbb{R})$ satisfies

$$LU_n + \frac{c^2}{4}U_n = 0 \text{ on } [0,2\pi]\times\mathbb{R}$$

and

$$\frac{\partial U_n}{\partial t}(0^+,x) - \frac{\partial U_n}{\partial t}(2\pi^-,x) = \begin{cases} \frac{1}{2\pi} & \text{if } n=0 \\ \frac{1}{\pi}\cos nx & \text{if } n>1. \end{cases}$$

Assuming by now that the function U_n has been constructed we explain why U is the fundamental solution. Integration by parts leads to

$$\int_{\mathbb{T}^2} \left(L^*\phi + \frac{c^2}{4}\phi\right) U_n = \int_0^{2\pi} \left(\frac{\partial U_n}{\partial t}(0^+,x) - \frac{\partial U_n}{\partial t}(2\pi^-,x)\right) \phi(0,x)dx$$

and so

$$\begin{aligned} \int_{\mathbb{T}^2} \left(L^*\phi + \frac{c^2}{4}\phi\right) U &= \frac{1}{2\pi}\int_0^{2\pi} \phi(0,x)dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} \phi(0,x)\cos nx dx \\ &= \phi(0,0) \end{aligned}$$

The last identity follows from the Fourier expansion of $\phi(0,\cdot)$.

Finally we give an explicit formula for U_n. For $n \geq 1$, U_n is the periodic extension of

$$U_n(t,x) = \frac{e^{-ct/2}}{\pi(1-e^{-c\pi})} \frac{\sin n^\gamma t \cos nx}{n^\gamma}, \ (t,x) \in [0,2\pi]\times\mathbb{R}.$$

For $n = 0$ the function U_0 does not depends on x and is defined as the periodic extension of solution of the boundary value problem

$$\begin{cases} u'' + cu' + \frac{c^2}{4}u = 0, \ t \in [0, 2\pi] \\ u(0) = u(2\pi), u'(0^+) = u'(2\pi^-) + \frac{1}{2\pi}. \end{cases}$$

Namely,

$$U_0(t) = \left[\frac{e^{-c\pi}}{(1 - e^{-c\pi})^2} + \frac{t}{2\pi(1 - e^{-c\pi})} \right] e^{-ct/2}, \ t \in [0, 2\pi].$$

■

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