Fastness and continuous dependence in front propagation in Fisher-KPP equations

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Dedicated to Paolo Marcellini on the occasion of his 60th birthday

Abstract

We investigate the continuous dependence of the minimal speed of propagation and the profile of the corresponding travelling wave solution of Fisher-type reaction-diffusion equations

$$\vartheta_t = (D(\vartheta)\vartheta_x)_x + f(\vartheta)$$

with respect to both the reaction term f and the diffusivity D.

We also introduce and discuss the concept of fast heteroclinic in this context, which allows to interpret the appearance of sharp heteroclinic in the case of degenerate diffusivity ($D(0) = 0$).

1 Introduction

It is well known that travelling wave solutions play an important role in the study of the asymptotic behavior of the solutions of initial value problems for evolution equations.

Keywords and phrases: reaction-diffusion equations, travelling wave solutions, wave speed, fast heteroclinic, continuous dependence, sharp solutions, degenerate diffusivity.

Mathematics subject classification: primary 35K57, secondary 34B40

[1] Partially supported by FQM 1268, Junta de Andalucía, Spain.

[2] Partially supported by DGIMTM 2005-03483, M.E.C., Spain.

In the last years a large number of papers appeared concerning the study of travelling wave solutions for reaction-diffusion equations of the form

$$\vartheta_t = (D(\vartheta)\vartheta_x)_x + f(\vartheta) \tag{RD}$$

where f is a so-called Fisher-KPP reaction term, i.e. a Lipschitz function $f : [0,1] \to \mathbb{R}$ satisfying $f(0) = f(1) = 0$ and $f(s) > 0$ for $s \in (0,1)$, and the diffusion term $D(s)$ is a C^1-function on $[0,1]$ with $D(s) > 0$ for $s \in]0,1]$. We refer to the monographs [3], [6] and [11] and the references there included.

Recall that a travelling wave solution (t.w.s.) for (RD) is a solution having a constant profile moving with a constant speed, i.e. a solution of the equation of the form $\vartheta(t,x) = u(x - ct)$ for some constant c. The function u is called the *profile* of the wave and the constant c is the *wave speed*.

The profile having speed c, connecting the stationary states 1 and 0, is a solution of the boundary value problem

$$\begin{cases} (D(u)u')' + cu' + f(u) = 0 \\ u(-\infty) = 1, \ u(+\infty) = 0 \end{cases} \tag{1}$$

where the constant c is a further unknown of the problem. A solution of (1) is usually called a *heteroclinic solution*.

When $D(0) > 0$ one says that (RD) is non-degenerate. As it is well-known (see, for instance, [2]), there exists a positive number $c^* := c^*(D,f)$ such that the boundary value problem (1) admits a solution if and only if $c \geq c^*$. Moreover, when $f'(0)$ exists, c^* satisfies the estimate

$$2\sqrt{D(0)f'(0)} \leq c^* \leq 2\sqrt{\sup_{s\in(0,1]} \frac{f(s)D(s)}{s}}. \tag{2}$$

When $D(0) = 0$ one says that (RD) is degenerate at zero. The treatment of the problem is a bit more complicated in this case but it is again possible to prove that there exists a solution of (1), i.e. a t.w.s. connecting the two stationary states having speed c, whenever $c > c^*$ and no solution exists if $c < c^*$. When $c = c^*$ a profile in the classical sense does not exist. Instead of it, a different kind of solution appears, called *sharp-type* heteroclinic, which reaches the equilibrium $u = 0$ at a finite time (see Section 5 for a precise definition). This type of solutions were analyzed in [6], [9].

The threshold value c^*, usually called *minimal speed of propagation*, and the profile u^* moving with speed c^* play a fundamental role since the solutions of the initial value problem for equation (RD) tend, in some sense, to u^* for large times (see [6], [7]). So, in the degenerate case the dynamic is usually said to exhibit the phenomenon of *finite speed of propagation*, since if the initial datum has compact support, then the solution of (RD) maintains compact support at any time (see [6] for a discussion on this matter).

In [1] a variational study of t.w.s. has been carried out in the case of constant diffusivity ($D(s) \equiv 1$). In particular the authors discussed the fastness of the decay at 0 of t.w.s., distinguishing between *fast solution* (those whose profile u belongs to the weighted Sobolev space $H^1(e^{ct})$), and the non-fast ones. The fast heteroclinic are minimizers of certain functionals. In such a paper it was proved that fast t.w.s. can appear only if $c = c^*$ and actually this occurs when $c^* > 2\sqrt{f'(0)}$. The question about the possible fastness of t.w.s. when $c^* = 2\sqrt{f'(0)}$ remained open.

The first aim of this paper is to investigate the continuous dependence on f and D of the minimal speed c^* and the profile u^* having speed c^*. More precisely, we show that taking a sequence of non-degenerate diffusivities $\{D_n\}_{n\geq 0}$, uniformly convergent in $[0,1]$ to D_0, and a sequence $\{f_n\}_{n\geq 0}$ such that $\{\frac{f_n(s)}{s}\}_{n\geq 0}$ uniformly converges to $\frac{f_0(s)}{s}$ on $(0,1]$, then $c^*(f_n, D_n) \to c^*(f_0, D_0)$ (see Corollary 13) and the corresponding sequence of heteroclinic $\{u^*_n\}_n$ converges (up to shifts) to u^*_0 uniformly on $\mathbb{R}$ and also in $C^2(\mathbb{R})$, endowed with the usual topology of the uniform convergence on compact sets of the first two derivatives (see Theorem 14). We refer to [4] for a study of the continuous dependence (and further regularity) of the minimal wave speed in a the special case $f(u) = u^m(1-u)$, $m \geq 1$. Moreover we also mention the paper [8] where the continuous dependence was established for bistable reaction-diffusion equations, that is for changing-sign reaction terms (note that in this case there is a unique admissible speed, instead of infinitely many speeds as in the Fisher-case).

A second aim for this research is to introduce and investigate the concept of fast heteroclinic in the case of non-constant diffusion. More in detail, in the case of non-degenerate diffusivity ($D(s) > 0,\ s \in [0,1]$), we show how in this context it is natural to define as *fast* any t.w.s. whose profile belongs to the space $H^1(e^{\frac{c}{D(0)}t})$ and we prove that such a t.w.s. exists whenever $c^*(D,f) > 2\sqrt{D(0)f'(0)}$ (see Corollary 17). Moreover, we tackle the question about the possible fastness of the t.w.s. when $c^* = 2\sqrt{D(0)f'(0)}$, showing that there is no fast t.w.s. both when f and D are sufficiently smooth (see Corollary 19), and when $D(u)f(u) \leq D(0)f'(0)u,\ u \in [0,1]$ (see Corollary 18).

The growth at $+\infty$ of the weight function $e^{\frac{c}{D(0)}t}$ becomes greater and greater as the value $D(0)$ approaches 0. So it naturally arises the question of how to interpret the appearance of sharp t.w.s. in the degenerate case (i.e. when $D(0) = 0$). To this purpose, we show that taking a sequence of non-degenerate diffusion terms $\{D_n\}_n$ uniformly convergent to a degenerate one D_0, then the corresponding sequence of heteroclinic solutions, moving with the corresponding minimal speed, converges to the sharp heteroclinic solution relative to D_0 in the space $H^1(e^{\alpha t})$ for every $\alpha > 0$. Roughly speaking, one can say that the t.w.s. become faster and faster till to converge, in such a Sobolev space, to a function definitely identically null from a time on (see Theorem 24).

The paper is divided in six sections. After introducing some notations and preliminary results in Section 2, in Section 3 we deal with the constant diffusion case. Sections 4 and 5 are devoted to the cases of non-constant diffusivity. Lastly, in Section 6 we use the variational setting to prove Theorem 8.

Acknowledgment: The authors would like to thank Rafael Ortega for pointing out the uniform convergence of the profiles when the speed is fixed.

2 Notations and preliminary results

We will denote by $BC(\mathbb{R})$ the space of the continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with the topology of the uniform convergence, and by $C^n(\mathbb{R})$ the usual space of continuous n^{th}-times differentiable functions endowed with the usual topology of the uniform convergence on compact sets of the first n^{th}-derivatives.

Let $\mathcal{F}$ denote the space of Lipschitz functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, $f(s) > 0$, $s \in (0, 1)$, and such that $f'(0)$ exists (finite), continued as null functions to the whole real line, endowed with the following topology:

$$f_n \to f_0 \text{ in } \mathcal{F} \quad \Leftrightarrow \quad \frac{f_n(s)}{s} \to \frac{f_0(s)}{s} \text{ uniformly in } (0, 1].$$

Of course, if $f_n \to f_0$ in $\mathcal{F}$, then $f_n \to f_0$ uniformly in $\mathbb{R}$.

By $H^1(e^{ct})$, $c > 0$, we will denote the weighted Sobolev space

$$H^1(e^{ct}) = \{u \in H^1_{loc}(\mathbb{R}) : \ e^{\frac{ct}{2}} u \in H^1(\mathbb{R})\},$$

endowed with the norm $\|u\| := \left(\int_{-\infty}^{+\infty} e^{ct} u'(t)^2 \, \mathrm{d}t\right)^{1/2}$. This norm is equivalent to the usual one as a consequence of a Poincaré type inequality:

$$\int_{-\infty}^{+\infty} e^{ct} u'(t)^2 \, \mathrm{d}t \geq \frac{c^2}{4} \int_{-\infty}^{+\infty} e^{ct} u(t)^2 \, \mathrm{d}t, \quad u \in H^1(e^{ct}). \tag{3}$$

Observe that the inclusion $H^1(e^{ct}) \subset L^2(e^{ct})$ is not compact. These spaces were studied in [1].

Given $f \in \mathcal{F}$, a heteroclinic solution of the equation

$$u'' + cu' + f(u) = 0 \tag{4}$$

is a solution of (4), satisfying the boundary conditions

$$\lim_{t\to-\infty} u(t) = 1 \quad \text{and} \quad \lim_{t\to+\infty} u(t) = 0.$$

The following result is well known.

Proposition 1 *Let* $0 \le u(t) \le 1$ *be a non-constant solution of equation* (4)*. Then* u *is a heteroclinic solution satisfying* $u'(t) < 0$*, for every* $t \in \mathbb{R}$*.*

It is also known that if a heteroclinc solution of (4) exists, it is unique up to a time-shift. Therefore when we fix $t_0 \in \mathbb{R}$ and $s_0 \in (0,1)$ then the heteroclinic solution satisfying $u(t_0) = s_0$, if it exists, is unique.

About the existence, it is known that there exists a threshold value $c^* = c^*(f) > 0$ such that (4) admits heteroclinic solutions if and only if $c \ge c^*$. Moreover,

$$2\sqrt{f'(0)} \le c^* \le 2\sqrt{\sup_{s\in(0,1]} \frac{f(s)}{s}}$$

(see [2] or [9]).

A solution u of (4) is called *fast* if $u \in H^1(e^{ct})$. The following result obtained in [1] concerns a variational characterization of the minimal speed.

Theorem 2 [1, Theorem 12] *Let* c^* *be the minimal speed for* (4)*. Then*

$$\frac{1}{(c^*)^2} = \inf\left\{\int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t : \ u \in H^1(e^t),\ \int_{-\infty}^{+\infty} e^t F(u(t))\,\mathrm{d}t = 1\right\}, \tag{5}$$

where $F(u) := \int_0^u f(s)\,\mathrm{d}s$*.*

Moreover, u *is a minimizer for* (5) *if and only if the function* $\tilde{u}(t) := u(c^*t)$ *is a fast heteroclinic for* (4) *with* $c = c^*$*.*

Condition $\int_{-\infty}^{+\infty} e^t F(u(t))\,\mathrm{d}t = 1$ in (5) is a sort of normalization. Indeed, if $u \in H^1(e^t)$ then $\int_{-\infty}^{+\infty} e^t F(u(t))\ \mathrm{d}t = a \in \mathbb{R}^+$. Putting $v(t) := u(t + \ln a)$, $v \in H^1(e^t)$, then $\int_{-\infty}^{+\infty} e^t F(v(t))\,\mathrm{d}t = 1$ and $\|v\| = \frac{\|u\|}{\sqrt{a}}$. Hence (5) can be rewritten as

$$\frac{1}{(c^*)^2} = \inf\left\{\frac{\int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t}{\int_{-\infty}^{+\infty} e^t F(u(t))\,\mathrm{d}t} \ : \ u \in H^1(e^t), u \ne 0.\right\} \tag{6}$$

Finally, let us observe that a fast solution has to be integrable in $[0, +\infty)$ as a consequence of Hölder's inequality.

The following result concerns the existence of fast heteroclinic solutions and summarizes some of the results proved in [1] (see Proposition 11 and Lemma 13)

Theorem 3 [1] *If* $c^* > 2\sqrt{f'(0)}$ *then the heteroclinic of* (4) *for* $c = c^*$ *is fast and*

$$\lim_{t \to +\infty} \frac{u'(t)}{u(t)} = -\frac{1}{2}\left(c^* + \sqrt{(c^*)^2 - 4f'(0)}\right).$$

Vice versa, if (4) *has a fast heteroclinic, then* $c = c^*$.

The next two Lemmas are technical results which will be used later. The first one gives an a priori bound for the heteroclinic solution of (4).

Lemma 4 *Let* u *be a heteroclinic solution of* (4). *Then,* $0 < -u'(t) \leq cu(t)$, *for every* $t \in \mathbb{R}$.

Proof. Define $y(t) := -\frac{u'(t)}{u(t)}$, $t \in \mathbb{R}$. Then, y is a positive solution of the equation

$$y' = -cy + y^2 + \frac{f(u(t))}{u(t)},$$

defined on the whole real line. Being $\frac{f(s)}{s} > 0$, y is a supersolution of the equation

$$y' = y(y - c)$$

and then, $0 < y(t) \leq c$ for every $t \in \mathbb{R}$, since any solution going over c blows up in a finite time. ■

The second lemma is a tool to obtain the uniform convergence from the convergence in some points.

Lemma 5 *Let* $\{w_n\}_{n \geq 0}$, $w_n : \mathbb{R} \to [0, 1]$, *be a sequence of continuous decreasing functions satisfying*

$$\lim_{t \to -\infty} w_n(t) = 1 \quad \text{and} \quad \lim_{t \to +\infty} w_n(t) = 0, \qquad n \geq 0.$$

Assume that $w_n(t) \to w_0(t)$ *for* t *in a dense subset of the interval* $(\alpha_0, \beta_0) := \{t \in \mathbb{R} \;/\; 0 < w_0(t) < 1\}$. *Then* $w_n \to w_0$ *uniformly on* $\mathbb{R}$.

Proof. We only have to prove that for every sequence $\{t_n\}_n \in \mathbb{R}$

$$w_n(t_n) - w_0(t_n) \to 0.$$

To this aim assume, by contradiction, the existence of a sequence $\{t_n\}_n$ such that

$$|w_n(t_n) - w_0(t_n)| > \varepsilon_0 > 0 \quad \text{for every } n \in \mathbb{N}, \tag{7}$$

for some ε_0 fixed. Taking a subsequence, we can assume that $t_n \to t_0 \in \overline{\mathbb{R}}$.

If $t_0 \geq \beta_0$ we can take $t^+ \in (\alpha_0, \beta_0)$ such that $w_0(t^+) < \frac{\varepsilon_0}{2}$ and $w_n(t^+) \to w_0(t^+)$. Then for n large enough we have $t_n > t^+$ and

$$|w_n(t_n) - w_0(t_n)| \leq w_n(t_n) + w_0(t_n) \leq w_n(t^+) + w_0(t^+) \to 2w_0(t^+) < \varepsilon_0,$$

in contradiction with (7). A similar argument can be applied if $t_0 \leq \alpha_0$.

Finally, if $\alpha_0 < t_0 < \beta_0$ then we take $t^- < t_0 < t^+$ with $w_0(t^-) - w_0(t^+) < \varepsilon_0$ and $w_n(t^\pm) \to w_0(t^\pm)$, respectively. For n large enough, we have $t_n \in (t^-, t^+)$ and then

$$w_n(t_n) - w_0(t_n) \leq w_n(t^-) - w_0(t^+) \to w_0(t^-) - w_0(t^+),$$

and

$$w_0(t_n) - w_n(t_n) \leq w_0(t^-) - w_n(t^+) \to w_0(t^-) - w_0(t^+).$$

Hence

$$\limsup |w_n(t_n) - w_0(t_n)| \leq w_0(t^-) - w_0(t^+) < \varepsilon_0,$$

in contradiction with (7). ■

3 Constant diffusion case

In this section we will deal with the case $D(s) \equiv 1$.

Firstly we discuss the possible fastness of the heteroclinic solutions of (4) for $c = c^*$ in the case $c^* = 2\sqrt{f'(0)}$. The following result provides a sufficient condition in order to assert the non-existence of fast heteroclinic.

Theorem 6 *Let $f \in \mathcal{F}$ be such that*

$$f(u) \leq f'(0)u \tag{8}$$

for every $u \in [0,1]$. Then $\frac{1}{(c^*)^2} = \frac{1}{4f'(0)}$ in (6) is not a minimum and there is no fast heteroclinic solution.

Proof. Suppose by contradiction that $\frac{1}{(c^*)^2} = \frac{1}{4f'(0)}$ is a minimum and let $u \in H^1(e^t)$ be a minimizer.

Assumption (8) implies that $2F(u) \le f'(0)u^2$ (where recall that $F(u) = \int_0^u f(s)\mathrm{d}s$), so by (3) we get

$$\frac{1}{4f'(0)} = \frac{\int_{-\infty}^{+\infty} e^t u'(t)^2\,\mathrm{d}t}{2\int_{-\infty}^{+\infty} e^t F(u(t))\,\mathrm{d}t} \ge \frac{\int_{-\infty}^{+\infty} e^t u'(t)^2 \mathrm{d}t}{f'(0)\int_{-\infty}^{+\infty} e^t u(t)^2\,\mathrm{d}t} \ge \frac{1}{4f'(0)}.$$

Then, all the inequalities in the previous expression are actually identities and

$$2F(u(t)) = f'(0)u(t)^2, \quad \forall t \in \mathbb{R}.$$

Since $u'(t) < 0$, differentiating we obtain

$$f(u(t)) = f'(0)u(t), \quad \forall t \in \mathbb{R}.$$

Then, by Theorem 2, u is a solution of the linear equation $u'' + u' + \frac{1}{4}u = 0$ for $t \ge 0$. Hence, u has the following analytic expression:

$$u(t) = e^{-\frac{1}{2}t}\left\{u(0) + (\frac{1}{2}u(0) + u'(0))t\right\}, \quad t \ge 0.$$

Therefore, one can easily verify that $e^t u^2(t)$ is not integrable in $\mathbb{R}$, that is, $u \notin H^1(e^t)$, which is a contradiction since u is a minimizer in (6).

■

Remark. Condition (8) is not optimal, indeed in view of the previous proof, it is enough to assume that

$$2F(u) \le f'(0)u^2, \quad t \in [0,1] \tag{9}$$

in order to obtain the conclusion of Theorem 6. However, neither this condition is optimal, since when the term f is sufficiently smooth, it is not necessary, as the following result states.

Theorem 7 *Suppose f is C^2 and $c = 2\sqrt{f'(0)}$. Then there is no fast heteroclinic solutions of equation* (4).

Proof. Split

$$f(u) = \frac{c^2}{4}u + u\,h(u)$$

where h is a C^1 function with $h(0) = 0$.

Assume that u is a fast heteroclinic solution of (4) and put $u(t) = v(t)e^{-\frac{c}{2}t}$. Then, $v \in L^2(\mathbb{R})$ and it verifies

$$v''(t) + h(u(t))v(t) = 0. \tag{10}$$

Now we will use the exponential decay of u. By Theorem 3, $\frac{u'(t)}{u(t)} \to -\frac{c}{2}$. So, for t large enough and some $0 < \beta < \frac{c}{2}$, u is a subsolution of the equation $x' = -\beta x$. Since it is positive, $u(t) \leq ke^{-\beta t}$ for some constant $k > 0$. Having in mind (10), also $\frac{v''}{v}$ has exponential decay, that is,

$$\frac{v''(t)}{v(t)} \leq ke^{-\beta t}$$

for some constant $k, \beta > 0$ (possibly different from the previous ones) and $t \geq 0$.

Putting $r(t) := -\frac{v'(t)}{v(t)}$, we have

$$r'(t) = r^2(t) - \frac{v''(t)}{v(t)} \geq r^2(t) - ke^{-\beta t}.$$

If $r(t_0) > \sqrt{k}e^{-\frac{\beta}{2}t_0}$ for some $t_0 > 0$, then r verifies for $t \geq t_0$

$$r'(t) \geq r^2 - \alpha^2, \quad r(t_0) > \alpha,$$

where $\alpha = \sqrt{k}e^{-\frac{\beta}{2}t_0}$. Therefore r reaches ∞ in a finite time. So $\dfrac{v'(t)}{v(t)} \geq -\sqrt{k}e^{-\frac{\beta}{2}t}$ and integrating between 0 and t we deduce $\ln\left(\dfrac{v(t)}{v(0)}\right) \geq -\sqrt{k}\dfrac{2}{\beta}$, that is,

$$v(t) \geq v(0)e^{-\frac{2\sqrt{k}}{\beta}},$$

which is in contradiction with $v \in L^2(\mathbb{R})$. ∎

Remark. The previous result is still true when $f \in \mathcal{F}$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$.

Now we analyze the continuous dependence on f of the minimal speed $c^*(f)$. The proof of the following theorem needs some further results involving the variational structure introduced in [1] and will be carried out in Section 6.

Theorem 8 *Let* $\{f_n\}_{n \geq 0} \in \mathcal{F}$ *be such that* $f_n \to f_0$ *in* $\mathcal{F}$. *Then*

$$c^*(f_n) \to c^*(f_0).$$

Another question is the continuous dependence of the profiles. Since they are unique up to a time-shift, we will prove the continuity of the profiles up a normalization:

Theorem 9 *Let $\{f_n\}_{n\geq 0}$ be as in Theorem 8. Given two real sequences $\{t_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$, such that $t_n \to t_0$, $s_n \to s_0$ and $s_n \in (0,1)$ $n \geq 0$, let u_n be the heteroclinic solution of* (4) *for $f = f_n$ and $c = c^*(f_n)$, satisfying $u_n(t_n) = s_n$, $n \geq 0$.*

Then, $u_n \to u_0$ in $BC(\mathbb{R}) \cap C^2(\mathbb{R})$.

Proof. From Lemma 4 we have an uniform bound for $u_n'(t)$. Using the differential equation (4) we obtain an uniform bound also for $\{u_n''\}_n$. Then Ascoli's Lemma and Theorem 8 allow us to prove that $\{u_n\}_n$ admits a subsequence uniformly convergent on compact subsets of $\mathbb{R}$ to a solution $\tilde{u}$ of (4) with $f = f_0$ and $c = c^*(f_0)$.

Using the uniform convergence on compact sets, we get $\tilde{u}(t_0) = s_0$, so $\tilde{u}$ is non-constant. By Proposition 1, $\tilde{u} \equiv u_0$. The proof of the uniform convergence on compact sets of the whole sequence is standard. The uniform convergence on all the real line is now a consequence of the monotonicity of the heteroclinic solutions and Lemma 5. ∎

Corollary 10 *Let $\{f_n\}_{n\geq 0} \subset \mathcal{F}$ such that $f_n \to f_0$ uniformly on* $[0,1]$ *and let $c > 0$ be fixed with $c \geq c^*(f_n)$, $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Given $\{t_n\}_n$ and $\{s_n\}_n$ as in Theorem 9, let u_n denote the heteroclinic solution of* (4) *for $f = f_n$ and such a c, which satisfies $u_n(t_n) = s_n$.*

Then, $u_n \to u_0$ in $BC(\mathbb{R}) \cap C^2(\mathbb{R})$. In particular, $c \geq c^*(f_0)$.

Proof. It is enough to note that the condition $f_n'(0) \to f_0'(0)$ is only used in the proof of Theorem 9 in order to show that $c^*(f_n) \to c^*(f_0)$. If we take c fixed, it is not necessary.

Remark. As a consequence of the previous result one can deduce that

"If $f_n \to f_0$ uniformly on $[0,1]$*, then* $\liminf c^*(f_n) \geq c^*(f_0)$*."*

So, the uniform convergence suffices to ensure the lower semi-continuity of $c^*(f)$ with respect to f.

However, the convergence of the derivatives at zero is needed to obtain convergence, as we can see in the following example.

Example. Let $f \in \mathcal{F}$ fixed and $a > 0$ such that $2\sqrt{a} > c^*(f)$. Define

$$f_n := \max\{g_n, f\}, \quad \text{where } g_n(s) := \min\{as, 1/n, a(1-s)\},\ n \in \mathbb{N}.$$

Since $2\sqrt{f'(0)} \le c^*(f) < 2\sqrt{a}$, then $f'(0) < a$, and $f(s) < as$ in a neighborhood of 0. Hence, $f_n'(0) = a$, $n \in \mathbb{N}$. So, we have

$$c^*(f_n) \ge 2\sqrt{f_n'(0)} = 2\sqrt{a} > c^*(f),$$

and $c^*(f_n) \nrightarrow c^*(f)$ although, as one can easily check, $f_n \to f$ uniformly on $[0,1]$.

The following two theorems are about the convergence of the profiles in the Sobolev spaces $H^1(e^{ct})$.

Theorem 11 *Under the same assumptions of Theorem 9, if moreover*

$$c_0^* := c^*(f_0) > 2\sqrt{f_0'(0)},$$

then $u_n \to u_0$ in $H^1(e^{c_0^* t})$.

The proof of this theorem uses a technical result on "uniform decay" to zero of the heteroclinic solutions.

Lemma 12 *Under the same assumptions of Theorem 11, for every $\alpha \in \left(\frac{c_0^*}{2}, \frac{c_0^*+\sqrt{(c_0^*)^2-4f_0'(0)}}{2}\right)$, there exist $K>0$, $n_0 \in \mathbb{N}$ such that $u_n(t) \le Ke^{-\alpha t}$, for $n \ge n_0$, $t \in \mathbb{R}$.*

Proof. Since α is between the two roots of the equation $r^2 - c_0^* r + f_0'(0) = 0$, we can define $\delta > 0$ by

$$\alpha^2 - c_0^*\alpha + f_0'(0) = -\delta.$$

Claim: *There exist $n_0 \in \mathbb{N}$ and $T > 0$ such that*

$$\alpha^2 - c_n^*\alpha + \frac{f_n(u_n(t))}{u_n(t)} < -\frac{\delta}{2}, \tag{11}$$

for $n \ge n_0$, $t \ge T$, and $c_n^* := c^*(f_n)$.

Let us suppose the Claim is true. Fix $n \ge n_0$ and consider $r_n(t) = -\frac{u_n'(t)}{u_n(t)}$, then it satisfies

$$r_n' = r_n^2 - c_n^* r_n + \frac{f_n(u_n(t))}{u_n(t)}$$

We are going to show that $r_n(t) \ge \alpha$ for all $t \ge T$. Indeed, suppose by contradiction there exists $\tilde{t} \in [T, +\infty)$ with $r_n(\tilde{t}) < \alpha$. Then, by (11), $r_n'(\tilde{t}) <$

0 and $r_n(t) < \alpha$ for all $t \geq \tilde{t}$. But if we take the limit as $t \to +\infty$ in equation (11) we obtain

$$\alpha^2 - c_n^* \alpha + f_n'(0) \leq -\frac{\delta}{2},$$

so, α is between the two roots of the equation $\xi^2 - c_n^* \xi + f_n'(0) = 0$ which, in particular, are different. Then, $c_n^* > 2\sqrt{f_n'(0)}$ and, by Theorem 3, we obtain that $r_n(t)$ converges to the upper root of this last parabola when $t \to +\infty$, which is impossible since $r_n(t) < \alpha,\ t \geq \tilde{t}$.

We conclude that $r_n(t) \geq \alpha$ and hence, $\frac{u_n'(t)}{u_n(t)} \leq -\alpha$, for $t \geq T$. So,

$$u_n(t) \leq u_n(T) e^{\alpha T} e^{-\alpha t}, \quad t \geq T.$$

It is enough to observe that $0 < u_n(t) < 1$ and to take $K = e^{\alpha T}$ in order to finish the proof. ∎

Proof of the Claim. Recalling that $f_n \to f_0$ in $\mathcal{F}$, there exists n_1 such that

$$\left| \frac{f_n(u_n(t)) - f_0(u_n(t))}{u_n(t)} \right| < \frac{\delta}{6}, \qquad \text{for every } n \geq n_1,\ t \in \mathbb{R}.$$

Take $\varepsilon > 0$ such that

$$\left| \frac{f_0(u)}{u} - f_0'(0) \right| < \frac{\delta}{6}, \qquad \text{when } 0 < u < \varepsilon.$$

By the uniform convergence of $\{u_n\}_n$ to u_0 in $\mathbb{R}$, there exists n_2 such that

$$|u_n(t) - u_0(t)| < \frac{\varepsilon}{2}, \qquad \text{for every } t \in \mathbb{R}, n \geq n_2.$$

Since $u_0(t) \to 0$ as $t \to +\infty$, there exists $T \in \mathbb{R}$ such that $|u_0(t)| < \frac{\varepsilon}{2}$, $t \geq T$. Hence, $|u_n(t)| < \varepsilon$ and then,

$$\left| \frac{f_0(u_n(t))}{u_n(t)} - f_0'(0) \right| < \frac{\delta}{6}, \qquad \text{for every } t \geq T, n \geq n_2.$$

Finally, since $c_n^* \to c_0^*$, we can find $n_3 \in \mathbb{N}$ such that

$$|c_n^* \alpha - c_0^* \alpha| < \frac{\delta}{6}.$$

Summarizing, (11) holds for $n \geq n_0 := \max\{n_1, n_2, n_3\}$ and $t \geq T$. ∎

Proof of Theorem 11. Put $c_n^* := c^*(f_n)$. We have to prove that

$$\int_{-\infty}^{+\infty} |u_n'(t) - u_0'(t)|^2 e^{c_0^* t}\, dt \to 0,\ n \to \infty.$$

In order to do that, we observe that, by Lemma 4 and Theorems 8 and 9 we obtain the existence of a constant $M > 0$ such that

$$|u_n'(t)| \le M u_n(t) \le M, \quad \forall t \in \mathbb{R},\, n \ge 0. \tag{12}$$

So, $\{u_n'\}_{n\ge 0}$ is uniformly bounded on $\mathbb{R}$. Hence, we can apply the Lebesgue's Dominated Convergence Theorem and obtain

$$\int_{-\infty}^{0} |u_n'(t) - u_0'(t)|^2 e^{c_0^* t}\, dt \to 0, \quad \text{as } n \to \infty.$$

Moreover, by Lemma 12 and (12), given $\alpha \in \left(\frac{c_0^*}{2}, \frac{c_0^* + \sqrt{(c_0^*)^2 - 4 f_0'(0)}}{2} \right)$,

$$|u_n'(t)|^2 e^{c_0^* t} \le M^2 K^2 e^{(c_0^* - 2\alpha)t} \quad \text{for } n \ge n_0, t \in \mathbb{R}.$$

Again by the Lebesgue's Dominated Convergence Theorem we can conclude

$$\int_{0}^{+\infty} |u_n'(t) - u_0'(t)|^2 e^{c_0^* t}\, dt \to 0, \quad \text{as } n \to \infty,$$

and the result is proved. ■

Remark. Note that $H^1(e^{at}) \not\subseteq H^1(e^{bt})$ even when $a > b$. However, since u_n and u_n', $n \ge 0$ are uniformly bounded, is easy to see that $u_n \to u_0$ in $H^1(e^{\alpha t})$ for any $\alpha \le c_0^*$.

Remark. Lemma 12 allows us to go a bit further. In fact, in the hypotheses of Theorem 9, $u_n \to u_0$ in $H^1(e^{\alpha t})$ for all $0 < \alpha < c_0^* + \sqrt{(c_0^*)^2 - 4 f_0'(0)}$.

4 Non-constant diffusivity: non-degenerate case

In this section we consider the reaction-diffusion equation (RD) with non-degenerate non-constant diffusivity, that is, for $D \in C^1[0,1]$ satisfying

$$D(s) > 0,\ \forall\, s \in [0,1].$$

The profile of a t.w.s. for (RD) having a constant speed c is a heteroclinic solution of equation

$$(D(u)u')' + cu' + f(u) = 0 \tag{13}$$

instead of (4). As in the case of constant diffusion, there is a minimal speed of propagation, $c^* = c^*(D, f) > 0$, such that a heteroclinic solution of (13) exists if and only if $c \geq c^*$ (see [6], [9]).

The usual approach for studying the heteroclinic solutions of (13), consists in reducing it to an equation of the type (4) by means of a change of variable (see e.g. [5]). To be precise, for any u heteroclinic solution of (13) the function

$$\eta_u(t) := \int_0^t \frac{1}{D(u(\xi))}\,\mathrm{d}\xi, \quad t \in \mathbb{R} \tag{14}$$

is a diffeomorphism from $\mathbb{R}$ onto itself (indeed $0 < \alpha_1 \leq D(u) \leq \alpha_2$, with $\alpha_1, \alpha_2 \in \mathbb{R}$). Then, $v(\tau)$ defined by $v(\eta_u(t)) = u(t)$ is a heteroclinic solution of

$$v'' + cv' + f(v)D(v) = 0. \tag{15}$$

On the other hand, if v is a heteroclinic solution of (15) then the function

$$\phi_v(\tau) := \int_0^\tau D(v(s))\,\mathrm{d}s\ , \quad \tau \in \mathbb{R}. \tag{16}$$

is again a diffeomorphism from $\mathbb{R}$ onto itself that gives a heteroclinic solution of (13), $u(t)$, by the way $v(\tau) = u(\phi_v(\tau))$.

As a consequence of the bijective correspondence between heteroclinic solutions of equations (13) and (15) it is clear that $c^*(D, f)$ for (13) coincides with $c^*(fD)$ for (15). The following corollary is an immediate consequence of Theorem 8.

Corollary 13 *Let $\{D_n\}_{n\geq 0}$ be a sequence of non-degenerate diffusion terms and $\{f_n\}_{n\geq 0} \subset \mathcal{F}$. Assume that D_n converges to D_0 uniformly in $[0, 1]$ and f_n converges to f_0 in $\mathcal{F}$ then*

$$c^*(D_n, f_n) \to c^*(D_0, f_0).$$

The following theorem sets up the continuous dependence of the profiles in this framework.

Theorem 14 *Let $\{D_n\}_{n\geq 0}$ be a sequence of non-degenerate diffusion terms converging to D_0 uniformly in $[0, 1]$. Consider $\{f_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$ and $\{u_n\}_{n\geq 0}$ as in Theorem 9. Then $u_n \to u_0$ in $BC(\mathbb{R}) \cap C^2(\mathbb{R})$.*

Proof. Let v_n denote the heteroclinic solution of (15) for $c = c^*(D_n, f_n)$ satisfying $v_n(0) = s_n$, $n \geq 0$. By Theorem 9, $v_n \to v_0$ in $BC(\mathbb{R}) \cap C^2(\mathbb{R})$, and taking

$$\phi_n(\tau) = \phi_{v_n}(\tau) + t_n, \;\; \eta_n(t) = \phi_n^{-1}(t)$$

we obtain

$$u_n(t) = v_n(\eta_n(t)) \quad \text{and} \quad u_n'(t) = \frac{v_n'(\eta_n(t))}{D(u_n(t))}.$$

Taking Lemma 5 into account, it remains to prove that $\eta_n \to \eta_0$ uniformly on compact sets of $\mathbb{R}$, in fact u'' can be expressed in term of the other derivatives using the differential equation. The convergence of η_n follows from the following lemma. ∎

Lemma 15 *Let $\{\phi_n\}_{n\geq 0}$, $\phi_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous and strict increasing functions. Suppose that $\phi_n \to \phi_0$ uniformly on compact sets. Then for any I compact interval, $I \subset \phi_0(\mathbb{R})$, there exists n_0 such that $I \subset \phi_n(\mathbb{R})$, $n \geq n_0$, and $\phi_n^{-1} \to \phi_0^{-1}$ uniformly on I.*

Proof. Since $\phi_0(\mathbb{R})$ is open, we can assume without loss of generality that I is not a point. Put $I = [a, b]$ with $a < b$. We will show the first statement by a contradiction argument. Since $\phi_n(\mathbb{R})$ is an interval, if the statement is false we can assume that, up to a subsequence,

$$a \leq \inf \; \phi_n(\mathbb{R}) \quad \text{or} \quad b \geq \sup \; \phi_n(\mathbb{R}).$$

In the first case, $a < \phi_n(t)$ for any $t \in \mathbb{R}$ and therefore $a \leq \phi_0(t)$. This is not possible because $\phi_0(\mathbb{R})$ is open. The other case is similar.

So, ϕ_0^{-1} and ϕ_n^{-1}, $n \geq n_0$, are well defined on I. Let us show the uniform convergence again by a contradiction argument. If this is not true, we can take a sequence $s_n \in I$ with

$$|\phi_n^{-1}(s_n) - \phi_0^{-1}(s_n)| > \varepsilon_0 \in (0, +\infty). \tag{17}$$

Up to a subsequence, $s_n \to s_0 \in I$ and $\phi_0^{-1}(s_n) \to \phi_0^{-1}(s_0)$. Let us denote $t_n := \phi_n^{-1}(s_n)$. We have three possibilities:

1. There exists a subsequence, relabelled t_n, $t_n \to +\infty$. In this case, given $k \in \mathbb{R}$, $t_n > k$ for n large enough. Since ϕ_n is strictly increasing, $s_n > \phi_n(k)$, n large enough, and so, $s_0 \geq \phi_0(k)$ for any $k \in \mathbb{R}$ which is impossible because $s_0 \in I$.

2. There exists a subsequence, relabelled t_n, $t_n \to -\infty$. A similar argument to the previous one gets to a contradiction.

3. There exists a subsequence, relabelled t_n, $t_n \to t_0 \in \mathbb{R}$. In this case one can see that $s_n = \phi_n(t_n) \to \phi_0(t_0)$. Hence, $s_0 = \phi_0(t_0)$ and then $t_n = \phi_n^{-1}(s_n) \to \phi_0^{-1}(s_0)$, which is in contradiction with (17). ■

As regards the concept of fast heteroclinic in the context of non-constant diffusion, by using the correspondence between the heteroclinic of (13) and (15) discussed above, we can prove the following result.

Proposition 16 *Assume D is non-degenerate. Let u be a heteroclinic solution of* (13) *and let v be the corresponding solution of* (15).

Then v is a fast heteroclinic solution if and only if $u \in H^1(e^{\frac{ct}{D(0)}})$.

Proof. Let v be a fast heteroclinic solution of (15). Then $v'(\tau) \to 0$ exponentially as $\tau \to +\infty$, hence

$$\lim_{\tau\to+\infty} \tau^\alpha[D(v(\tau))-D(0)] = \frac{-1}{\alpha} \lim_{\tau\to+\infty} \tau^{\alpha+1}\, D'(v(\tau))v'(\tau) = 0\,, \quad \text{for every } \alpha > 0.$$

Therefore $\int_0^\infty [D(v(\tau)) - D(0)]\, \mathrm{d}\tau < +\infty$, and

$$\lim_{\tau\to+\infty} e^{\frac{c}{D(0)}\phi_v(\tau)-c\tau} = \lim_{\tau\to+\infty} e^{\frac{c}{D(0)}[\phi_v(\tau)-D(0)\tau]} = e^{\frac{c}{D(0)}\int_0^\infty [D(v(\tau))-D(0)]\mathrm{d}\tau} \in \mathbb{R}. \tag{18}$$

Hence,

$$\int_0^\infty e^{\frac{c}{D(0)}\phi_v(\tau)}\, (v'(\tau))^2\, \mathrm{d}\tau \le K \int_0^\infty e^{c\tau}\, (v'(\tau))^2\, \mathrm{d}\tau < +\infty$$

for some constant $K > 0$.

Thus, by the change of variable $t = \phi_v(\tau)$ we get

$$\int_0^\infty e^{\frac{c}{D(0)}t}\, (u'(t))^2\, \mathrm{d}t = \int_0^\infty e^{\frac{c}{D(0)}t} v'(\eta_u(t))^2(\eta_u'(t))^2\, \mathrm{d}t =$$

$$\int_0^\infty e^{\frac{c}{D(0)}\phi_v(\tau)} \frac{(v'(\tau))^2}{D(v(\tau))}\, \mathrm{d}\tau \le \frac{K}{m}\int_0^\infty e^{c\tau}(v'(\tau))^2\, \mathrm{d}\tau < +\infty$$

where $m := \min_{v\in[0,1]} D(v) > 0$.

The finiteness of $\int_{-\infty}^0 e^{\frac{c}{D(0)}t}\, (u'(t))^2\, \mathrm{d}t$ is a consequence of Lemma 4.

Viceversa, let $u \in H^1(e^{\frac{c}{D(0)}t})$ be a solution of (13). Since the limit in (18) is a positive real value, we get also

$$\lim_{t\to+\infty} e^{c\eta_u(t)-\frac{ct}{D(0)}} \in \mathbb{R}.$$

So,
$$\int_0^{\infty} e^{c\eta_u(t)}\ (u'(t))^2\ \mathrm{d}t \le K_1 \int_0^{\infty} e^{\frac{ct}{D(0)}}\ (u'(t))^2\ \mathrm{d}t < +\infty$$
for some constant $K_1 > 0$. So, by making the same change of variable as above, we obtain
$$\int_0^{\infty} e^{c\tau}(v'(\tau))^2\ \mathrm{d}\tau = \int_0^{\infty} e^{c\eta_u(t)}(u'(t))^2 D(u(t))\ \mathrm{d}t \le K_2 \int_0^{\infty} e^{c\eta_u(t)}(u'(t))^2\ \mathrm{d}t < +\infty$$
for some constant K_2. The finiteness of $\displaystyle\int_{-\infty}^{0} e^{c\tau}\ (v'(\tau))^2\ \mathrm{d}\tau$ is again a consequence of Lemma 4. ∎

According to the previous result, in the case of non-degenerate diffusivity it is natural to define *fast heteroclinic solution* of (13) any heteroclinic solution belonging to the space $H^1(e^{\frac{c}{D(0)}t})$. Moreover, as for the existence or non-existence of fast heteroclinic solutions, the previous result allows us to deduce the following criteria, immediate consequences of Theorems 3, 6, 7.

Corollary 17 *Fast heteroclinic solutions for* (13) *can appear only for* $c = c^*(D, f)$. *Moreover, if*
$$c^*(D, f) > 2\sqrt{D(0)f'(0)}$$
then the heteroclinic solution of (13) *with* $c = c^*(D, f)$ *is fast.*

Corollary 18 *Let* $D \in C^1[0, 1]$ *be a non-degenerate diffusion term and* $f \in \mathcal{F}$. *Suppose that*
$$D(u)f(u) \le D(0)f'(0)u,\ u \in [0, 1].$$
Then, (13) *has no fast heteroclinic solutions.*

Corollary 19 *Suppose* D *and* f *are* C^2 *and let* $c^*(D, f) = 2\sqrt{D(0)f'(0)}$. *Then there is no fast heteroclinic of equation* (13).

The following theorem concerns the convergence in $H^1(e^{\alpha t})$.

Theorem 20 *Under the same hypotheses of Theorem 14, if moreover*
$$c_0^* := c^*(D_0, f_0) > 2\sqrt{D_0(0)f_0'(0)},$$
then $u_n \to u_0$ *in* $H^1(e^{\frac{c_0^*}{D_0(0)}t})$.

The proof proceeds just as that of Theorem 11, but using Lemma 21 below instead of Lemma 12. ∎

Lemma 21 *In the hypothesis of Theorem 20, given*

$$\alpha \in \left(\frac{c_0^*}{2}, \frac{c_0^* + \sqrt{(c_0^*)^2 - 4D_0(0) f_0'(0)}}{2} \right),$$

there exist $K > 0$, $n_0 \in \mathbb{N}$ *such that* $u_n(t) \le K e^{-\frac{\alpha}{D_0(0)} t}$, *for* $n \ge n_0$, $t \in \mathbb{R}$.

Proof. To prove this result we will need the following
Claim: *Given* $\varepsilon > 0$, *there exist* $n_0 \in \mathbb{N}$ *and* $T > 0$ *such that*

$$\left| \frac{1}{D_n(u_n(t))} - \frac{1}{D_0(0)} \right| < \varepsilon, \quad t \ge T, \; n \ge n_0.$$

Suppose the Claim is proved. Denote $\tilde{u}_n(t) := u_n(t+T)$ and let $\tilde{v}_n(\tau)$ be the heteroclinic solution of (15) with $D \equiv D_n$, $f \equiv f_n$ and $c \equiv c_n^*$, associated to $\tilde{u}_n$. Let $\epsilon > 0$ be such that $\beta := \dfrac{\alpha}{1 + \epsilon D_0(0)} > \dfrac{c_0^*}{2}$. By Lemma 12, there exist $\tilde{K} > 0$ and $n_0 \in \mathbb{N}$ such that

$$\tilde{v}_n(\tau) \le \tilde{K} e^{-\beta \tau}, \quad n \ge n_0, \tau \in \mathbb{R}.$$

Denoting $\tilde{\eta}_n(t) := \int_0^t \frac{1}{D_n(\tilde{u}_n(s))} \mathrm{d}s$, the claim sets up

$$\tilde{\eta}_n(t) - \frac{t}{D_0(0)} \le \varepsilon t, \qquad \text{for } n \ge n_0, \; t \ge 0.$$

Hence,

$$\tilde{u}_n(t) = \tilde{v}_n(\tilde{\eta}_n(t)) \le \tilde{K} e^{-\beta(\frac{1}{D_0(0)} + \varepsilon)t} = \tilde{K} e^{-\frac{\alpha}{D_0(0)} t}, \qquad \text{for } t \ge 0, n \ge n_0.$$

and so, putting $K = \max\{1, \tilde{K} e^{\frac{\alpha}{D_0(0)} T}\}$ and recalling that $0 \le u_n(t) \le 1$, we deduce

$$u_n(t) \le K e^{-\frac{\alpha}{D_0(0)} t}, \quad n \ge n_0, t \in \mathbb{R}.$$

■

Proof of the Claim. Since $D_n \to D_0$ uniformly on $[0,1]$ and $D_0(s) > 0$ for $s \in [0,1]$, there exists $C > 0$ such that $D_n(s) \ge C$, $s \in [0,1]$, $n \ge 0$. So, we have to prove that $|D_n(u_n(t)) - D_0(0)| < C^2 \varepsilon$.

Indeed, by the uniform convergence, given ε there exists $n_1 \in \mathbb{N}$ such that

$$|D_n(s) - D_0(s)| < \frac{C^2}{2} \varepsilon, \quad s \in [0,1], \; n \ge n_1,$$

and, by the continuity of D_0, there exists $\delta > 0$ such that

$$|D_0(s) - D_0(0)| < \frac{C^2}{2}\varepsilon, \quad 0 < s < \delta.$$

Moreover, by Theorem 14, and having in mind that $u_0(t) \to 0$ as $t \to \infty$, there exist $n_2 \in \mathbb{N}$ and $T \in \mathbb{R}$ such that $u_n(t) < \delta,\ n \geq n_2,\ t \geq T$.

It is enough to take $n \geq n_0 := \max(n_1, n_2)$ and $t \geq T$ to have the result.

■

Remark. Analogous remarks as those stated at the end of the previous section hold.

5 The degenerate case: sharp t.w.s.

The purpose of this section is to deal with t.w.s. for equation (RD) when the diffusivity D is degenerate at zero, that is, $D(0) = 0$ but $D(s) > 0,\ s \in (0, 1]$, and $f \in \mathcal{F}$. We will restrict in all this section to the case $D'(0) > 0$ since other cases are far from our aims.

It is known (see [6], [9], [10]) that there exists a threshold value $c^* = c^*(D, f)$ such that there is a classical t.w.s. if $c > c^*$ and no one when $c < c^*$. However, when $c = c^*$ another type of t.w.s. appear that are called *sharp* t.w.s. The profile of this kind of t.w.s. is called *sharp heteroclinic solution*. To be precise, a sharp heteroclinic solution of equation (13) is a function u such that there exists $\ell_u \in \mathbb{R}$ with

1. $u \in C^2(-\infty, \ell_u) \cap C^1(-\infty, \ell_u]$ and satisfies (13) on $(-\infty, \ell_u)$.
2. $u(-\infty) = 1$, $u(\ell_u) = 0$ and $u \equiv 0$ in $[\ell_u, +\infty)$.
3. $u'(t) < 0$ on $t \in (-\infty, \ell_u)$ and $u'(\ell_u^-) = \frac{-c}{D'(0)}$.

In order to carry on the analysis of sharp heteroclinic solutions we use the same change of variable as in the non-degenerate case to reduce equation (13) to (15). The following proposition implies that $c^*(D, f) = c^*(Df)$ also in this case.

Proposition 22 *Let v be a heteroclinic solution of* (15) *with $c = c^*(Df)$. Then ϕ_v defined in (16) maps $\mathbb{R}$ into an interval $(-\infty, \ell)$ and the function defined as*

$$u(t) = \begin{cases} v(\phi_v^{-1}(t)) & \textit{if } t \in (-\infty, \ell) \\ 0 & \textit{elsewhere} \end{cases} \tag{19}$$

is a sharp heteroclinic solution of (13) *with* $c = c^*(D, f)$.

Reciprocally, let u *be a sharp heteroclinic solution of* (13) *with* $c = c^*(D, f)$. *Then* η_u *defined by (14) map* $(-\infty, \ell)$ *into* $\mathbb{R}$ *and* $v(\tau) = u(\eta_u^{-1}(\tau))$ *is a heteroclinic solution for* (15) *with* $c = c^*(D, f)$.

Proof. First notice that put $\tilde{f}(v) := D(v)f(v)$, being $\tilde{f}'(0) = 0$ and since the minimal speed is always positive, as a consequence of Theorem 3 we get that a heteroclinic solution v of (15) is fast if and only if $c = c^*(\tilde{f})$ and if and only if $\frac{v'(t)}{v(t)} \to -c$ as $t \to +\infty$.

Since $D(0) = 0$ there exists a positive number $\delta > 0$ such that $D(\sigma) < (D'(0) + 1)\sigma$ for every $\sigma \in (0, \delta)$. So, given a heteroclinic solution v of (15) with $c = c^*(Df)$, if $\bar{t}$ is such that $v(\bar{t}) = \delta$, we get

$$\int_{\bar{t}}^{\infty} D(v(s))\mathrm{d}s \leq (D'(0) + 1) \int_{\bar{t}}^{\infty} v(s)\mathrm{d}s < +\infty$$

since v is fast. Therefore, defining $\ell = \int_0^\infty D(v(s))\mathrm{d}s < +\infty$ and u as in (19), u is a sharp heteroclinic solution of (13) with $c = c^*(Df)$. Indeed, we will only compute $u'(\ell^-)$; the other properties can be easily verified. Since $u(\phi_v(\tau)) = v(\tau)$ one has $v'(\tau) = u'(\phi_v(\tau))D(v(\tau))$, so

$$u'(t) = \frac{v'(\phi_v^{-1}(t))}{D(v(\phi_v^{-1}(t)))}.$$

All we need to conclude is to take the limit as $t \to \ell$.
Vice versa, let u be a sharp heteroclinic solution solution of (13) with $c = c^*(D, f)$ and let $\bar{\xi}$ such that

$$u(\xi) < \min\{\delta, (u'(\ell_u^-) - 1)(\xi - \ell_u)\} \quad \text{for every } \xi \in (\bar{\xi}, \ell_u).$$

We have

$$D(u(\xi)) < (D'(0)+1)\, u(\xi) < (D'(0)+1)\, (u'(\ell_u^-)-1)\, (\xi-\ell_u), \quad \text{for } \xi \in (\bar{\xi}, \ell_u)$$

hence

$$\eta_u(\ell_u) = \int_0^{\ell_u} \frac{1}{D(u(\xi))}\, \mathrm{d}\xi = +\infty.$$

This prove that η_u maps $(-\infty, \ell_u)$ into $\mathbb{R}$. Then defining $v(\tau) = u(\eta_u^{-1}(\tau))$, v is a heteroclinic solution of (15) for such a value of c. Let show that this heteroclinic is fast. As before, $\dfrac{v'(\eta_v(t))}{D(v(\eta_v(t))} = u'(t)$. Then $\frac{v'(\tau)}{D(v(\tau))} \to \frac{-c}{D'(0)}$ as $\tau \to +\infty$ and the conclusion holds since $\frac{D(v(\tau))}{v(\tau)} \to D'(0)$. ∎

Remark. Let us observe that when we fix $t_0 \in \mathbb{R}$ and $s_0 \in (0,1)$ the sharp heteroclinic solution of (13) satisfying $u(t_0) = s_0$ is again unique.

Now we can prove the following convergence result.

Theorem 23 *Let $\{D_n\}_{n\geq 1}$ be a sequence of positive (non-degenerate) diffusion terms that converges uniformly on $[0,1]$ to a degenerate one D_0 (i.e. with $D_0(0) = 0$), such that $D_0'(0) > 0$.*

Given $\{f_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$ and $\{u_n\}_{n\geq 0}$ as in Theorem 9, then $c^*(D_n, f_n) \to c^*(D_0, f_0)$ and the sequence $\{u_n\}_n$ uniformly converges on $\mathbb{R}$ to the sharp solution u_0. Moreover, the sequence of the derivatives $\{u_n'\}_{n\geq 1}$ converges uniformly to u_0' and $\{u_n''\}_{n\geq 1}$ converges uniformly to u_0'' on compact subsets of $(-\infty, \ell_{u_0})$.

Proof. The convergence of $c^*(D_n, f_n)$ to $c^*(D_0, f_0)$ follows from Theorem 8 and Proposition 22. The proof of the uniform convergence on compact subsets of $(-\infty, \ell_{u_0})$ of u, u' and u'' is similar to that of Theorem 14. Let observe that the corresponding sequence of diffeomorphisms $(\phi_n)_n$ converges uniformly on compact sets to ϕ_0 that is not a diffeomorphism, but Lemma 15 can be applied obtaining the uniform convergence on compact subsets of $(-\infty, \ell_{u_0})$. This convergence is enough in order to apply Lemma 5. ∎

Observe that in Theorem 20, as smaller is the value of $D_0(0)$, greater is the weight with respect to which convergence exists. This fact leads us to ask ourselves if in the setting of Theorem 23 the convergence of the profile holds in $H^1(e^{\alpha t})$ for every positive α. The answer is affirmative, as the following result states.

Theorem 24 *Under the same assumptions of Theorem 23, assume further the existence of a value $\epsilon_0 > 0$ such that*

$$D_n(u) \geq \epsilon_0 u \quad \textit{for any } u \in [0,1] \textit{ and } n \geq 0. \tag{20}$$

Then, $u_n \to u_0$ in $H^1(e^{\alpha t})$, for every $\alpha > 0$.

Proof. Let $c_n^* := c^*(D_n, f_n)$ and v_n be the corresponding solution of (15). Using the environment of Theorem 23 that is formally the same as Theorem 14, we have $u_0(\phi_0(\tau)) = v_0(\tau)$, then $\lim_{\tau\to\infty} \phi_n(\tau) = \ell_{u_0}$, so

$$\ell_{u_0} = t_0 + \int_0^\infty D_0(v_0(s))\, \mathrm{d}s \tag{21}$$

Since $D_n \to D_0$ uniformly in $[0,1]$ one obtains $D_0 \circ v_n \to D_0 \circ v_0$ using also that D_0 is uniformly continuous. So

$$D_n(v_n(t)) \leq D_0(v_0(t)) + \delta_n \text{ for any } t \in \mathbb{R}, \tag{22}$$

where $\delta_n \to 0$, and consequently using (21)

$$\phi_n(t) = \phi_{v_n}(t) + t_n \leq \int_0^t D_0(v_0(t))dt + \delta_n t + t_n \leq \ell_{u_0} + \delta_n t + t_n - t_0. \tag{23}$$

On the other hand, observe that $\eta_n(\ell_{u_0}) \to +\infty$. Indeed, if there exists a subsequence satisfying $\eta_n(\ell_{u_0}) \leq \tau_0$ for some constant $\tau_0 > 0$, then since ϕ_n is the inverse of η_n, $\ell_{u_0} \leq \phi_n(\tau_0)$ and taking the limit as $n \to +\infty$ we obtain $\ell_{u_0} \leq \phi_0(\tau_0)$, that is a contradiction.

To finish these preliminary claims, observe that u_n' is uniformly bounded, since

$$|u_n'(t)| = \frac{|v_n'(\eta(t))|}{D_n(v_n(\eta_n(t)))} \leq \frac{|v_n'(\eta_n(t))|}{\epsilon_0 v_n(\eta(t))}$$

for any $t \in (-\infty, \ell_{u_0})$ and this is bounded by Lemma 4 and Theorem 23.

Now we have to prove that

$$\int_{-\infty}^{\infty} e^{\alpha t}(u_n'(t) - u_0'(t))^2 \text{ dt} \to 0.$$

First of all, we note that

$$\int_{-\infty}^{\infty} e^{\alpha t}(u_n'(t) - u_0'(t))^2 \text{ dt} = \int_{-\infty}^{\ell_{u_0}} e^{\alpha t}(u_n'(t) - u_0'(t))^2 \text{ dt} + \int_{\ell_{u_0}}^{\infty} e^{\alpha t} u_n'(t)^2 \text{ dt}.$$

The first part tend to zero by the Dominated Convergence Theorem, since u_n' is uniformly bounded. As for the second one, we make the change of variable $t = \phi_n(\tau)$ obtaining

$$\int_{\ell_{u_0}}^{\infty} e^{\alpha t} u_n'(t)^2 \text{ dt} = \int_{\eta_n(\ell_{u_0})}^{\infty} e^{\alpha \phi_n(\tau)} \frac{(v_n'(\tau))^2}{D_n(v_n(\tau))} \text{ d}\tau.$$

By the assumptions and (23) we get

$$\frac{e^{\alpha(\ell_{u_0}+t_n-t_0)}}{\epsilon_0} \int_{\eta_n(\ell_{u_0})}^{+\infty} e^{\alpha\delta_n\tau} \frac{v_n'(\tau)^2}{v_n(\tau)} \text{ d}\tau \leq \frac{c_n^* e^{\alpha(\ell_{u_0}+t_n-t_0)}}{\epsilon_0} \int_{\eta_n(\ell_{u_0})}^{+\infty} e^{\alpha\delta_n\tau} |v_n'(\tau)| \text{ d}\tau.$$

Now using Holder inequality we estimate the last integral by

$$\frac{c_n^* e^{\alpha(\ell_{u_0}+t_n-t_0)}}{\epsilon_0} \left(\int_{\eta_n(\ell_{u_0})}^{+\infty} e^{(2\alpha\delta_n - c_0^*)\tau} d\tau \right)^{\frac{1}{2}} \left(\int_{\eta_n(\ell_{u_0})}^{+\infty} e^{c_0^*\tau} v_n'(\tau)^2 d\tau \right)^{\frac{1}{2}}$$

The first term is clearly bounded, the second one can be easily calculated and tends to zero since $\eta_n(\ell_{u_0}) \to \infty$. Finally, the last term is bounded since by Theorem 20 we have $v_n \to v_0$ in $H^1(e^{c_0^* t})$. This concludes the proof. ■

6 Proof of Theorem 8

Let us do the change of variable $\tilde{u}(t) = u(\frac{t}{c})$ transforming (4) in

$$u'' + u' + \lambda f(u) = 0, \tag{24}$$

where $\lambda = \frac{1}{c^2}$. This change of variable puts in equivalence the two equations (4) and (24) so this last equation has a heteroclinic solution if and only if $0 < \lambda \leq \lambda^*$ where

$$\lambda^* = \lambda^*(f) = \frac{1}{(c^*(f))^2} \tag{25}$$

is just the infimum in (6).

From the variational point of view, equation (24) allows to consider only fixed space $H^1(e^t)$ instead of the space $H^1(e^{ct})$ of the previous sections. A fast heteroclinic here means a heteroclinic solution (24) that belongs to $H^1(e^t)$. So, a function u is a minimizer for (6) if and only if is a fast heteroclinic solution of equation (24), and it can exist only when $\lambda = \lambda^*$.

Moreover, we have

$$\lambda^* \leq \frac{1}{4f'(0)} \tag{26}$$

and when such an inequality is strict (in the sense of the extended real numbers), then the corresponding heteroclinic is fast, as a consequence of Theorem 3.

The following Lemma is a technical result whose proof can be picked out from [1] but we prefer to give it here for the sake of clarity.

Lemma 25 *Let $f \in \mathcal{F}$ and $\{u_n\}_n \subset H^1(e^t)$ be a sequence with $0 \leq u_n(t) \leq 1$, for every $t \in \mathbb{R}$, $n \in \mathbb{N}$, and such that*

$$\frac{1}{2}\int_{-\infty}^{+\infty} e^t u_n'(t)^2 \, \mathrm{d}t \to \lambda^* < \frac{1}{4f'(0)}, \qquad \int_{-\infty}^{+\infty} e^t F(u_n(t)) \, \mathrm{d}t \to 1.$$

Then, $u_n \to u_0$ in $H^1(e^t)$ and u_0 is a minimizer in (5).

Proof.

Let u_0 denote the weak limit of a subsequence of $\{u_n\}_n$. Since the minimizer in (5) is unique, there is no loss of generality in supposing $\{u_n\}_n$ weakly convergent.

First we note that $\{f'(0)\int_{-\infty}^{+\infty} e^t \frac{u_n(t)^2}{2} \, \mathrm{d}t\}_n$ is convergent. Indeed, observe that

$$\int_{-\infty}^{+\infty} e^t F(u_n(t)) \, \mathrm{d}t = f'(0)\int_{-\infty}^{+\infty} e^t \frac{u_n(t)^2}{2} \, \mathrm{d}t + \int_{-\infty}^{+\infty} e^t \tilde{F}(u_n(t)) \, \mathrm{d}t,$$

where $\tilde{F}(u) = F(u) - f'(0)\frac{u^2}{2}$. By applying [1, Lemma 8], we deduce that $\int_{-\infty}^{+\infty} e^t \tilde{F}(u_n(t))\,\mathrm{d}t \to \int_{-\infty}^{+\infty} e^t \tilde{F}(u_0(t))\,\mathrm{d}t$. So

$$f'(0)\int_{-\infty}^{+\infty} e^t \frac{u_n(t)^2}{2}\,\mathrm{d}t \to 1 - \int_{-\infty}^{+\infty} e^t \tilde{F}(u_0(t))\,\mathrm{d}t.$$

On the other hand, since $\lambda^* < \frac{1}{4f'(0)}$, using (3) we obtain that

$$\int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t - \lambda^* f'(0)\int_{-\infty}^{+\infty} e^t \frac{u(t)^2}{2}\,\mathrm{d}t$$

is a positive convex function, vanishing only at $u \equiv 0$, whose square root provides an equivalent norm to the usual one in the space $H^1(e^t)$. The weak semicontinuity of the norm implies that

$$\lambda^* - \lambda^*\left(1 - \int_{-\infty}^{+\infty} e^t \tilde{F}(u_0(t))\,\mathrm{d}t\right) \geq \int_{-\infty}^{+\infty} e^t \frac{u_0'(t)^2}{2}\,\mathrm{d}t - \lambda^* f'(0)\int_{-\infty}^{+\infty} e^t \frac{u_0(t)^2}{2}\,\mathrm{d}t, \tag{27}$$

and hence

$$0 \geq \int_{-\infty}^{+\infty} e^t \frac{u_0'(t)^2}{2}\,\mathrm{d}t - \lambda^* \int_{-\infty}^{+\infty} e^t F(u_0(t))\,\mathrm{d}t.$$

Since the left side term is nonnegative by (6), the previous inequality, and so (27), is in fact an equality. Hence, u_0 is a minimizer in (5). Finally, the convergence of the norms, together with the weak convergence, implies the strong convergence in the Hilbert space $H^1(e^t)$. ■

Now we can give the proof of Theorem 8.

Proof. Of course, in the present notations, we have to prove that $\lambda^*(f_n) \to \lambda^*(f_0)$.

Put $\epsilon_n := \sup_{u\in(0,1]} \frac{1}{u}|f_n(u) - f_0(u)|$, by the assumption we have $\epsilon_n \to 0$ and

$$|F_n(u) - F_0(u)| \leq \varepsilon_n \frac{u^2}{2}, \quad u \in [0,1], \tag{28}$$

where, obviously, $F_n(u) = \int_0^u f_n(s)\,ds$.

Given $\varepsilon > 0$, consider $u \in H^1(e^t)$ such that

$$\int_{-\infty}^{+\infty} e^t F_0(u(t))\,\mathrm{d}t = 1 \;\text{ and }\; \lambda^*(f_0) \leq \int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t \leq \lambda^*(f_0) + \varepsilon.$$

By (28) we get

$$\left|\int_{-\infty}^{+\infty} e^t F_n(u(t))\,\mathrm{d}t - \int_{-\infty}^{+\infty} e^t F_0(u(t))\,\mathrm{d}t\right| \leq \varepsilon_n \int_{-\infty}^{+\infty} e^t \frac{u(t)^2}{2}\,\mathrm{d}t,$$

and by (3) we have $\int_{-\infty}^{+\infty} e^t F_n(u(t))\,\mathrm{d}t \to 1$. Using (6) we obtain

$$\lambda^*(f_n) \le \frac{\int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t}{\int_{-\infty}^{+\infty} e^t F_n(u(t))\,\mathrm{d}t} \to \int_{-\infty}^{+\infty} e^t \frac{u'(t)^2}{2}\,\mathrm{d}t \le \lambda^*(f_0) + \varepsilon,$$

and then,

$$\limsup_{n\to\infty} \lambda^*(f_n) \le \lambda^*(f_0).$$

To show the other inequality, assume by contradiction the existence of a subsequence, relabelled $\{f_n\}_n$, such that $\lambda^*(f_n) < \lambda^*(f_0) - \varepsilon$ for some $\varepsilon > 0$. Since $f_n'(0) \to f_0'(0)$, inequality (26) is strict for n large enough and so $\lambda^*(f_n)$ is achieved at some u_n.

Again by (28), we have

$$\left|\int_{-\infty}^{+\infty} e^t F_n(u_n(t))\,\mathrm{d}t - \int_{-\infty}^{+\infty} e^t F_0(u_n(t))\,\mathrm{d}t\right| \le \varepsilon_n \int_{-\infty}^{+\infty} e^t \frac{u_n(t)^2}{2}\,\mathrm{d}t \le$$

$$\le 4\varepsilon_n \int_{-\infty}^{+\infty} e^t \frac{u_n'(t)^2}{2}\,\mathrm{d}t = 4\varepsilon_n \lambda^*(f_n) \to 0, \quad n \to \infty.$$

Since $\int_{-\infty}^{+\infty} e^t F_n(u_n(t))\,\mathrm{d}t = 1$, $\int_{-\infty}^{+\infty} e^t F_0(u_n(t))\,\mathrm{d}t \to 1$, $n \to \infty$. Therefore, by (6),

$$\lambda^*(f_0) \le \frac{\int_{-\infty}^{+\infty} e^t \frac{u_n'(t)^2}{2}\,\mathrm{d}t}{\int_{-\infty}^{+\infty} e^t F_0(u_n(t))\,\mathrm{d}t} = \frac{\lambda^*(f_n)}{\int_{-\infty}^{+\infty} e^t F_0(u_n(t))\,\mathrm{d}t} \to \alpha < \lambda^*(f_0),$$

for a certain α, which is impossible. ∎

Corollary 26 *Under the same assumptions of Theorem 8, suppose moreover that inequality* (26) *is strict for* f_0.

Then the infimum in (6) is attained for f_n, n *large enough, and the sequence of minimizers converges in* $H^1(e^t)$ *to the minimizer for* f_0.

Proof. Since (26) is strict for f_0 we deduce that it is strict for f_n too, for n large enough and we have a minimizer u_n in (6) for f_n. Such minimizers satisfy $\int_{-\infty}^{+\infty} e^t F_n(u_n(t))\,\mathrm{d}t = 1$. Using (27) as previously we obtain

$$\int_{-\infty}^{+\infty} e^t F_0(u_n(t))\,\mathrm{d}t \to 1,$$

and we can apply Lemma 25. ∎

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