THE CLASSIFICATION OF COMPLETE STABLE AREA-STATIONARY SURFACES IN THE HEISENBERG GROUP $\mathbb{H}^1$

ANA HURTADO, MANUEL RITORÉ, AND CÉSAR ROSALES

Abstract. We prove that any C^2 complete, orientable, connected, stable area-stationary surface in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ is either a Euclidean plane or congruent to the hyperbolic paraboloid $t = xy$.

1. Introduction

Minimal surfaces in Euclidean 3-space are area-stationary, a condition which is equivalent, by the Euler-Lagrange equation, to have mean curvature zero. An important question for such a variational problem is the classification of global minimizers. Hence is natural to consider the second variation. Minimal surfaces with non-negative second variation of the area are called *stable minimal surfaces*. It is well-known that minimal graphs are stable minimal surfaces (in fact area-minimizing by a standard calibration argument). A complete minimal graph must be a plane by the classical Bernstein's Theorem [6]. Bernstein result was later extended by do Carmo and Peng [18], and Fischer-Colbrie and Schoen [21], who proved that a complete stable oriented minimal surface in $\mathbb{R}^3$ must be a plane. The proof in [21] follows from more general results for 3-manifolds of non-negative scalar curvature. Non existence of non-orientable complete stable minimal surfaces in $\mathbb{R}^3$ has been proved by Ros [36].

A similar analysis of the variational properties of area-minimizing surfaces is also of great interest in some special spaces, such as the three-dimensional Heisenberg group $\mathbb{H}^1$. This is the simplest model of a sub-Riemannian space and of a Carnot group. It is also the local model of any 3-dimensional pseudo-hermitian manifold. For background on $\mathbb{H}^1$ we refer the reader to Section 2 and [8].

Area-stationary surfaces of class C^2 in $\mathbb{H}^1$ are well understood. It is known [10], [34] that, outside the singular set of the points where the tangent plane is horizontal, such a surface is ruled by characteristic horizontal segments. Moreover, based on the description of the singular set given by Cheng, Hwang, Malchiodi and Yang [10], and on a first variation formula of the area moving the singular set [33], Ritoré and Rosales [34] proved that a C^2 surface Σ immersed in $\mathbb{H}^1$ is area-stationary if and only if its mean curvature is zero and the characteristic segments in Σ meet orthogonally the singular curves when they exist. A similar result was independently obtained for area-minimizing t-graphs (Euclidean graphs over the plane $t = 0$) by Cheng, Hwang, and Yang [11]. Furthermore, the classification of C^2 complete, connected, orientable, area-stationary surfaces with non-empty singular set

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was achieved in [34]: the only examples are, modulo congruence, non-vertical Euclidean planes, the hyperbolic paraboloid $t = xy$, and the classical left-handed minimal helicoids in $\mathbb{R}^3$. Though some results for complete area-stationary surfaces with empty singular set have been proved, see for example [33, Thm. 5.4], [9], [24] and [34, Prop. 6.16], a detailed description of such surfaces seems far from being established. This gives us an additional motivation for the study of second order minima of the area in $\mathbb{H}^1$.

As in the Euclidean case, we define a *stable area-stationary* surface in $\mathbb{H}^1$ as a C^2 area-stationary surface with non-negative second derivative of the area under compactly supported variations. These surfaces have been considered in previous papers in connection with some Bernstein type problems in $\mathbb{H}^1$. Let us describe some related works.

In [10], a classification of all the complete C^2 solutions to the minimal surfaces equation for t-graphs in $\mathbb{H}^1$ is given. In [34], this classification was refined by showing that the only complete area-stationary t-graphs are Euclidean non-vertical planes or those congruent to the hyperbolic paraboloid $t = xy$. By means of a calibration argument it is also proved in [34] that they are all area-minimizing.

In [13] and [4] the Bernstein problem for *intrinsic graphs* in $\mathbb{H}^1$ was studied. The notion of intrinsic graph is the one used by Franchi, Serapioni and Serra Cassano in [23]. Geometrically, an intrinsic graph is a normal graph over some Euclidean vertical plane with respect to the left invariant Riemannian metric g in $\mathbb{H}^1$ defined in Section 2. A C^1 intrinsic graph has empty singular set. Examples of C^2 complete area-stationary intrinsic graphs different from vertical Euclidean planes were found in [13]. So a natural question is to study complete area-minimizing intrinsic graphs. A remarkable difference with respect to the case of the t-graphs is the existence of complete C^2 area-stationary intrinsic graphs which are not area-minimizing, see [13]. In [4], Barone, Serra Cassano and Vittone classified complete C^2 area-stationary intrinsic graphs. Then they computed the second variation formula of the area for such graphs to establish that the only stable ones are the Euclidean vertical planes. An interesting calibration argument, also given in [4], yields that the vertical planes are in fact area-minimizing surfaces in $\mathbb{H}^1$.

In the interesting paper [15], it is proven that C^2 complete stable area-stationary Euclidean graphs with empty singular set must be vertical planes. This is done by showing that if such a graph is different from a vertical plane then it contains a particular example of unstable surfaces called *strict graphical strips*. From the geometrical point of view, a graphical strip is a C^2 surface given by the union of a family of horizontal lines L_t passing through and filling a vertical segment so that the angle function of the horizontal projection of L_t is a monotonic function. The graphical strip is strict if the angle function is strictly monotonic. If the angle function is constant we have a piece of a vertical plane. We would like to remark that there are examples of complete area-stationary surfaces with empty singular set which do not contain a graphical strip, such as the sub-Riemannian catenoids $t^2 = \lambda^2\,(x^2 + y^2 - \lambda^2)$, $\lambda \neq 0$. Hence the main result in [15] does not apply to general surfaces.

The following natural step is to consider complete stable surfaces in $\mathbb{H}^1$. In fact, all the aforementioned results leave open the existence of stable examples different from intrinsic graphs or Euclidean graphs with empty singular set. The purpose of the present paper is to classify complete stable area-stationary surfaces in $\mathbb{H}^1$ with empty singular set or not. In Theorem 6.1 we prove the following result

> *The only complete, orientable, connected, stable area-stationary surfaces in $\mathbb{H}^1$ of class C^2 are the Euclidean planes and the surfaces congruent to the hyperbolic paraboloid $t = xy$.*

In particular, this result provides the classification of all the complete C^2 orientable area-minimizing surfaces in $\mathbb{H}^1$.

In order to prove Theorem 6.1 we compute the second derivative of the area for some compactly supported variations of a C^2 area-stationary surface Σ by means of Riemannian geodesics. In Theorem 3.7, variations of a portion Σ' of the regular part of Σ in the direction of $vN + wT$, where N is the unit normal to Σ and T is the Reeb vector field in $\mathbb{H}^1$, will be considered. Here v, w are assumed to have compact support in Σ, but not on Σ'. Hence the boundary of Σ' is moving along the variation. In Proposition 3.11 variations in the direction of wT of a C^2 area-stationary surface Σ with singular curves of class C^3 will be taken. Here w has compact support near the singular curves, and it is constant along the characteristic curves of Σ. Both types of variations will be combined to produce global ones in Proposition 5.2. Second variation formulas of the area for variations *supported in the regular set* have appeared in several contexts. In [10], such a formula was obtained for C^3 surfaces inside a 3-dimensional pseudo-hermitian manifold. In [4], a second variation formula was proved for variations by intrinsic graphs of class C^2. In [12], it is computed the second derivative of the area associated to a C^2 variation of a C^2 surface along Euclidean straight lines.

Once we have the second variation formula we proceed into two steps. First we prove in Theorem 4.7 that a C^2 complete oriented stable area-stationary surface with empty singular set must be a vertical plane. In fact, for such a surface Σ, the second derivative of the area for a compactly supported variation as in Theorem 3.7 is given by

$$\mathcal{I}(u,u) = -\int_\Sigma u\,\mathcal{L}(u),$$

where u is the normal component of the variation, and $\mathcal{L}$ is the hypoelliptic operator on Σ given in (3.44). By analogy with the Riemannian situation [3] we refer to $\mathcal{I}$ as the *index form* associated to Σ and to $\mathcal{L}$ as the *stability operator* of Σ. In Proposition 3.12 we see that the stability condition for Σ implies that $\mathcal{I}(u,u) \geqslant 0$ for any $u \in C_0(\Sigma)$ which is also C^1 along the characteristic lines. Then we choose the function $u := |N_h|$, where N is the Riemannian unit normal to Σ for the left invariant Riemannian metric g on $\mathbb{H}^1$ defined in Section 2, N_h is the horizontal projection of N, and the modulus is computed with respect to the metric g. We see in Proposition 4.6 that this function u satisfies

$$\mathcal{L}(u) \geqslant 0,$$

and the inequality is strict in pieces of Σ which are not contained inside Euclidean vertical planes. In such a case we produce a compactly supported non-negative function v in Σ so that inequality $\mathcal{I}(v,v) < 0$ still holds. To construct the function v we use the Jacobi vector field on Σ associated to the family of horizontal straight lines ruling Σ and which is studied in Lemma 4.5. Observe that the function $|N_h|$ is associated to the variational vector field induced by the surfaces equidistant to Σ in the Carnot-Carathéodory distance, see [1]. Hence, our construction of the test function v is, in spirit, similar to that in the Euclidean case, where the equivalent test function is $u \equiv 1$. Using Fischer-Colbrie's results [20], a stable minimal surface in $\mathbb{R}^3$ is conformally a compact Riemann surface minus a finite number of points, so that a logarithmic cut-off function v of $u \equiv 1$ has compact support and yields instability unless the surface is a plane. We remark that the function $|N_h|$ was already used as a test function in [4], [13] and [15].

In the second step of the proof of Theorem 6.1 we consider a complete area-stationary surface Σ with non-empty singular set. From the classification in [34], we conclude that Σ must be a non-vertical plane, congruent to the hyperbolic paraboloid $t = xy$, or congruent to a left-handed helicoid, see Proposition 5.1 for a precise statement. The first two types of surfaces are t-graphs and then they are area-minimizing by a calibration argument [34]. For the third type we will combine our second variation formulas in Theorem 3.7 and Proposition 3.11 to produce the stability inequality $\mathcal{Q}(u) \geqslant 0$, where $\mathcal{Q}$ is the quadratic form defined in (5.8). The construction of appropriate test functions with $\mathcal{Q}(u) < 0$ will prove the instability of the helicoids. It is interesting to observe that $\mathcal{Q}(u) \geqslant 0$ for functions u with support in the regular part of the helicoids.

In the Heisenberg groups $\mathbb{H}^n$, with $n \geqslant 5$, there is no counterpart to Theorem 4.7, as some examples have been constructed in [4] of complete area-minimizing intrinsic graphs different from Euclidean vertical hyperplanes. For $n = 2, 3, 4$ it is still unknown if similar examples can be obtained.

We would like to mention that examples of area-minimizing surfaces in $\mathbb{H}^1$ with low Euclidean regularity have been obtained in [11], [32], [35] and [31]. Hence our results are optimal in the class of C^2 area-stationary surfaces.

Finally, the techniques in this paper can be employed to prove classification results for complete stable area-stationary surfaces under a volume constraint in the first Heisenberg group [37], and inside the sub-Riemannian three-sphere [28].

We have organized this paper as follows: the next section contains some background material in several subsections. In the third one we recall known facts about area-stationary surfaces and we compute second variation formulas for the area. The fourth and fifth sections treat complete stable surfaces without and with singular points, respectively. In the sixth section we state and prove the main result.

After the distribution of this paper we were informed by Prof. Nicola Garofalo that Theorem 4.7 was proven, for the case of embedded surfaces, by Danielli, Garofalo, Nhieu and Pauls in late 2006, [14].

2. Preliminaries

In this section we gather some previous results that will be used throughout the paper. We have organized it in several parts.

2.1. **The Heisenberg group.**

The *Heisenberg group* $\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3, *)$, where the product $*$ is defined, for any pair of points $[z,t]$, $[z',t'] \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, by

$$[z,t] * [z',t'] := [z+z', t+t'+\operatorname{Im}(z\overline{z}')], \qquad (z = x+iy).$$

For $p \in \mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q) = p * q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X := \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y := \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T := \frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar distribution generated by X and Y. The *horizontal projection* of a tangent vector U onto $\mathcal{H}$ will be denoted by U_h. A vector field U is *horizontal* if $U = U_h$.

We denote by $[U,V]$ the Lie bracket of two C^1 vector fields U and V on $\mathbb{H}^1$. Note that $[X,T] = [Y,T] = 0$, while $[X,Y] = -2T$, so that $\mathcal{H}$ is a bracket-generating distribution. Moreover, by Frobenius theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega := -y\,dx + x\,dy + dt$.

2.2. **The left invariant metric.**

We shall consider on $\mathbb{H}^1$ the Riemannian metric $g = \langle \cdot\,, \cdot \rangle$ so that $\{X, Y, T\}$ is an orthonormal basis at every point. The restriction of g to $\mathcal{H}$ coincides with the usual sub-Riemannian metric in $\mathbb{H}^1$. Let D be the Levi-Civita connection associated to g. From Koszul formula and the Lie bracket relations we get

$$
\begin{aligned}
&D_X X = 0, \qquad && D_Y Y = 0, \qquad && D_T T = 0, \\
&D_X Y = -T, && D_X T = Y, && D_Y T = -X, \\
&D_Y X = T, && D_T X = Y, && D_T Y = -X.
\end{aligned} \tag{2.1}
$$

For any tangent vector U on $\mathbb{H}^1$ we define $J(U) := D_U T$. Then we have $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$, so that $J^2 = -\text{Id}$ when restricted to $\mathcal{H}$. It is also clear that

$$
\big\langle J(U), V \big\rangle + \big\langle U, J(V) \big\rangle = 0, \tag{2.2}
$$

for any pair of tangent vectors U and V. The involution $J : \mathcal{H} \to \mathcal{H}$ together with the 1-form $\omega = -y\,dx + x\,dy + dt$, provides a pseudo-hermitian structure on $\mathbb{H}^1$, see [7, Sect. 6.4].

Let R be the Riemannian curvature tensor of g defined for tangent vectors U, V, W by

$$
R(U,V)W = D_V D_U W - D_U D_V W + D_{[U,V]} W.
$$

From (2.1) and the Lie bracket relations we can obtain the following identities

$$
\begin{aligned}
&R(X,Y)\,X = -3Y, \qquad && R(X,Y)\,Y = 3X, \qquad && R(X,Y)\,T = 0, \\
&R(X,T)\,X = T, && R(X,T)\,Y = 0, && R(X,T)\,T = -X, \\
&R(Y,T)\,X = 0, && R(Y,T)\,Y = T, && R(Y,T)\,T = -Y.
\end{aligned} \tag{2.3}
$$

We denote by Ric the Ricci curvature in $(\mathbb{H}^1, g)$ defined, for any pair of tangent vectors U and V, as the trace of the map $W \mapsto R(U,W)V$. These equalities can be checked by taking into account (2.3)

$$
\begin{aligned}
&\text{Ric}(X,Y) = 0, \qquad && \text{Ric}(X,T) = 0, \qquad && \text{Ric}(Y,T) = 0, \\
&\text{Ric}(X,X) = -2, && \text{Ric}(Y,Y) = -2, && \text{Ric}(T,T) = 2.
\end{aligned} \tag{2.4}
$$

2.3. **Horizontal curves and Carnot-Carathéodory distance.** Let $\gamma : I \to \mathbb{H}^1$ be a piecewise C^1 curve defined on a compact interval $I \subset \mathbb{R}$. The *length* of γ is the usual Riemannian length $L(\gamma) := \int_I |\dot\gamma(\varepsilon)|\,d\varepsilon$, where $\dot\gamma$ is the tangent vector of γ. A *horizontal curve* γ in $\mathbb{H}^1$ is a C^1 curve whose tangent vector always lies in the horizontal distribution. For two given points in $\mathbb{H}^1$ we can find, by Chow's connectivity theorem [25, Sect. 1.2.B], a horizontal curve joining these points. The *Carnot-Carathéodory distance* d_{cc} between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points. The topology associated to d_{cc} coincides with the usual topology in $\mathbb{R}^3$, see [5, Cor. 2.6].

2.4. **Geodesics and Jacobi fields in** $(\mathbb{H}^1, g)$**.** A *geodesic* in $(\mathbb{H}^1, g)$ is a C^2 curve γ such that the covariant derivative of the tangent vector field $\dot\gamma$ vanishes along γ.

Let $\gamma(s) = (x(s), y(s), t(s))$. Dots will indicate derivatives with respect to s. We write $\dot\gamma = \dot x\,X + \dot y\,Y + (\dot t - \dot x y + x\dot y)\,T$. Then γ is a geodesic in $(\mathbb{H}^1, g)$ if and only if

$$
\begin{aligned}
\ddot x &= 2\,\big\langle \dot\gamma, T \big\rangle\,\dot y, \\
\ddot y &= -2\,\big\langle \dot\gamma, T \big\rangle\,\dot x, \\
\frac{d}{ds}\,&\big\langle \dot\gamma, T \big\rangle = 0.
\end{aligned}
$$

Let λ be the constant $\dot t - \dot x y + x\dot y = \langle \dot\gamma, T\rangle$. An easy integration shows that the geodesic with initial conditions $(x(0), y(0), t(0)) = (x_0, y_0, t_0)$ and $(\dot x(0), \dot y(0), \dot t(0)) = (A, B, C)$ is given by

$$\begin{aligned} x(s) &= x_0 + As\, f(2\lambda s) + Bs\, g(2\lambda s), \\ y(s) &= y_0 - As\, g(2\lambda s) + Bs\, f(2\lambda s), \\ t(s) &= t_0 + \lambda s + (A^2+B^2)s^2\, h(2\lambda s) + (Ax_0 + By_0)s\, g(2\lambda s) \\ &\qquad + (Ay_0 - Bx_0)s\, f(2\lambda s), \end{aligned} \tag{2.5}$$

where f, g and h are the real analytic functions

$$f(x) := \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \quad g(x) := \begin{cases} \dfrac{1-\cos(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad h(x) := \begin{cases} \dfrac{x-\sin(x)}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

In particular, we have

$$\exp_p(sv) = p + sv, \qquad \text{for } p \in \mathbb{H}^1 \text{ and } v \in \mathcal{H}_p \text{ or } v \,||\, T_p, \tag{2.6}$$

which is a horizontal or vertical straight line. Here $\exp_p$ denotes the exponential map of $(\mathbb{H}^1, g)$ at p.

In the next result we construct Riemannian Jacobi fields associated to C^1 families of Riemannian geodesics.

Lemma 2.1. *Let $\alpha : I \to \mathbb{H}^1$ be a C^1 curve defined on some open interval $I \subseteq \mathbb{R}$. For any C^1 vector field U along α we consider the map $F : I \times \mathbb{R} \to \mathbb{H}^1$ given by $F(\varepsilon, s) := \exp_{\alpha(\varepsilon)}(s\, U_{\alpha(\varepsilon)})$. Then, the variational vector field $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(\varepsilon, s)$ is C^∞ along the geodesic $\gamma_\varepsilon(s) := F(\varepsilon, s)$. As a consequence, $[\dot\gamma_\varepsilon, V_\varepsilon] = 0$ and V_ε satisfies the Jacobi equation*

$$V_\varepsilon'' + R(\dot\gamma_\varepsilon, V_\varepsilon)\dot\gamma_\varepsilon = 0, \tag{2.7}$$

where the prime $'$ denotes the covariant derivative along the geodesic γ_ε. Moreover, if γ_ε is a horizontal straight line, then

$$V_\varepsilon'' - 3\big\langle V_\varepsilon, J(\dot\gamma_\varepsilon)\big\rangle\, J(\dot\gamma_\varepsilon) + |\dot\gamma_\varepsilon|^2 \,\big\langle V_\varepsilon, T\big\rangle\, T = 0. \tag{2.8}$$

Remark 2.2. The classical proofs in Riemannian geometry of $[\dot\gamma_\varepsilon, V_\varepsilon] = 0$ and the fact that V_ε satisfies the Jacobi equation do not apply directly in our setting since we only suppose that F is a C^1 map.

Proof of Lemma 2.1. Let $(x_0(\varepsilon), y_0(\varepsilon), t_0(\varepsilon))$ and $(A(\varepsilon), B(\varepsilon), C(\varepsilon))$ be the Euclidean coordinates of $\alpha(\varepsilon)$ and $U_{\alpha(\varepsilon)}$, respectively. By using the expression of the Riemannian geodesics in (2.5), we see that the map $F(\varepsilon, s)$ can be written as

$$\begin{aligned} x(\varepsilon, s) &= x_0(\varepsilon) + A(\varepsilon)s\, f(2\lambda(\varepsilon)s) + B(\varepsilon)s\, g(2\lambda(\varepsilon)s), \\ y(\varepsilon, s) &= y_0(\varepsilon) - A(\varepsilon)s\, g(2\lambda(\varepsilon)s) + B(\varepsilon)s\, f(2\lambda(\varepsilon)s), \\ t(\varepsilon, s) &= t_0(\varepsilon) + \lambda(\varepsilon)s + (A^2+B^2)(\varepsilon)s^2\, h(2\lambda(\varepsilon)s) + (A(\varepsilon)x_0(\varepsilon) + B(\varepsilon)y_0(\varepsilon))s\, g(2\lambda(\varepsilon)s) \\ &\qquad + (A(\varepsilon)y_0(\varepsilon) - B(\varepsilon)x_0(\varepsilon))s\, f(2\lambda(\varepsilon)s), \end{aligned}$$

where $\lambda(\varepsilon) := C(\varepsilon) - A(\varepsilon)y_0(\varepsilon) + B(\varepsilon)x_0(\varepsilon)$. Observe that the functions $x_0(\varepsilon)$, $y_0(\varepsilon)$, $t_0(\varepsilon)$, $A(\varepsilon)$, $B(\varepsilon)$, $C(\varepsilon)$ and $\lambda(\varepsilon)$ are C^1. A direct computation of $(\partial F/\partial\varepsilon)(\varepsilon, s)$ shows that $V_\varepsilon(s)$ is C^∞ along the geodesic $\gamma_\varepsilon(s)$.

On the other hand, we can check that for all $k \in \mathbb{N}$ and any of the Euclidean components $\phi(\varepsilon, s)$ of $F(\varepsilon, s)$, the partial derivatives $\partial^{k+1}\phi/\partial\varepsilon\,\partial^k s$ exist and are continuous functions. In particular, it follows from the classical Schwarz's theorem that $\partial^2\phi/\partial\varepsilon\partial s = \partial^2\phi/\partial s\partial\varepsilon$ and $\partial^3\phi/\partial\varepsilon\partial s^2 = \partial^3\phi/\partial s\partial\varepsilon\partial s$. Now, the classical proofs in [17, p. 68 and p. 111] can be traced to prove that $[\dot\gamma_\varepsilon, V_\varepsilon] = 0$ and that V_ε satisfies the Jacobi equation. Finally, to get

(2.8) from (2.7) it suffices to use (2.3) to obtain $R(w,v)w = -3\big\langle v, J(w)\big\rangle J(w) + |w|^2\big\langle v, T\big\rangle T$ provided w is a horizontal vector. $\square$

2.5. **Geometry of surfaces in $\mathbb{H}^1$.** Unless explicitly stated we shall consider surfaces with empty boundary. Let Σ be a C^1 surface immersed in $\mathbb{H}^1$. The *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent plane $T_p\Sigma$ coincides with $\mathcal{H}_p$. As Σ_0 is closed and has empty interior in Σ, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. It was proved in [16, Lem. 1], see also [2, Thm. 1.2], that, for a C^2 surface, the Hausdorff dimension of Σ_0 with respect to the Riemannian distance on $\mathbb{H}^1$ is less than or equal to one. In particular, the Riemannian area of Σ_0 vanishes. If N is a unit normal vector to Σ in $(\mathbb{H}^1, g)$, then we can describe the singular set as $\Sigma_0 = \{p \in \Sigma; N_h(p) = 0\}$, where $N_h = N - \big\langle N, T\big\rangle T$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal Gauss map* ν_h and the *characteristic vector field* Z, by

$$\nu_h := \frac{N_h}{|N_h|}, \qquad Z = J(\nu_h). \tag{2.9}$$

As Z is horizontal and orthogonal to ν_h, we conclude that Z is tangent to Σ. Hence Z_p generates $T_p\Sigma \cap \mathcal{H}_p$. The integral curves of Z in $\Sigma - \Sigma_0$ will be called (*oriented*) *characteristic curves* of Σ. They are both tangent to Σ and horizontal. If we define

$$S := \big\langle N, T\big\rangle\, \nu_h - |N_h|\, T, \tag{2.10}$$

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$. Moreover, for any $p \in \Sigma - \Sigma_0$ we have the orthonormal basis of $T_p\mathbb{H}^1$ given by $\{Z_p, (\nu_h)_p, T_p\}$. From here we deduce the following identities on $\Sigma - \Sigma_0$

$$|N_h|^2 + \big\langle N, T\big\rangle^2 = 1, \qquad (\nu_h)^\top = \big\langle N, T\big\rangle\, S, \qquad T^\top = -|N_h|\, S, \tag{2.11}$$

where $U^\top$ stands for the projection of a vector field U onto the tangent plane to Σ.

Given a C^1 immersed surface Σ with a unit normal vector N, we define the *area* of Σ by

$$A(\Sigma) := \int_\Sigma |N_h|\, d\Sigma, \tag{2.12}$$

where $d\Sigma$ is the Riemannian area element on Σ. If Σ is a C^2 surface bounding a set Ω, then $A(\Sigma)$ coincides with all the notions of perimeter of Ω and area of Σ introduced by other authors, see [22, Prop. 2.14], [30, Thm. 5.1] and [22, Cor. 7.7].

Finally, for a C^2 immersed surface Σ with a unit normal vector N, we denote by B the Riemannian shape operator of Σ with respect to N. It is defined for any vector W tangent to Σ by $B(W) = -D_W N$. The Riemannian mean curvature of Σ is $-2H_R = \operatorname{div}_\Sigma N$, where $\operatorname{div}_\Sigma$ denotes the Riemannian divergence relative to Σ.

2.6. **Isometries and dilations.** By a *horizontal isometry* of $\mathbb{H}^1$ we mean an isometry of $(\mathbb{H}^1, g)$ leaving invariant the horizontal distribution. These isometries preserve the area defined in (2.12). Examples of such isometries are the left translations and the Euclidean rotations about the t-axis. We say that two surfaces Σ_1 and Σ_2 are *congruent* if there is a horizontal isometry ϕ such that $\phi(\Sigma_1) = \Sigma_2$.

In the Heisenberg group $\mathbb{H}^1$ there is a one-parameter group of C^∞ *dilations* $\{\delta_\lambda\}_{\lambda\in\mathbb{R}}$ given in coordinates (x, y, t) by

$$\delta_\lambda(x, y, t) = (e^\lambda x, e^\lambda y, e^{2\lambda} t). \tag{2.13}$$

From (2.13) it is easy to check that any δ_λ preserves the horizontal and the vertical distributions. The behaviour of the area with respect to δ_λ is contained in the formula

$$A(\delta_\lambda(\Sigma)) = e^{3\lambda} A(\Sigma). \tag{2.14}$$

For a proof of (2.14) see [34, Proof of Thm. 4.12].

2.7. **A weak Riemannian divergence theorem.** Let Σ be a C^2 Riemannian surface. For any integer $r \geqslant 1$ we denote by $C_0^r(\Sigma)$ and $C^r(\Sigma)$ the spaces of functions of class C^r with or without compact support in Σ. For $r \geqslant 1$ let $L^r(\Sigma)$ be the corresponding space of integrable functions with respect to the Riemannian measure $d\Sigma$. Let U be a C^1 tangent vector field on Σ. Given a continuous function f on Σ, a continuous vector field V on Σ, and a point $p \in \Sigma$, we define $U_p(f) = (f \circ \alpha)'(0)$ and $(D_U V)(p) = V'_{\alpha(s)}(0)$. Here α is the integral curve of U with $\alpha(0) = p$, while the primes denote derivatives of functions depending on s and covariant derivatives along $\alpha(s)$. We say that f and V are C^1 in the U-direction if $U(f)$ and $D_U V$ are well defined and they are continuous on Σ. We also set

$$\operatorname{div}_\Sigma(fU) := f \operatorname{div}_\Sigma U + U(f), \tag{2.15}$$

where $\operatorname{div}_\Sigma U$ stands for the Riemannian divergence of U. Note that these definitions coincide with the classical ones when $f \in C^1(\Sigma)$ and V is a C^1 vector field on Σ. In the same way we can introduce derivatives of higher order in the U-direction.

Now we extend the classical Riemannian divergence theorem in Σ to certain vector fields with compact support which are not C^1 on Σ. First we need an approximation result.

Lemma 2.3. *Let Σ be a C^2 Riemannian surface. Consider a C^1 tangent vector field U on Σ such that $U_p \neq 0$ for any $p \in \Sigma$. Then, for any function $f \in C_0(\Sigma)$ which is also C^1 in the U-direction, there is a compact set $K \subseteq \Sigma$ and a sequence of functions $\{f_\varepsilon\}_{\varepsilon>0}$ in $C_0^1(\Sigma)$ such that the supports of f and f_ε are contained in K for any $\varepsilon > 0$, and*

(i) *$\{f_\varepsilon\} \to f$ in $L^r(\Sigma)$ for any integer $r \geqslant 1$,*
(ii) *$\{U(f_\varepsilon)\} \to U(f)$ in $L^r(\Sigma)$ for any integer $r \geqslant 1$.*

Proof. Let $p \in \Sigma$. By using the local flow of U in Σ and that $U_p \neq 0$, we can find a local C^1 chart $(D, \phi = (x, y))$ of Σ around p such that $K = \overline{D}$ is compact and the restriction of U to D coincides with the basic vector field ∂_y. This means that $U(h) = (\partial(h \circ \phi^{-1})/\partial y) \circ \phi$ for any function h which is C^1 in the U-direction. To finish the proof it suffices, by a standard partition of unity argument, to prove the claim when the support of f is contained in D. Let $D' = \phi(D)$ and $g = f \circ \phi^{-1}$. We have $g \in C_0(D')$ and $\partial g/\partial y = U(f) \circ \phi^{-1} \in C_0(D')$. From the standard regularization by convolution in $\mathbb{R}^2$, see for instance [19, Sect. 4.2.1], we can find a sequence $\{g_\varepsilon\}_{\varepsilon>0}$ in $C_0^\infty(\mathbb{R}^2)$ such that $\{g_\varepsilon\} \to g$ and $\{\partial g_\varepsilon/\partial y\} \to \partial g/\partial y$ uniformly in $\mathbb{R}^2$, while the supports of g_ε are contained in D' for any $\varepsilon > 0$. It follows that the family $\{f_\varepsilon\}_{\varepsilon>0}$ with $f_\varepsilon = g_\varepsilon \circ \phi$ satisfies $\{f_\varepsilon\} \to f$ and $\{U(f_\varepsilon)\} \to U(f)$ uniformly in D, while the support of f_ε is contained in $D \subset K$ for any $\varepsilon > 0$. Clearly $\{f_\varepsilon\}_{\varepsilon>0}$ proves the lemma. ☐

Lemma 2.4. *Let Σ be a C^2 Riemannian surface. Consider a C^1 tangent vector field U on Σ such that $U_p \neq 0$ for any $p \in \Sigma$. Then, for any $f \in C_0(\Sigma)$ which is also C^1 in the U-direction, we have*

$$\int_\Sigma \operatorname{div}_\Sigma(fU)\, d\Sigma = 0.$$

Proof. By definition (2.15) it follows that $\operatorname{div}_\Sigma(fU) \in L^1(\Sigma)$ since f has compact support and $U(f)$ is continuous. By Lemma 2.3 we can find a sequence $\{f_\varepsilon\}_{\varepsilon>0}$ in $C_0^1(\Sigma)$ such that $\{f_\varepsilon\} \to f$ and $\{U(f_\varepsilon)\} \to U(f)$ in $L^1(\Sigma)$, while the supports of f_ε and f are contained in the same compact set $K \subseteq \Sigma$ for any $\varepsilon > 0$. In particular, we deduce $\{f_\varepsilon \operatorname{div}_\Sigma U\} \to f \operatorname{div}_\Sigma U$ in $L^1(\Sigma)$ since $\operatorname{div}_\Sigma U$ is continuous. By using the Riemannian divergence theorem for C^1 vector fields with compact support, we obtain

$$0 = \int_\Sigma \operatorname{div}_\Sigma(f_\varepsilon U)\, d\Sigma = \int_\Sigma f_\varepsilon \operatorname{div}_\Sigma U\, d\Sigma + \int_\Sigma U(f_\varepsilon)\, d\Sigma, \qquad \varepsilon > 0.$$

Letting $\varepsilon \to 0$ in the previous equality the claim is proven. $\square$

3. Stable surfaces. Second variation formulas of the area

In this section we define stable surfaces and we show that they satisfy an analytical inequality by means of a second variation formula for the area functional defined in (2.12). We first introduce the appropriate variational background.

Let Σ be a C^2 oriented surface immersed in $\mathbb{H}^1$ with singular set Σ_0. By a *variation* of Σ we mean a C^1 map $\varphi : I \times \Sigma \to \mathbb{H}^1$, where I is an open interval containing the origin, satisfying the following properties:

(i) $\varphi(0,p) = p$ for any $p \in \Sigma$,
(ii) the set $\Sigma_s = \{\varphi(s,p);\, p \in \Sigma\}$ is a C^1 surface immersed in $\mathbb{H}^1$ for any $s \in I$,
(iii) the map $\varphi_s : \Sigma \to \Sigma_s$ given by $\varphi_s(p) = \varphi(s,p)$ is a diffeomorphism for any $s \in I$.

We say that the variation is *compactly supported* if there is a compact set $K \subseteq \Sigma$ such that $\varphi_s(p) = p$ for any $s \in I$ and $p \in \Sigma - K$. If, in addition, the set K is contained inside $\Sigma - \Sigma_0$ then the variation is *nonsingular.* The area functional associated to the variation is $A(s) := A(\Sigma_s)$. Note that only the deformation over the compact set K contributes to the change of area. We say that Σ is *area-stationary* if $A'(0) = 0$ for any compactly supported variation. We say that Σ is *stable* (resp. *stable under non-singular variations*) if it is area-stationary and $A''(0) \geqslant 0$ for any compactly supported (resp. non-singular) variation of Σ. Finally by an *area-minimizing* surface in $\mathbb{H}^1$ we mean a C^2 orientable surface Σ such that any compact region $M \subset \Sigma$ satisfies $A(M) \leqslant A(M')$ for any other C^1 compact surface M' in $\mathbb{H}^1$ with $\partial M = \partial M'$. Clearly any area-minimizing surface is stable.

Remark 3.1. Consider a C^1 vector field U with compact support on Σ. For any $s \in \mathbb{R}$ we denote $\varphi_s(p) = \exp_p(sU_p)$, where $\exp_p$ is the exponential map of $(\mathbb{H}^1, g)$ at p. It is easy to see that, for s small enough, $\{\varphi_s\}_s$ defines a compactly supported variation of Σ. In case the support of U is contained in $\Sigma - \Sigma_0$ then the induced variation is nonsingular. This was the point of view used in [34] to define variations of a C^2 surface. In particular, our notion of area-stationary surface implies the one introduced in [34, Sect. 4].

It is clear that stability is preserved under left translations and vertical rotations since they are horizontal isometries in $\mathbb{H}^1$. In the next result we prove that any dilation δ_λ as defined in (2.13) satisfies the same property.

Lemma 3.2. *Let Σ be a C^2 immersed oriented surface in $\mathbb{H}^1$. Then Σ is stable* (*resp. stable under non-singular variations*) *if and only if the same holds for $\delta_\lambda(\Sigma)$.*

Proof. Let $\Sigma_\lambda = \delta_\lambda(\Sigma)$. Take a compactly supported variation $\{\varphi_s\}_{s\in I}$ of Σ_λ. By using that the family of dilations is a one-parameter group of diffeomorphisms we can see that $\{\psi_s\}_{s\in I}$ with $\psi_s = \delta_{-\lambda} \circ \varphi_s \circ \delta_\lambda$ provides a compactly supported variation of Σ. Moreover, the variation $\{\psi_s\}_{s\in I}$ is nonsingular if and only if $\{\varphi_s\}_{s\in I}$ is nonsingular. By (2.14) we get

$$A(\Sigma_s) = A(\psi_s(\Sigma)) = A(\delta_{-\lambda}((\Sigma_\lambda)_s)) = e^{-3\lambda} A((\Sigma_\lambda)_s).$$

From here it is easy to deduce that if Σ is stable (resp. stable under non-singular variations) then the same holds for Σ_λ. To prove the reverse statement it suffices to change the roles of Σ and Σ_λ. $\square$

3.1. **Area-stationary surfaces.** In this part of the section we gather some facts about area-stationary surfaces in $\mathbb{H}^1$ that will be useful in the sequel.

Let Σ be a C^2 immersed surface in $\mathbb{H}^1$ with a unit normal vector N. We define the *mean curvature* of Σ as in [33] and [34], by the equality

$$-2H(p) = (\operatorname{div}_\Sigma \nu_h)(p), \qquad p \in \Sigma - \Sigma_0, \tag{3.1}$$

where ν_h is the horizontal Gauss map defined in (2.9) and $\operatorname{div}_\Sigma U$ stands for the divergence relative to Σ of a C^1 vector field U. We say that Σ is a *minimal surface* if the mean curvature vanishes on $\Sigma - \Sigma_0$.

In the following proposition we recall some features about area-stationary and minimal surfaces in $\mathbb{H}^1$ involving the structure of the regular and the singular set, see [10, Sect. 3], [34, Sect. 4] and the references therein. Similar results also hold in other sub-Riemannian spaces, see [27] and [29].

Proposition 3.3. *Let Σ be a C^2 immersed oriented minimal surface in $\mathbb{H}^1$ with singular set Σ_0. Then we have*

(i) *Any characteristic curve of Σ is a segment of a horizontal straight line.*
(ii) *Σ_0 consists of isolated points and C^1 curves with non-vanishing tangent vector (singular curves).*
(iii) *If Γ is a singular curve and $p \in \Gamma$, then there is a neighborhood B of p in Σ such that $B - \Gamma$ is the union of two disjoint domains B^+ and B^- contained in $\Sigma - \Sigma_0$. Moreover, the vector fields Z and ν_h extend continuously to p from B^+ and B^- in such a way that $Z_p^+ = -Z_p^-$ and $(\nu_h)_p^+ = -(\nu_h)_p^-$.*
(iv) *If Σ is any C^2 immersed oriented surface, then Σ is area-stationary if and only if Σ is minimal and the characteristic curves meet orthogonally the singular curves.*

Now we prove a regularity result for minimal surfaces in $\mathbb{H}^1$. Given a C^2 surface Σ in $\mathbb{H}^1$ with unit normal vector N, it is clear that the vector field $D_Z N$ is well defined on $\Sigma - \Sigma_0$ and it is continuous. By using the ruling property of minimal surfaces in Proposition 3.3 (i) we can obtain more regularity for N in the Z-direction.

Lemma 3.4. *Let Σ be a C^2 immersed oriented surface in $\mathbb{H}^1$. If Σ is minimal then, in $\Sigma - \Sigma_0$, the normal vector N is C^∞ in the direction of the characteristic field Z.*

Proof. Take $p \in \Sigma - \Sigma_0$. Let γ be the characteristic curve through p. Consider a C^1 curve $\alpha : (-\varepsilon_0, \varepsilon_0) \to \Sigma - \Sigma_0$ transverse to γ with $\alpha(0) = p$. Define $F(\varepsilon, s) := \alpha(\varepsilon) + s\, Z_{\alpha(\varepsilon)}$. By using (2.6) and Lemma 2.1 we get that $V(s) := (\partial F/\partial \varepsilon)(0, s)$ is a C^∞ Jacobi field along γ. Since both $\dot\gamma(s)$ and $V(s)$ are C^∞ and linearly independent for s small enough, the unit normal N to Σ along γ is given by

$$N = \pm \frac{\dot\gamma \times V}{|\dot\gamma \times V|},$$

where $\times$ denotes the cross product in $(\mathbb{H}^1, g)$. We conclude that N is a C^∞ vector field along γ. □

3.2. **Second variation of the area.** In this part of the section we provide some formulas for the second derivative of the area functional associated to some variations of an area-stationary surface. We first give some preliminary computations.

Lemma 3.5. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 immersed surface with unit normal vector N and singular set Σ_0. Consider a point $p \in \Sigma - \Sigma_0$, the horizontal Gauss map ν_h and the characteristic* *field Z defined in* (2.9)*. For any $v \in T_p\mathbb{H}^1$ we have*

$$D_v N_h = (D_v N)_h - \langle N, T\rangle\, J(v) - \langle N, J(v)\rangle\, T, \tag{3.2}$$

$$v\,(|N_h|) = \langle D_v N, \nu_h\rangle - \langle N, T\rangle\,\langle J(v), \nu_h\rangle, \tag{3.3}$$

$$v(\langle N, T\rangle) = \langle D_v N, T\rangle + \langle N, J(v)\rangle, \tag{3.4}$$

$$D_v \nu_h = |N_h|^{-1}\left(\langle D_v N, Z\rangle - \langle N, T\rangle\,\langle J(v), Z\rangle\right) Z + \langle Z, v\rangle\, T. \tag{3.5}$$

Proof. Equalities (3.2) and (3.3) are easily obtained since $N_h = N - \langle N, T\rangle\, T$. The proof of (3.4) is immediate. Let us show that (3.5) holds. As $|\nu_h| = 1$ and $\{Z_p, (\nu_h)_p, T_p\}$ is an orthonormal basis of $T_p\mathbb{H}^1$, we get

$$D_v \nu_h = \langle D_v \nu_h, Z\rangle\, Z + \langle D_v \nu_h, T\rangle\, T.$$

Note that $\langle D_v \nu_h, T\rangle = -\langle \nu_h, J(v)\rangle = \langle Z, v\rangle$ by (2.2). On the other hand, by using (3.2) and the fact that Z is tangent and horizontal, we deduce

$$\langle D_v \nu_h, Z\rangle = |N_h|^{-1}\langle D_v N_h, Z\rangle = |N_h|^{-1}\big(\langle D_v N, Z\rangle - \langle N, T\rangle\langle J(v), Z\rangle\big),$$

and the proof follows. □

Remark 3.6. In $\Sigma - \Sigma_0$ we can consider the orthonormal basis $\{Z, S\}$ defined in (2.9) and (2.10). By using the definition of mean curvature in (3.1) we have

$$-2H = \operatorname{div}_\Sigma \nu_h = \langle D_Z \nu_h, Z\rangle + \langle D_S \nu_h, S\rangle.$$

By (3.5) we get $D_Z \nu_h = T - |N_h|^{-1}\langle B(Z), Z\rangle Z$, and that $D_S \nu_h$ is proportional to Z. It follows that, in $\Sigma - \Sigma_0$

$$2H = |N_h|^{-1}\,\langle B(Z), Z\rangle, \tag{3.6}$$

$$D_Z \nu_h = T - (2H)\, Z, \tag{3.7}$$

where B is the Riemannian shape operator of Σ. On the other hand, the vector $D_Z Z$ is orthogonal to Z and T since $|Z| = 1$ and $\langle J(Z), Z\rangle = 0$. It follows that $D_Z Z$ is proportional to ν_h. From (3.7) we obtain

$$D_Z Z = \langle D_Z Z, \nu_h\rangle\, \nu_h = 2H\, \nu_h. \tag{3.8}$$

The second derivative of the area for non-singular variations of a minimal surface in $\mathbb{H}^1$ has appeared in several contexts, see [10, Prop. 6.1], [4, Sect. 3.2], [12, Sect. 14], [31, Proof of Thm. 3.5] and [26, Thm. E]. In the next theorem we compute the second derivative of the area functional for some *non-singular variations by Riemannian geodesics* of a C^2 minimal surface (maybe with non-empty boundary) in $\mathbb{H}^1$.

Theorem 3.7. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 immersed minimal surface with boundary $\partial\Sigma$ and singular set Σ_0. Consider the C^1 vector field $U = vN + wT$, where N is a unit normal vector to Σ and v, $w \in C^1_0(\Sigma - \Sigma_0)$. If $u = \langle U, N\rangle$, then the second derivative of the area for the variation induced by U is given by*

$$A''(0) = \int_\Sigma |N_h|^{-1}\left\{Z(u)^2 - \left(|B(Z) + S|^2 - 4|N_h|^2\right) u^2\right\} d\Sigma + \int_\Sigma \operatorname{div}_\Sigma(\xi Z)\, d\Sigma + \int_\Sigma \operatorname{div}_\Sigma(\mu Z)\, d\Sigma. \tag{3.9}$$

Here $\{Z, S\}$ is the orthonormal basis in (2.9) *and* (2.10)*, B is the Riemannian shape operator of Σ, the functions ξ and μ are defined by*

$$\xi = \langle N, T\rangle\,(1 - \langle B(Z), S\rangle)\, u^2, \tag{3.10}$$

$$\mu = |N_h|^2\,\left(\langle N, T\rangle\,(1 - \langle B(Z), S\rangle)\, w^2 - 2\langle B(Z), S\rangle\, vw\right), \tag{3.11}$$

and the divergence terms are understood in the sense of (2.15).

In particular, if $\partial\Sigma$ *is empty, then*

$$A''(0)=\int_\Sigma |N_h|^{-1}\left\{Z(u)^2-\left(|B(Z)+S|^2-4|N_h|^2\right)u^2\right\}d\Sigma. \tag{3.12}$$

Proof. We will follow closely the arguments in [38, §9]. Let $\varphi_s(p)=\exp_p(sU_p)$, for s small, be the variation induced by U. Then any $\Sigma_s=\varphi_s(\Sigma)$ is a C^1 immersed oriented surface. We extend the vector U along the variation by setting $U(\varphi_s(p))=(d/dt)|_{t=s}\,\varphi_t(p)$. Let N be a continuous vector field along the variation whose restriction to any Σ_s is a unit normal vector. By using (2.12), the coarea formula, and that the Riemannian area of Σ_0 vanishes, we have

$$A(s)=A(\Sigma_s)=\int_{\Sigma_s}|N_h|\,d\Sigma_s=\int_{\Sigma-\Sigma_0}(|N_h|\circ\varphi_s)\,|\text{Jac}\,\varphi_s|\,d\Sigma, \tag{3.13}$$

where $\text{Jac}\,\varphi_s$ is the Jacobian determinant of the diffeomorphism $\varphi_s:\Sigma\to\Sigma_s$.

We can suppose that $|N_h|(\varphi_s(p))>0$ whenever $p\in\Sigma-\Sigma_0$ and $|s|<s_0$. Take a point $p\in\Sigma-\Sigma_0$ and consider the orthonormal basis $\{e_1,e_2\}$ of $T_p\Sigma$ given by $e_1=Z_p$ and $e_2=S_p$. Let γ be the Riemannian geodesic defined by $\gamma(s)=\varphi_s(p)=\exp_p(sU_p)$. Denote by $N(s)$ the unit normal to Σ_s at $\gamma(s)$. Let $\alpha_i:(-\varepsilon_0,\varepsilon_0)\to\Sigma-\Sigma_0$ be a C^1 curve such that $\alpha_i(0)=p$ and $\dot\alpha_i(0)=e_i$. We define the C^1 map $F_i:(-\varepsilon_0,\varepsilon_0)\times\mathbb{R}\to\mathbb{H}^1$ given by $F_i(\varepsilon,s)=\varphi_s(\alpha_i(\varepsilon))=\exp_{\alpha_i(\varepsilon)}(sU_{\alpha_i(\varepsilon)})$. By using Lemma 2.1 we deduce that $E_i(s)=(\partial F_i/\partial\varepsilon)(0,s)=e_i(\varphi_s)$ is a C^∞ Jacobi vector field along γ with $[\dot\gamma,E_i]=0$ and $E_i(0)=e_i$. Therefore, we have the following identities along γ

$$D_UD_UE_i=-R(U,E_i)U, \tag{3.14}$$

$$D_UE_i=D_{E_i}U. \tag{3.15}$$

On the other hand, it is clear that $\{E_1(s),E_2(s)\}$ provide a basis of the tangent space to Σ_s at $\gamma(s)$. In particular $|\text{Jac}\,\varphi_s|=(|E_1|^2\,|E_2|^2-\langle E_1,E_2\rangle^2)^{1/2}(s)$, and so $|\text{Jac}\,\varphi_s|$ is C^∞ along γ. Moreover, we have $N(s)=\pm|E_1\times E_2|^{-1}\,(E_1\times E_2)(s)$, which is C^∞ on γ. Here $\times$ is the cross product in $(\mathbb{H}^1,g)$. We conclude that $|N_h|(s)$ is C^∞ along γ as well. Thus we can apply the classical result of differentiation under the integral sign to deduce, from (3.13), that

$$A''(0)=\int_{\Sigma-\Sigma_0}\{|N_h|''(0)+2\,|N_h|'(0)\,|\text{Jac}\,\varphi_s|'(0)+|N_h|\,|\text{Jac}\,\varphi_s|''(0)\}\,d\Sigma, \tag{3.16}$$

where we have used that $\varphi_0(p)=p$ for any $p\in\Sigma$, and so $|\text{Jac}\,\varphi_0|=1$.

Now we compute the different terms in (3.16). The calculus of $|\text{Jac}\,\varphi_s|'(0)$ and $|\text{Jac}\,\varphi_s|''(0)$ is found in [38, §9] for C^2 variations of a C^1 surface in Euclidean space. The arguments can be generalized to any Riemannian manifold for a C^1 variation obtained when we leave from a C^2 surface by geodesics. As $U=vN+wT$ on Σ, we deduce, by using $\text{div}_\Sigma T=0$ and the third equality in (2.11), that

$$|\text{Jac}\,\varphi_s|'(0)=\text{div}_\Sigma U=(-2H_R)\,v-|N_h|\,S(w)=-\langle B(S),S\rangle\,v-|N_h|\,S(w). \tag{3.17}$$

To get the second equality we have taken into account (3.6) to obtain

$$2H_R=-\,\text{div}_\Sigma N=\langle B(Z),Z\rangle+\langle B(S),S\rangle=\langle B(S),S\rangle.$$

On the other hand, it is known that

$$\begin{aligned}|\mathrm{Jac}\,\varphi_s|''(0) =&(\mathrm{div}_\Sigma U)^2 + \sum_{i=1}^2 |(D_{e_i} U)^\perp|^2 \\ &- \sum_{i=1}^2 \big\langle R(U,e_i)U, e_i\big\rangle - \sum_{i,j=1}^2 \big\langle D_{e_i}U, e_j\big\rangle \big\langle D_{e_j}U, e_i\big\rangle.\end{aligned}$$

Hence from (3.17), equality

$$D_e U = e(v)\,N - v\,B(e) + e(w)\,T + w\,J(e), \tag{3.18}$$

and equation (2.3), we get

$$\begin{aligned}|\mathrm{Jac}\,\varphi_s|''(0) =&|\nabla_\Sigma v|^2 + \big\langle N,T\big\rangle^2 |\nabla_\Sigma w|^2 - 2|N_h|\,Z(v)\,w - 2|N_h|\,\big\langle B(Z),S\big\rangle\,Z(w)\,v \\ &+ 2\big\langle N,T\big\rangle\,Z(v)\,Z(w) + 2\big\langle N,T\big\rangle\,S(v)\,S(w) \\ &- (\mathrm{Ric}(N,N) + |B|^2 - \big\langle B(S),S\big\rangle^2)\,v^2 - 4\big\langle N,T\big\rangle\,vw,\end{aligned} \tag{3.19}$$

where ∇_Σ is the gradient relative to Σ, Ric is the Ricci tensor in $(\mathbb{H}^1, g)$, and $|B|^2$ is the squared norm of the Riemannian shape operator of Σ.

Let us compute $|N_h|'(0)$ and $|N_h|''(0)$. From (3.3) and (2.2) it follows that

$$|N_h|'(s) = U(|N_h|) = \big\langle D_U N, \nu_h\big\rangle - \big\langle N,T\big\rangle\,\big\langle J(U),\nu_h\big\rangle = \big\langle D_U N, \nu_h\big\rangle + \big\langle N,T\big\rangle\,\big\langle U,Z\big\rangle.$$

Note that $U = uN - (|N_h|w)S$. Then $\big\langle U,Z\big\rangle = 0$ and $D_U N = -\nabla_\Sigma u + (|N_h|w)B(S)$ on $\Sigma - \Sigma_0$. By the second equality in (2.11) we obtain

$$|N_h|'(0) = \big\langle D_U N, \nu_h\big\rangle = -\big\langle N,T\big\rangle\,S(u) + |N_h|\,\big\langle N,T\big\rangle\,\big\langle B(S),S\big\rangle\,w. \tag{3.20}$$

We also deduce the following

$$\begin{aligned}|N_h|''(0) =&\big\langle D_U D_U N, \nu_h\big\rangle + \big\langle D_U N, D_U \nu_h\big\rangle \\ &+ U(\big\langle N,T\big\rangle)\,\big\langle U,Z\big\rangle + \big\langle N,T\big\rangle\,U(\big\langle U,Z\big\rangle) \\ =&\big\langle D_U D_U N, \nu_h\big\rangle + \big\langle D_U N, D_U \nu_h\big\rangle + \big\langle N,T\big\rangle\,\big\langle U, D_U Z\big\rangle,\end{aligned} \tag{3.21}$$

since $\big\langle U,Z\big\rangle = 0$ and $D_U U = 0$ on $\Sigma - \Sigma_0$. We can compute $D_U \nu_h$ from (3.5). By using that $D_U N = -\nabla_\Sigma u + (|N_h|w)B(S)$ and $J(U) = (|N_h|v)Z$ on $\Sigma - \Sigma_0$, we get

$$D_U \nu_h = -|N_h|^{-1}\,(Z(u) - |N_h|\,\big\langle B(Z),S\big\rangle\,w + |N_h|\,\big\langle N,T\big\rangle\,v)\,Z,$$

and so

$$\begin{aligned}\big\langle D_U N, D_U \nu_h\big\rangle =&|N_h|^{-1}\,Z(u)^2 + \big\langle N,T\big\rangle\,Z(u)\,v - 2\big\langle B(Z),S\big\rangle\,Z(u)\,w \\ &- |N_h|\,\big\langle N,T\big\rangle\,\big\langle B(Z),S\big\rangle\,vw + |N_h|\,\big\langle B(Z),S\big\rangle^2\,w^2.\end{aligned} \tag{3.22}$$

Now we compute $D_U Z$. The coordinates of this vector with respect to the orthonormal basis $\{Z, \nu_h, T\}$ are given by

$$\big\langle D_U Z, Z\big\rangle = 0, \quad \big\langle D_U Z, \nu_h\big\rangle = |N_h|^{-1}\,Z(u) - \big\langle B(Z),S\big\rangle\,w + \big\langle N,T\big\rangle\,v, \quad \big\langle D_U Z, T\big\rangle = -|N_h|\,v.$$

The previous equalities and the fact that $U = vN + wT$ on $\Sigma - \Sigma_0$ imply that

$$\big\langle U, D_U Z\big\rangle = Z(u)\,v - |N_h|\,(1 + \big\langle B(Z),S\big\rangle)\,vw. \tag{3.23}$$

It remains to compute $D_U D_U N$. Note that $\{E_1, E_2, N\}$ provides an orthonormal basis of $T_p\mathbb{H}^1$. As a consequence

$$D_U D_U N = \sum_{i=1}^2 \big\langle D_U D_U N, E_i\big\rangle\,E_i + \big\langle D_U D_U N, N\big\rangle\,N.$$

As $\langle N, E_i\rangle = 0$ along γ we get

$$\begin{aligned}\langle D_U D_U N, E_i\rangle &= -2\langle D_U N, D_U E_i\rangle - \langle N, D_U D_U E_i\rangle \\ &= -2\langle D_U N, D_{E_i} U\rangle + \langle N, R(U, E_i)U\rangle.\end{aligned}$$

The second equality follows from (3.15) and (3.14). Now recall that $e_1 = Z_p$ and $e_2 = S_p$. It follows that

$$\begin{aligned}\langle D_U D_U N, \nu_h\rangle = &- 2\langle N, T\rangle \langle D_U N, D_S U\rangle + \langle N, T\rangle \langle N, R(U, S)U\rangle \\ &+ |N_h| \langle D_U D_U N, N\rangle.\end{aligned} \tag{3.24}$$

By taking into account (3.18) and that $D_U N = -\nabla_\Sigma u + (|N_h|\, w)B(S)$, we obtain

$$\begin{aligned}\langle D_U N, D_S U\rangle = &- |N_h|^2 \langle B(S), S\rangle\, S(w)\, w - \langle N, T\rangle\, Z(u)\, w \\ &+ |N_h|\, S(u)\, S(w) + B(S)(u)\, v \\ &+ |N_h| \langle N, T\rangle \langle B(Z), S\rangle\, w^2 - |N_h|\, |B(S)|^2\, vw.\end{aligned} \tag{3.25}$$

On the other hand, we use (2.3) so that, after a straightforward computation, we conclude

$$\langle R(U, S)U, N\rangle = |N_h|\, (v + \langle N, T\rangle\, w)\, w. \tag{3.26}$$

Moreover, since $|N|^2 = 1$ on $\Sigma - \Sigma_0$ we have

$$\langle D_U D_U N, N\rangle = -|D_U N|^2 = -|\nabla_\Sigma u|^2 + 2|N_h|\, B(S)(u)\, w - |N_h|^2\, |B(S)|^2\, w^2. \tag{3.27}$$

By substituting (3.25), (3.26) and (3.27) into (3.24) we get $\langle D_U D_U N, \nu_h\rangle$. From (3.24), (3.22) and (3.23), after simplifying, equality (3.21) becomes

$$\begin{aligned}|N_h|''(0) = &|N_h|^{-1}\, Z(u)^2 - |N_h|\, |\nabla_\Sigma u|^2 + 2\langle N, T\rangle\, (Z(u)\, v - B(S)(u)\, v) \\ &+ 2\big(\langle N, T\rangle^2 + |N_h|^2 \langle B(Z), S\rangle - \langle B(Z), S\rangle\big)\, Z(u)\, w \\ &+ 2|N_h|^2 \langle B(S), S\rangle \big(S(u)\, w + \langle N, T\rangle\, S(w)\, w\big) - 2|N_h| \langle N, T\rangle\, S(u)\, S(w) \\ &+ 2|N_h| \langle N, T\rangle \big(|B(S)|^2 - \langle B(Z), S\rangle\big)\, vw \\ &+ \big(|N_h| \langle N, T\rangle^2 (1 - \langle B(Z), S\rangle)^2 - |N_h|^3 \langle B(S), S\rangle^2\big)\, w^2.\end{aligned} \tag{3.28}$$

Now, since $u = v + \langle N, T\rangle w$, we have $\nabla_\Sigma u = \nabla_\Sigma v + w\, \nabla_\Sigma(\langle N, T\rangle) + \langle N, T\rangle\, \nabla_\Sigma w$. By (3.4) and (2.11) it is easy to see that

$$\begin{aligned}Z(\langle N, T\rangle) &= |N_h|\, (\langle B(Z), S\rangle - 1), \\ S(\langle N, T\rangle) &= |N_h|\, \langle B(S), S\rangle.\end{aligned} \tag{3.29}$$

This allows us to compute the term $|N_h|\, |\nabla_\Sigma u|^2$ in (3.28). At this moment, we use (3.28), (3.20), (3.17) and (3.19) so that, after simplifying, we get that

$$|N_h|''(0) + 2\, |N_h|'(0)\, |\mathrm{Jac}\, \varphi_s|'(0) + |N_h|\, |\mathrm{Jac}\, \varphi_s|''(0)$$

is equal to

$$\begin{aligned}&|N_h|^{-1}\, Z(u)^2 + 2\langle N, T\rangle\, (1 - \langle B(Z), S\rangle)\, (Z(v)\, v + Z(w)\, w) \\ &+ 2(\langle N, T\rangle^2 - \langle B(Z), S\rangle)\, (Z(w)\, v + Z(v)\, w) + q_1\, v^2 \\ &+ 2|N_h| \langle N, T\rangle\, (\langle B(Z), S\rangle - 3)\, vw - |N_h|(1 - \langle B(Z), S\rangle)^2\, w^2,\end{aligned} \tag{3.30}$$

where q_1 is the function given by

$$q_1 = |N_h|\, \big(\langle B(S), S\rangle^2 - \mathrm{Ric}(N, N) - |B|^2\big).$$

In order to obtain (3.9) from (3.16) and (3.30), we apply Lemma 3.9 below. We deduce the following

$$\begin{aligned}
&|N_h|''(0) + 2\,|N_h|'(0)\,|\mathrm{Jac}\,\varphi_s|'(0) + |N_h|\,|\mathrm{Jac}\,\varphi_s|''(0) \\
&\quad =|N_h|^{-1}\,Z(u)^2 + \mathrm{div}_\Sigma(\rho Z) \\
&\quad\quad + \big\{q_1 + \big(\langle B(Z),S\rangle - 1\big)\big(\langle N,T\rangle\,q_2 + Z(\langle N,T\rangle)\big) + \langle N,T\rangle\,Z(\langle B(Z),S\rangle)\big\}\,v^2 \\
&\quad\quad + \big\{\big(\langle B(Z),S\rangle - 1\big)\big(\langle N,T\rangle\,q_2 + Z(\langle N,T\rangle)\big) + \langle N,T\rangle\,Z(\langle B(Z),S\rangle) \\
&\quad\quad -|N_h|\,(1-\langle B(Z),S\rangle)^2\big\}\,w^2 + 2\Big\{|N_h|\,\langle N,T\rangle\,(\langle B(Z),S\rangle - 3) - Z(\langle N,T\rangle^2) \\
&\quad\quad -\langle N,T\rangle^2\,q_2 + \langle B(Z),S\rangle\,q_2 + Z(\langle B(Z),S\rangle)\Big\}\,vw,
\end{aligned}$$

where ρ is the function

$$\langle N,T\rangle\,(1-\langle B(Z),S\rangle)\,(v^2+w^2) + 2\,(\langle N,T\rangle^2 - \langle B(Z),S\rangle)\,vw.$$

A straightforward computation using (3.29), (3.34), the identities

$$\begin{aligned}
&\mathrm{Ric}(N,N) = 2 - 4|N_h|^2 \quad \text{(it follows from (2.4))}, \\
&|B|^2 = \langle B(Z),Z\rangle^2 + \langle B(S),S\rangle^2 + 2\langle B(Z),S\rangle^2 = \langle B(S),S\rangle^2 + 2\langle B(Z),S\rangle^2, \\
&B(Z) = \langle B(Z),Z\rangle\,Z + \langle B(Z),S\rangle\,S = \langle B(Z),S\rangle\,S,
\end{aligned}$$

and that $u = v + \langle N,T\rangle w$, gives us

$$\begin{aligned}
&|N_h|''(0) + 2\,|N_h|'(0)\,|\mathrm{Jac}\,\varphi_s|'(0) + |N_h|\,|\mathrm{Jac}\,\varphi_s|''(0) \\
&\qquad =|N_h|^{-1}\,Z(u)^2 - |N_h|^{-1}\,(|B(Z)+S|^2 - 4|N_h|^2)\,u^2 \\
&\qquad\quad + \mathrm{div}_\Sigma(\xi Z) + \mathrm{div}_\Sigma(\mu Z),
\end{aligned}$$

where ξ and μ are the functions given in (3.10) and (3.11).

Finally, suppose that $\partial\Sigma$ is empty. Then ξ and μ are continuous functions with compact support in $\Sigma - \Sigma_0$ and they are also C^1 in the Z-direction by Lemma 3.10. Hence the integrals of $\mathrm{div}_\Sigma(\xi Z)$ and $\mathrm{div}_\Sigma(\mu Z)$ vanish by virtue of the divergence theorem in Lemma 2.4. This proves (3.12). □

Remark 3.8. The divergence terms in (3.9) need not vanish if $\partial\Sigma$ is nonempty. In the proof of Proposition 5.2 we will show that these terms play an important role.

Lemma 3.9. *Let Σ be a C^2 immersed oriented surface in $\mathbb{H}^1$ and $\phi \in C^1(\Sigma)$. Then, in the regular set $\Sigma - \Sigma_0$, we have*

$$\mathrm{div}_\Sigma(\phi\, Z) = Z(\phi) + q_2\,\phi,$$

where q_2 is the function given by

$$q_2 = |N_h|^{-1}\langle N,T\rangle\,(1+\langle B(Z),S\rangle).$$

Proof. Clearly we have

$$\mathrm{div}_\Sigma(\phi\, Z) = (\mathrm{div}_\Sigma Z)\,\phi + Z(\phi). \tag{3.31}$$

Note that

$$\mathrm{div}_\Sigma Z = \langle D_Z Z, Z\rangle + \langle D_S Z, S\rangle = \langle D_S Z, S\rangle,$$

since $|Z|^2 = 1$. We compute the components of $D_S Z$ in the orthonormal basis $\{Z, \nu_h, T\}$. Observe that $D_S Z$ is orthogonal to Z. By using (3.5) and that $J(S) = \langle N,T\rangle Z$, we get

$$\begin{aligned}
\langle D_S Z, \nu_h\rangle &= -\langle Z, D_S \nu_h\rangle = |N_h|^{-1}\,(\langle B(Z),S\rangle + \langle N,T\rangle^2), \\
\langle D_S Z, T\rangle &= -\langle Z, J(S)\rangle = -\langle N,T\rangle.
\end{aligned}$$

From here we deduce

$$D_S Z = |N_h|^{-1} \left(\left\langle B(Z), S \right\rangle + 1 - |N_h|^2 \right) \nu_h - \left\langle N, T \right\rangle T. \tag{3.32}$$

As a consequence, we obtain

$$\operatorname{div}_\Sigma Z = \left\langle D_S Z, S \right\rangle = |N_h|^{-1} \left\langle N, T \right\rangle (1 + \left\langle B(Z), S \right\rangle). \tag{3.33}$$

The proof finishes by substituting (3.33) into (3.31). □

Lemma 3.10. *Let Σ be a C^2 immersed oriented minimal surface in $\mathbb{H}^1$. Then, in the regular set $\Sigma - \Sigma_0$, we have*

(i) *The functions $\left\langle N, T \right\rangle$ and $|N_h|$ are C^∞ in the Z-direction.*
(ii) *The vector fields ν_h and S are C^∞ in the Z-direction.*
(iii) *The function $\left\langle B(Z), S \right\rangle$ is C^∞ in the Z-direction, and*

$$Z(\left\langle B(Z), S \right\rangle) = 4|N_h| \left\langle N, T \right\rangle - 2|N_h|^{-1} \left\langle N, T \right\rangle \left\langle B(Z), S \right\rangle (1 + \left\langle B(Z), S \right\rangle). \tag{3.34}$$

Proof. Recall that N is C^∞ in the Z-direction by Lemma 3.4. This implies (i). Assertions (ii) and (iii) follow from (i) by the definition of ν_h and S in (2.9) and (2.10). To compute $Z(\left\langle B(Z), S \right\rangle)$ note that

$$Z(\left\langle B(Z), S \right\rangle) = Z(-\left\langle D_Z N, S \right\rangle) = -\left\langle D_Z D_Z N, S \right\rangle - \left\langle D_Z N, D_Z S \right\rangle.$$

It is clear that $D_Z N$ is tangent to Σ. On the other hand, $D_Z S$ is proportional to N. This comes from the fact that $\left\langle D_Z S, Z \right\rangle = -\left\langle S, D_Z Z \right\rangle = 0$ by (3.8), whereas $\left\langle D_Z S, S \right\rangle = 0$. Therefore we have

$$\left\langle D_Z N, D_Z S \right\rangle = 0, \tag{3.35}$$

$$Z(\left\langle B(Z), S \right\rangle) = -\left\langle D_Z D_Z N, S \right\rangle = \left\langle N, D_Z D_Z S \right\rangle. \tag{3.36}$$

It remains to compute $D_Z D_Z S$. From (3.32) we see that $D_S Z$ is C^∞ in the Z-direction. As a consequence $[Z, S] = D_Z S - D_S Z$ is also C^∞ in the Z direction, and $D_Z [Z, S] = D_Z D_Z S - D_Z D_S Z$. Thus equation (3.36) becomes

$$\begin{aligned} Z(\left\langle B(Z), S \right\rangle) &= \left\langle N, D_Z [Z, S] \right\rangle + \left\langle N, D_Z D_S Z \right\rangle \\ &= \left\langle N, D_Z [Z, S] \right\rangle + \left\langle N, D_S D_Z Z \right\rangle - \left\langle N, R(Z, S) Z \right\rangle + \left\langle N, D_{[Z,S]} Z \right\rangle \\ &= \left\langle N, D_Z [Z, S] \right\rangle - \left\langle N, R(Z, S) Z \right\rangle + \left\langle N, D_{[Z,S]} Z \right\rangle, \end{aligned} \tag{3.37}$$

where R is the Riemannian curvature tensor and we have used (3.8) to get $D_S D_Z Z = 0$. Now, observe that

$$\left\langle [Z, S], N \right\rangle = \left\langle D_Z S, N \right\rangle - \left\langle D_S Z, N \right\rangle = -\left\langle S, D_Z N \right\rangle + \left\langle Z, D_S N \right\rangle = 0,$$

which implies that $[Z, S]$ is tangent to Σ. Therefore, we deduce

$$\begin{aligned} \left\langle N, D_Z [Z, S] \right\rangle &= \left\langle B(Z), [Z, S] \right\rangle = \left\langle B(Z), D_Z S \right\rangle - \left\langle B(Z), D_S Z \right\rangle = -\left\langle B(Z), D_S Z \right\rangle, \\ \left\langle N, D_{[Z,S]} Z \right\rangle &= -\left\langle D_{[Z,S]} N, Z \right\rangle = \left\langle B(Z), [Z, S] \right\rangle = -\left\langle B(Z), D_S Z \right\rangle, \end{aligned}$$

where we have used (3.35). If we put this information into (3.37), we obtain

$$Z(\left\langle B(Z), S \right\rangle) = -2\left\langle B(Z), D_S Z \right\rangle - \left\langle N, R(Z, S) Z \right\rangle. \tag{3.38}$$

To compute the first term above we take into account (3.32). After simplifying, we get

$$\left\langle B(Z), D_S Z \right\rangle = |N_h|^{-1} \left\langle N, T \right\rangle \left\langle B(Z), S \right\rangle (1 + \left\langle B(Z), S \right\rangle). \tag{3.39}$$

For the second term, we apply (2.3) so that, after a straightforward calculus, we conclude

$$\left\langle N, R(Z, S) Z \right\rangle = -4|N_h| \left\langle N, T \right\rangle. \tag{3.40}$$

The proof finishes by substituting (3.39) and (3.40) into (3.38). □

In the next result we compute the second derivative of the area for some *vertical variations* of an area-stationary surface Σ whose singular curves $(\Sigma_0)_c$ are C^3 (in [34, Prop. 4.20] we proved that they are always C^2). We suppose that the variation is constant along the characteristic curves of a tubular neighborhood around $(\Sigma_0)_c$. By a tubular neighborhood of radius $\varepsilon > 0$ we mean the union of all the characteristic segments of length 2ε centered at $(\Sigma_0)_c$.

Proposition 3.11. *Let Σ be a C^2 immersed oriented area-stationary surface in $\mathbb{H}^1$ such that the singular curves $(\Sigma_0)_c$ of Σ are of class C^3. Let $\varphi_r(p) := \exp_p(rw(p)T_p)$, for r small, be the vertical variation of Σ induced by a function $w \in C_0^2(\Sigma)$. Suppose that there is a tubular neighborhood E_0 of $\operatorname{supp}(w) \cap (\Sigma_0)_c$ where $Z(w) = 0$. Then, there is a tubular neighborhood E of $\operatorname{supp}(w) \cap (\Sigma_0)_c$ such that*

$$\frac{d^2}{dr^2}\bigg|_{r=0} A(\varphi_r(E)) = \int_{(\Sigma_0)_c} S(w)^2\, dl,$$

where S is any continuous extension of the vector field S defined in (2.10) to $(\Sigma_0)_c$ and dl denotes the Riemannian length element.

Proof. We can restrict ourselves to a neighborhood of a single singular curve Γ. We consider a parameterization $\Gamma(\varepsilon) = (x(\varepsilon), y(\varepsilon), t(\varepsilon))$ by arc-length. By Proposition 3.3 the area-stationary surface Σ can be parameterized in a neighborhood of $\operatorname{supp}(w) \cap \Gamma$ by

$$(\varepsilon, s) \mapsto \Gamma(\varepsilon) + sJ(\dot{\Gamma}(\varepsilon)),$$

so that the curves with ε constant are the characteristic curves of Σ. In Euclidean coordinates we have

$$\begin{aligned} x(\varepsilon, s) &= x(\varepsilon) - s\dot{y}(\varepsilon), \\ y(\varepsilon, s) &= y(\varepsilon) + s\dot{x}(\varepsilon), \\ t(\varepsilon, s) &= t(\varepsilon) - s\,(x\dot{x} + y\dot{y})(\varepsilon). \end{aligned}$$

As $Z(w) = 0$ we get that w is a function of ε alone. The deformation $\varphi_r(p) = \exp_p(rw(p)T_p)$ consists on changing the t-coordinate of the above parameterization by

$$t(\varepsilon, s) + rw(\varepsilon).$$

A simple computation shows that the tangent space to the surface $\Sigma_r := \varphi_r(\Sigma)$ is generated by the vectors

$$-\dot{y}X + \dot{x}Y, \qquad (\dot{x} - s\ddot{y})\,X + (\dot{y} + s\ddot{x})\,Y + \big(s(-2 + sh) + r\dot{w}\big)\,T, \tag{3.41}$$

where x, y and t are the coordinates of Γ, dots represent derivatives with respect to ε, and $h = h(\varepsilon) = (\dot{x}\ddot{y} - \dot{y}\ddot{x})(\varepsilon)$ is the Euclidean geodesic curvature of the xy-projection of Γ.

Hence the singular points of Σ_r corresponds to the zero set of $F(\varepsilon, s, r) := s(-2+sh(\varepsilon)) + r\dot{w}(\varepsilon)$. Observe that F is a C^1 function since the singular curves are assumed to be of class C^3 and $w \in C^2$. As $(\partial F/\partial s)(\varepsilon, 0, 0) = -2$, we can apply the Implicit Function Theorem and a compactness argument to show that there are positive values ε_0, s_0, r_0, and a C^1 function $s : (-\varepsilon_0, \varepsilon_0) \times (-r_0, r_0) \to (-s_0, s_0)$ with $s(\varepsilon, 0) = 0$ satisfying $F(\varepsilon, s(\varepsilon, r), r) = 0$. Here $\varepsilon_0 > 0$ is taken so that $\operatorname{supp}(w) \cap \Gamma \subset [-\varepsilon_0, \varepsilon_0]$. We define $E := F((-\varepsilon_0, \varepsilon_0) \times (-s_0, s_0))$.

On the other hand, a computation using (3.41) shows that

$$|(N_h)_r|\, d\Sigma_r = |s(-2 + sh(\varepsilon)) + r\dot{w}(\varepsilon)|\, d\varepsilon\, ds.$$

Hence we have

$$A(\varphi_r(E)) = \int_{-\varepsilon_0}^{\varepsilon_0} \bigg\{ \int_{-s_0}^{s_0} |s(-2 + sh(\varepsilon)) + r\dot{w}(\varepsilon)|\, ds \bigg\}\, d\varepsilon.$$

Denote by $f_\varepsilon(r)$ the integral between brackets. As $(\partial F/\partial s)(\varepsilon,0,0)<0$ we deduce

$$f_\varepsilon(r)=\int_{-s_0}^{s(\varepsilon,r)}\big(s(-2+sh(\varepsilon))+r\dot w(\varepsilon)\big)\,ds+\int_{s(\varepsilon,r)}^{s_0}\big(s(2-sh(\varepsilon))-r\dot w(\varepsilon)\big)\,ds.$$

Taking derivatives with respect to r we obtain

$$f_\varepsilon'(r)=\int_{-s_0}^{s(\varepsilon,r)}\dot w(\varepsilon)\,ds-\int_{s(\varepsilon,r)}^{s_0}\dot w(\varepsilon)\,ds=2\dot w(\varepsilon)s(\varepsilon,r).$$

Taking derivatives again we have

$$f_\varepsilon''(r)=2\dot w(\varepsilon)\,\frac{\partial s}{\partial r}(\varepsilon,r).$$

Since $(\partial s/\partial r)(\varepsilon,0)=\dot w(\varepsilon)/2$ we conclude

$$f_\varepsilon''(0)=\dot w(\varepsilon)^2,$$

and so

$$\frac{d^2}{dr^2}\bigg|_{r=0}A(\varphi_r(E))=\int_{-\varepsilon_0}^{\varepsilon_0}\dot w(\varepsilon)^2\,d\varepsilon.$$

By Proposition 3.3 we know that the vector field S defined in (2.10) extends continuously to Γ as a unit tangent vector to Γ. Then $\dot w(\varepsilon)^2=S(w)^2$ and the claim follows. □

3.3. **A stability criterion for stable surfaces in $\mathbb{H}^1$.** Here we obtain a useful criterion to check if a given area-stationary surface is unstable. First we need a definition. Let Σ be a C^2 oriented minimal surface immersed in $\mathbb{H}^1$. For two functions $u,v\in C_0(\Sigma-\Sigma_0)$ which are also C^1 in the Z-direction, we denote

$$\mathcal{I}(u,v):=\int_\Sigma |N_h|^{-1}\left\{Z(u)\,Z(v)-\big(|B(Z)+S|^2-4|N_h|^2\big)\,uv\right\}\,d\Sigma, \tag{3.42}$$

where $\{Z,S\}$ is the orthonormal basis in (2.9) and (2.10), and B is the Riemannian shape operator of Σ. The expression (3.42) defines a symmetric bilinear form, which we call the *index form* associated to Σ by analogy with the Riemannian situation, see [3].

Proposition 3.12. *Let Σ be a C^2 immersed oriented area-stationary surface in $\mathbb{H}^1$ with singular set Σ_0. If Σ is stable under non-singular variations then the index form defined in* (3.42) *satisfies $\mathcal{I}(u,u)\geqslant 0$ for any function $u\in C_0(\Sigma-\Sigma_0)$ which is also C^1 in the direction of the characteristic field Z.*

Proof. Let N be the unit normal vector to Σ. Take $u\in C_0^1(\Sigma-\Sigma_0)$ and consider the vector field $U=uN$. Note that Σ is a minimal surface since it is area-stationary. Hence Theorem 3.7 implies that the second derivative of the area for the variation induced by U is $A''(0)=\mathcal{I}(u,u)$. As Σ is stable under non-singular variations we deduce that

$$\mathcal{I}(u,u)\geqslant 0,\qquad \text{for any } u\in C_0^1(\Sigma-\Sigma_0). \tag{3.43}$$

Now fix a function $u\in C_0(\Sigma-\Sigma_0)$ which is also C^1 in the Z-direction. By using Lemma 2.3 and that Σ_0 has vanishing Riemannian area, we can find a compact set $K\subseteq\Sigma-\Sigma_0$ and a sequence of functions $\{u_\varepsilon\}_{\varepsilon>0}$ in $C_0^1(\Sigma-\Sigma_0)$ such that $\{u_\varepsilon\}\to u$ in $L^2(\Sigma)$, $\{Z(u_\varepsilon)\}\to Z(u)$ in $L^2(\Sigma)$, while the supports of u_ε and u are contained in K for any $\varepsilon>0$. From here it is not difficult to check that $\{|N_h|^{-1/2}Z(u_\varepsilon)\}\to|N_h|^{-1/2}Z(u)$, $\{(|N_h|^{-1}f_1)^{1/2}u_\varepsilon\}\to(|N_h|^{-1}f_1)^{1/2}u$ and $\{(|N_h|^{-1}f_2)^{1/2}u_\varepsilon\}\to(|N_h|^{-1}f_2)^{1/2}u$ in $L^2(\Sigma)$, where $f_1=|B(Z)+S|^2$ and $f_2=4|N_h|^2$. It follows that $\lim_{\varepsilon\to 0}\mathcal{I}(u_\varepsilon,u_\varepsilon)=\mathcal{I}(u,u)$, so that inequality (3.43) proves the claim. □

Remark 3.13. As in [12, Thm. 15.2] and [31, Thm. 3.5, Cor. 3.7] the previous result can be seen as a Poincaré type inequality for stable surfaces in $\mathbb{H}^1$.

3.4. **Integration by parts. The stability operator in $\mathbb{H}^1$.** In Riemannian geometry the index form of a minimal surface can be expressed in terms of a second order elliptic operator defined on the surface, see [3]. In this part of the section we prove a similar property for the index form (3.42) of a minimal surface in $\mathbb{H}^1$ which involves a hypoelliptic second order differential operator on the surface.

Proposition 3.14 (Integration by parts I)**.** *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 immersed surface with unit normal vector N and singular set Σ_0. Consider two functions $u \in C_0(\Sigma - \Sigma_0)$ and $v \in C(\Sigma - \Sigma_0)$ which are C^1 and C^2 in the Z-direction, respectively. Then we have*

$$\mathcal{I}(u,v) = -\int_\Sigma u\,\mathcal{L}(v)\,d\Sigma,$$

where $\mathcal{I}$ is the index form defined in (3.42)*, and $\mathcal{L}$ is the second order differential operator*

$$\mathcal{L}(v) := |N_h|^{-1}\big\{Z(Z(v)) + 2\,|N_h|^{-1}\,\big\langle N,T\big\rangle\,\big\langle B(Z),S\big\rangle\,Z(v) + (|B(Z)+S|^2 - 4|N_h|^2)\,v\big\}. \tag{3.44}$$

Proof. Along this proof we shall denote $q = |B(Z)+S|^2 - 4|N_h|^2$. First note that in $\Sigma - \Sigma_0$ the hypotheses about u and v ensure that $|N_h|^{-1}Z(v)$ and $|N_h|^{-1}Z(v)u$ are C^1 in the Z-direction. Suppose proved that

$$\mathcal{L}(v) = \operatorname{div}_\Sigma(|N_h|^{-1}Z(v)\,Z) + |N_h|^{-1}qv. \tag{3.45}$$

In such a case, we would apply the divergence theorem in Lemma 2.4 in order to get

$$\begin{aligned} 0 = \int_\Sigma \operatorname{div}_\Sigma(|N_h|^{-1}Z(v)\,u\,Z)\,d\Sigma &= \int_\Sigma u\,\operatorname{div}_\Sigma(|N_h|^{-1}Z(v)\,Z)\,d\Sigma + \int_\Sigma |N_h|^{-1}Z(u)\,Z(v)\,d\Sigma. \\ &= \int_\Sigma u\,\mathcal{L}(v)\,d\Sigma + \mathcal{I}(u,v), \end{aligned}$$

and this would finish the proof.

To obtain (3.45) observe that

$$\operatorname{div}_\Sigma(|N_h|^{-1}Z(v)\,Z) = |N_h|^{-1}Z(v)\,\operatorname{div}_\Sigma Z + Z(|N_h|^{-1}Z(v)). \tag{3.46}$$

The computation of $\operatorname{div}_\Sigma Z$ is given in (3.33). On the other hand, we have

$$\begin{aligned} Z(|N_h|^{-1}Z(v)) &= |N_h|^{-1}Z(Z(v)) + Z(|N_h|^{-1})\,Z(v) \\ &= |N_h|^{-1}Z(Z(v)) - |N_h|^{-2}Z(|N_h|)\,Z(v) \\ &= |N_h|^{-1}Z(Z(v)) + |N_h|^{-2}\big\langle N,T\big\rangle\,(\big\langle B(Z),S\big\rangle - 1)\,Z(v), \end{aligned} \tag{3.47}$$

where we have used (3.3) to compute $Z(|N_h|)$. To deduce (3.45) it suffices to simplify in (3.46) after substituting the information of (3.33) and (3.47). ☐

Remark 3.15. If Σ is a minimal surface then the functional $\mathcal{L}$ in (3.44) provides a Sturm-Liouville differential operator along any of the characteristic segments of Σ.

As a direct consequence of Propositions 3.14 and 3.12 we deduce

Corollary 3.16. *Let Σ be a C^2 immersed oriented area-stationary surface in $\mathbb{H}^1$. If Σ is stable under non-singular variations then we have*

$$-\int_\Sigma u\,\mathcal{L}(u)\,d\Sigma \geqslant 0,$$

for any function $u \in C_0(\Sigma - \Sigma_0)$ which is also C^2 in the Z-direction.

Finally, with the same technique as in Proposition 3.14 we can prove the following lemma.

Lemma 3.17 (Integration by parts II)**.** *Let $\Sigma\subset\mathbb{H}^1$ be a C^2 immersed surface with unit normal vector N and singular set Σ_0. Consider two functions $u\in C_0(\Sigma-\Sigma_0)$ and $v\in C(\Sigma-\Sigma_0)$ which are C^1 and C^2 in the Z-direction, respectively. Then we have*

$$\int_\Sigma |N_h|\,\big\{Z(u)\,Z(v)+u\,Z(Z(v))+2\,|N_h|^{-1}\,\big\langle N,T\big\rangle\,u\,Z(v)\big\}\,d\Sigma=0.$$

Proof. Observe that in $\Sigma-\Sigma_0$

$$\begin{aligned}\operatorname{div}_\Sigma(|N_h|\,u\,Z(v)\,Z)&=u\,Z(v)\,\big\{Z(|N_h|)+|N_h|\,\operatorname{div}_\Sigma Z\big\}\\&\quad+|N_h|\,Z(u)\,Z(v)+|N_h|\,u\,Z(Z(v)),\end{aligned}$$

and that the function in the left-hand side has vanishing integral by Lemma 2.4. On the other hand, (3.3) gives us

$$Z(|N_h|)=\big\langle N,T\big\rangle-\big\langle N,T\big\rangle\,\big\langle B(Z),S\big\rangle, \tag{3.48}$$

which together with (3.33) implies $Z(|N_h|)+|N_h|\operatorname{div}_\Sigma Z=2\,\big\langle N,T\big\rangle$. The result follows. □

Remark 3.18. Some other integration by parts formulas in $\mathbb{H}^1$ can be found in [12, Sect. 10].

4. Complete stable surfaces with empty singular set

In this section we provide the classification of C^2 complete stable surfaces in $\mathbb{H}^1$ with empty singular set. Recall that if $\Sigma_0=\emptyset$ then Σ is area-stationary if and only if Σ is minimal by Proposition 3.3 (iv). We say that an immersed surface Σ in $\mathbb{H}^1$ is *complete* if it is complete in the Riemannian manifold $(\mathbb{H}^1,g)$. For a C^2 complete area-stationary surface Σ with $\Sigma_0=\emptyset$ the characteristic curves are straight lines by Proposition 3.3 (i). In particular Σ cannot be compact. Some classification results for area-stationary surfaces with empty singular set can be found in [33, Thm. 5.4], [9] and [34, Prop. 6.16]. Note also that for such surfaces to be stable is equivalent to be stable under non-singular variations.

In Euclidean three-space the description of complete stable area-stationary surfaces can be obtained by means of a logarithmic cut-off of the function $u=1$ associated to the variation by level surfaces of the distance function, see [18]. In $\mathbb{H}^1$ the vector field induced by the family of equidistants for the Carnot-Carathéodory distance d_{cc} to a C^2 surface with empty singular set coincides, up to a sign, with the horizontal Gauss map ν_h, see [1, Thms. 1.1 and 1.2]. This leads us to use the stability condition in Proposition 3.12 with a test function of the form $f=u|N_h|$, where f is continuous with compact support on the surface and C^1 in the direction of the characteristic field Z. We first compute the index form for these type of functions.

Lemma 4.1. *Let $\Sigma\subset\mathbb{H}^1$ be a C^2 immersed minimal surface in $\mathbb{H}^1$ with unit normal vector N and singular set Σ_0. Then, for any function $f\in C_0(\Sigma-\Sigma_0)$ which is also C^1 in the Z-direction, we have*

$$\mathcal{I}(f|N_h|,f|N_h|)=\int_\Sigma |N_h|\,\big\{Z(f)^2-\mathcal{L}(|N_h|)\,f^2\big\}\,d\Sigma, \tag{4.1}$$

where $\mathcal{I}$ is the index form in (3.42)*, and $\mathcal{L}$ is the differential operator in* (3.44)*.*

Proof. Along this proof we shall denote $w=f|N_h|$ and $q=|B(Z)+S|^2-4|N_h|^2$. Note that w is C^1 in the Z-direction and $Z(w)=fZ(|N_h|)+|N_h|Z(f)$. If we introduce w in the index form we obtain

$$\mathcal{I}(w,w)=\int_\Sigma\big\{|N_h|\,Z(f)^2+|N_h|^{-1}Z(|N_h|)^2\,f^2+Z(f^2)\,Z(|N_h|)-|N_h|\,q\,f^2\big\}\,d\Sigma. \tag{4.2}$$

On the other hand, we know from Lemma 3.10 (i) that $|N_h|$ is C^∞ in the Z-direction. Therefore, we can apply Proposition 3.14 with $u = f^2|N_h|$ and $v = |N_h|$, so that we get

$$\begin{aligned}-\int_\Sigma |N_h|\,\mathcal{L}(|N_h|)\,f^2\,d\Sigma &= \int_\Sigma \big\{|N_h|^{-1}Z(|N_h|)^2\,f^2 + Z(f^2)\,Z(|N_h|) - |N_h|\,q\,f^2\big\}\,d\Sigma \\ &= \mathcal{I}(w,w) - \int_\Sigma |N_h|\,Z(f)^2\,d\Sigma,\end{aligned}$$

where in the second equality we have used (4.2). This proves the claim. □

Remark 4.2. Some other versions of (4.1) for variations of a C^2 surface Σ with associated vector field $f\nu_h$, $f \in C^2_0(\Sigma - \Sigma_0)$, can be found in [13, Lem. 3.9] and [15, Thm. 3.4]. See also [4, Sect. 3.2] and [31, Thm. 3.5] for the case of an intrinsic graph associated to a function with less regularity than C^2.

In the next lemma we particularize (4.1) for $f = uv^{-1}$. This type of test functions will be used to prove Theorem 4.7.

Lemma 4.3. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 immersed minimal surface in $\mathbb{H}^1$ with unit normal vector N and singular set Σ_0. Consider two functions $u \in C_0(\Sigma - \Sigma_0)$ and $v \in C(\Sigma - \Sigma_0)$ which are C^1 and C^2 in the Z-direction, respectively. If v never vanishes, then*

$$\begin{aligned}\mathcal{I}(uv^{-1}|N_h|,uv^{-1}|N_h|) &= \int_\Sigma |N_h|\,v^{-2}Z(u)^2\,d\Sigma \\ &+ \int_\Sigma |N_h|\,u^2\Big\{Z(v^{-1})^2 - \frac{1}{2}\,Z(Z(v^{-2})) - |N_h|^{-1}\,\big\langle N,T\big\rangle\,Z(v^{-2})\Big\}\,d\Sigma \\ &- \int_\Sigma |N_h|\,\mathcal{L}(|N_h|)\,(uv^{-1})^2\,d\Sigma,\end{aligned} \tag{4.3}$$

where $\mathcal{I}$ is the index form in (3.42)*, and $\mathcal{L}$ is the differential operator in* (3.44)*.*

Proof. From (4.1) we only have to compute

$$\int_\Sigma |N_h|\,Z(uv^{-1})^2\,d\Sigma.$$

Since

$$Z(uv^{-1})^2 = v^{-2}Z(u)^2 + u^2Z(v^{-1})^2 + \frac{1}{2}\,Z(u^2)\,Z(v^{-2}),$$

and Lemma 3.17 implies

$$\int_\Sigma \frac{1}{2}\,|N_h|\,Z(u^2)\,Z(v^{-2})\,d\Sigma = -\int_\Sigma |N_h|\,u^2\Big\{\frac{1}{2}\,Z(Z(v^{-2})) + |N_h|^{-1}\,\big\langle N,T\big\rangle\,Z(v^{-2})\Big\}\,d\Sigma,$$

we see that (4.3) holds. □

The previous lemmas suggest that, for a function $u = f|N_h|$, the stability condition in Proposition 3.12 is more restrictive if $\mathcal{L}(|N_h|) > 0$. Thus it is interesting to compute $\mathcal{L}(|N_h|)$ and to study its sign.

Lemma 4.4. *Let Σ be a C^2 immersed minimal surface in $\mathbb{H}^1$ with unit normal vector N. Consider the basis $\{Z,S\}$ defined in* (2.9) *and* (2.10)*. Let B be the Riemannian shape operator of Σ. Then, in the regular set $\Sigma - \Sigma_0$, we have*

$$\mathcal{L}(|N_h|) = 4\big(|N_h|^{-2}\,\big\langle B(Z),S\big\rangle - 1\big), \tag{4.4}$$

where $\mathcal{L}$ is the second order operator in (3.44)*.*

Proof. From Lemma 3.10 (i) we know that $|N_h|$ is C^∞ in the Z-direction. We must compute $Z(|N_h|)$ and $Z(Z(|N_h|))$. By (3.48) we have

$$Z(|N_h|) = \langle N,T\rangle - \langle N,T\rangle\,\langle B(Z),S\rangle,$$

and so

$$Z(Z(|N_h|)) = Z(\langle N,T\rangle) - Z(\langle N,T\rangle)\,\langle B(Z),S\rangle - \langle N,T\rangle\, Z(\langle B(Z),S\rangle).$$

Now we use (3.29) and (3.34), so that we get

$$\begin{aligned}Z(Z(|N_h|)) &= -5|N_h| + 4|N_h|^3 + 2|N_h|^{-1}\,\langle B(Z),S\rangle \\ &\quad + 2|N_h|^{-1}\,\langle B(Z),S\rangle^2 - 3|N_h|\,\langle B(Z),S\rangle^2.\end{aligned} \tag{4.5}$$

By substituting (3.48) and (4.5) into (3.44), we obtain

$$\begin{aligned}\mathcal{L}(|N_h|) &= -5 - \langle B(Z),S\rangle^2 + 4|N_h|^{-2}\,\langle B(Z),S\rangle - 2\langle B(Z),S\rangle + |B(Z)+S|^2 \\ &= 4(|N_h|^{-2}\,\langle B(Z),S\rangle - 1),\end{aligned}$$

where in the second equality we have applied that Σ is minimal, and so $B(Z) = \langle B(Z),S\rangle S$ by (3.6). This proves (4.4). □

In the next result we show some properties of the Jacobi field associated to the family of characteristic segments of a minimal surface in $\mathbb{H}^1$. This will allows us to study the sign of $\mathcal{L}(|N_h|)$ and to construct suitable test functions to introduce in (4.3) when Σ is a complete minimal surface with empty singular set.

Lemma 4.5. *Let $\Sigma\subset\mathbb{H}^1$ be a C^2 immersed minimal surface with unit normal N and singular set Σ_0. Consider an integral curve $\Gamma : I\to\Sigma-\Sigma_0$ of the vector field S in (2.10). We define the map $F : I\times I'\to\Sigma-\Sigma_0$ by $F(\varepsilon,s) := \Gamma(\varepsilon) + s\,Z_{\Gamma(\varepsilon)}$. Let $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(\varepsilon,s)$. Then V_ε is a C^∞ Jacobi vector field along $\gamma_\varepsilon(s) := F(\varepsilon,s)$. Moreover, we have*

(i) *The vertical component of V_ε is given by $\langle V_\varepsilon,T\rangle(s) = a_\varepsilon s^2 + b_\varepsilon s + c_\varepsilon$, with*

$$b_\varepsilon^2 - 4a_\varepsilon c_\varepsilon = -|N_h|^2(\Gamma(\varepsilon))\,\mathcal{L}(|N_h|)(\Gamma(\varepsilon)).$$

(ii) *V_ε is always orthogonal to γ_ε and never vanishes along γ_ε.*

(iii) *The function $v_\varepsilon(s) := |\langle V_\varepsilon,T\rangle(s)|^{1/2}$ satisfies*

$$Z(v_\varepsilon^{-1})^2 - \frac{1}{2}\,Z(Z(v_\varepsilon^{-2})) - |N_h|^{-1}\langle N,T\rangle\,Z(v_\varepsilon^{-2}) = \frac{1}{4\,|V_\varepsilon|\,|N_h|}\,\mathcal{L}(|N_h|),$$

along any segment $\gamma_\varepsilon(s)$ where $\langle V_\varepsilon,T\rangle(s)$ never vanishes.

Proof. To simplify the notation we will avoid the subscript ε along the proof. We will use primes for both the derivative of functions depending on s and the covariant derivative along $\gamma(s)$. By Proposition 3.3 (i) the curve γ is a characteristic curve of Σ. It follows from (2.6) and Lemma 2.1 that V is a C^∞ Jacobi field along γ with $[\dot\gamma,V]=0$. Note that

$$\langle V,T\rangle' = \langle V',T\rangle + \langle V,T'\rangle = -2\langle V,\nu_h\rangle, \tag{4.6}$$

since $T' = J(Z) = -\nu_h$, and

$$\langle V',T\rangle = \langle D_Z V,T\rangle = \langle D_V Z,T\rangle = -\langle Z,J(V)\rangle = \langle J(Z),V\rangle = -\langle V,\nu_h\rangle. \tag{4.7}$$

If we derive again in (4.6) then we obtain

$$\langle V,T\rangle'' = -2\langle V',\nu_h\rangle - 2\langle V,\nu_h'\rangle = -2(\langle V',\nu_h\rangle + \langle V,T\rangle), \tag{4.8}$$

since $\nu_h' = D_Z\nu_h = T$ by (3.7) and the fact that Σ is minimal. Hence

$$(-1/2)\langle V,T\rangle''' = (\langle V',\nu_h\rangle + \langle V,T\rangle)' = \langle V'',\nu_h\rangle + 2\langle V',T\rangle + \langle V,T'\rangle = 0, \tag{4.9}$$

where we have used the Jacobi equation (2.8), equality (4.7), and that $T' = -\nu_h$. To simplify (4.8) we compute $\langle V', \nu_h\rangle$. By (3.5) and the fact that V is tangent to Σ, we deduce

$$\begin{aligned}\langle V', \nu_h\rangle = \langle D_V Z, \nu_h\rangle = -\langle Z, D_V \nu_h\rangle &= -|N_h|^{-1}\left(\langle D_V N, Z\rangle - \langle N, T\rangle \langle J(V), Z\rangle\right)\\ &= |N_h|^{-1}\left(\langle B(Z), V\rangle + \langle N, T\rangle \langle V, \nu_h\rangle\right),\end{aligned}$$

and so, after substituting into (4.8), we get

$$\langle V, T\rangle'' = -2|N_h|^{-1}\left(\langle B(Z), V\rangle + \langle N, T\rangle \langle V, \nu_h\rangle + |N_h|\,\langle V, T\rangle\right). \tag{4.10}$$

From (4.9) we conclude that $\langle V, T\rangle(s)$ is a polynomial of degree at most two. Write

$$\langle V, T\rangle(s) = as^2 + bs + c. \tag{4.11}$$

Denote $p = \Gamma(\varepsilon)$. As $V(0) = S_p$, it is easy to check by (4.6) and (4.10), that

$$\begin{aligned}c &= \langle V, T\rangle(0) = -|N_h|(p),\\ b &= \langle V, T\rangle'(0) = -2\,\langle V, \nu_h\rangle(p) = -2\,\langle N, T\rangle(p),\\ a &= (1/2)\,\langle V, T\rangle''(0) = -|N_h|^{-1}\left(\langle B(Z), S\rangle + \langle N, T\rangle^2 - |N_h|^2\right)(p).\end{aligned}$$

In particular, it follows from (4.4) that

$$b^2 - 4ac = -4\left(\langle B(Z), S\rangle - |N_h|^2\right)(p) = -|N_h|^2(p)\,\mathcal{L}(|N_h|)(p),$$

which proves assertion (i) in the statement.

To prove assertion (ii), observe that

$$\langle V, \dot\gamma\rangle' = \langle V', \dot\gamma\rangle + \langle V, \dot\gamma'\rangle = \langle D_V Z, Z\rangle + \langle V, D_Z Z\rangle = 0,$$

by (3.8). This implies that $\langle V, \dot\gamma\rangle = 0$ along γ since $V(0) = S_p$. Hence there is a C^1 function $f : I' \to \mathbb{R}$ such that $V = fS$ along γ. Clearly $|f| = |V|$, and so $\langle V, T\rangle = \pm|V|\,|N_h|$. By (4.11) the vector V vanishes at most two times along γ. Suppose that $s_0 \in I'$ is the first positive value where $V(s_0) = 0$. Note that the sign of $f/|V|$ is constant along a small interval $(s_0 - \delta, s_0)$. By (4.6) and (4.10) we get $\langle V, T\rangle(s_0) = \langle V, T\rangle'(s_0) = \langle V, T\rangle''(s_0) = 0$. By using L'Hôpital's rule twice, we deduce

$$\pm|N_h|(\gamma(s_0)) = \lim_{s\uparrow s_0} \frac{\langle V, T\rangle}{|V|}(s) = \lim_{s\uparrow s_0} \frac{|V|\,\langle V, T\rangle'}{\langle V, V'\rangle}(s) = \lim_{s\uparrow s_0} \frac{|V|'\,\langle V, T\rangle' + |V|\,\langle V, T\rangle''}{|V'|^2 + \langle V, V''\rangle}(s).$$

The numerator tends to zero since $|V|' = \langle V/|V|, V'\rangle \leqslant |V'| \leqslant M$ on $(s_0 - \delta, s_0)$. The denominator goes to $|V'(s_0)|^2$, which is positive; otherwise, the Jacobi field V would be identically zero along γ. It follows that $|N_h|(\gamma(s_0)) = 0$, a contradiction since $\gamma(s_0) \in \Sigma - \Sigma_0$.

To prove (iii) let us suppose that $\langle V, T\rangle$ never vanishes along γ. Then it is clear that $v = |\langle V, T\rangle|^{1/2} = (-\langle V, T\rangle)^{1/2}$ since $V(0) = S_p$. In particular, we get $f = |V| > 0$ along γ. Now we derive $v = (-\langle V, T\rangle)^{1/2} = (f|N_h|)^{1/2}$ with respect to s. By taking into account

(4.6) and (4.10), we obtain

$$\begin{aligned}
Z(v^{-1}) &= \frac{1}{2}\,(-\big\langle V,T\big\rangle^{-3/2})\,\big\langle V,T\big\rangle' = -v^{-3}\,\big\langle V,\nu_h\big\rangle = \frac{-\big\langle N,T\big\rangle}{f^{1/2}\,|N_h|^{3/2}},\\
Z(v^{-2}) &= \big\langle V,T\big\rangle^{-2}\,\big\langle V,T\big\rangle' = -2v^{-4}\,\big\langle V,\nu_h\big\rangle = \frac{-2\big\langle N,T\big\rangle}{f|N_h|^2},\\
Z(Z(v^{-2})) &= \big(\big\langle V,T\big\rangle^{-2}\big\langle V,T\big\rangle'\big)' = -2\big\langle V,T\big\rangle^{-3}(\big\langle V,T\big\rangle')^2 + \big\langle V,T\big\rangle^{-2}\big\langle V,T\big\rangle''\\
&= \frac{8\big\langle N,T\big\rangle^2}{f|N_h|^3} - \frac{2}{f^2|N_h|^3}\,\big(\big\langle B(Z),V\big\rangle + \big\langle N,T\big\rangle\,\big\langle V,\nu_h\big\rangle + |N_h|\,\big\langle V,T\big\rangle\big)\\
&= \frac{8\big\langle N,T\big\rangle^2}{f|N_h|^3} - \frac{2}{f|N_h|^3}\,\big(\big\langle B(Z),S\big\rangle + \big\langle N,T\big\rangle^2 - |N_h|^2\big).
\end{aligned}$$

After simplifying, we conclude by (4.4) that

$$\begin{aligned}
Z(v^{-1})^2 - \frac{1}{2}\,&Z(Z(v^{-2})) - |N_h|^{-1}\big\langle N,T\big\rangle\,Z(v^{-2})\\
&= \frac{1}{f|N_h|}\,\big(|N_h|^{-2}\big\langle B(Z),S\big\rangle - 1\big) = \frac{1}{4f|N_h|}\,\mathcal{L}(|N_h|),
\end{aligned}$$

which proves the claim. □

Proposition 4.6. *Let Σ be a C^2 complete, oriented, area-stationary surface immersed in $\mathbb{H}^1$ with empty singular set. Then the operator $\mathcal{L}$ defined in* (3.44) *satisfies $\mathcal{L}(|N_h|) \geqslant 0$ on Σ. Moreover, $\mathcal{L}(|N_h|)(p) = 0$ for a point $p \in \Sigma$ if and only if $\big\langle N,T\big\rangle = 0$ and $\big\langle B(Z),S\big\rangle = 1$ along the characteristic line of Σ passing through p. As a consequence, $\mathcal{L}(|N_h|) \equiv 0$ on Σ if and only if any connected component of Σ is a Euclidean vertical plane.*

Proof. Take a point $p \in \Sigma$. Let $\Gamma : I \to \Sigma$ be the integral curve through p of the vector field S in (2.10). We define the map $F : I \times \mathbb{R} \to \mathbb{H}^1$ by $F(\varepsilon,s) := \Gamma(\varepsilon) + s\,Z_{\Gamma(\varepsilon)}$. By the completeness of Σ and Proposition 3.3 (i), any $\gamma_\varepsilon(s) := F(\varepsilon,s)$ is a characteristic curve of Σ. In particular, $F(I \times \mathbb{R}) \subseteq \Sigma$.

Let $V(s) := (\partial F/\partial\varepsilon)(0,s)$. By using Lemma 4.5 we deduce that, along the complete line $\gamma(s) := \gamma_0(s)$, the vectors $V(s)$ and $\dot\gamma(s)$ generate the tangent plane to Σ at $\gamma(s)$. Since Σ has empty singular set, it follows that the function $\big\langle V,T\big\rangle(s) = as^2 + bs + c$ never vanishes along $\gamma(s)$. In case $a = 0$ we must have $b = 0$ (otherwise we would find a root of as^2+bs+c). In case $a \neq 0$ we must have $b^2 - 4ac < 0$. Anyway, we get $b^2 - 4ac \leqslant 0$ and so $\mathcal{L}(|N_h|)(p) \geqslant 0$ by Lemma 4.5 (i).

Observe that $\mathcal{L}(|N_h|)(p) = 0$ if and only if $a = b = 0$. This is equivalent to that $\big\langle V,T\big\rangle$ is constant along γ. It follows from (4.6) and (4.10) that $\big\langle N,T\big\rangle = 0$ and $\big\langle B(Z),S\big\rangle = 1$ along γ. Conversely, if $\big\langle N,T\big\rangle = 0$ and $\big\langle B(Z),S\big\rangle = 1$ along γ then (4.4) implies that $\mathcal{L}(|N_h|) = 0$ along γ.

Finally, if $\mathcal{L}(|N_h|) \equiv 0$ on Σ then $\big\langle N,T\big\rangle \equiv 0$ on Σ. By [34, Prop. 6.16] we conclude that any connected component of Σ must be a Euclidean vertical plane. Conversely, it is not difficult to see that $\mathcal{L}(|N_h|) \equiv 0$ holds for any Euclidean vertical plane. □

Now we are ready to prove the main result of this section.

Theorem 4.7. *Let Σ be a C^2 complete, oriented, connected, area-stationary surface immersed in $\mathbb{H}^1$ with empty singular set. If Σ is not a Euclidean vertical plane then Σ is unstable.*

Proof. Let N be the unit normal vector to Σ. We can find $p \in \Sigma$ such that $\langle N, T\rangle(p) \neq 0$. Otherwise Σ would be a Euclidean vertical plane by [34, Prop. 6.16]. By using Proposition 3.3 (i) and the completeness of Σ, we can parameterize Σ, around the characteristic line containing p, by the map $F : I \times \mathbb{R} \to \Sigma$ given by $F(\varepsilon, s) = \Gamma(\varepsilon) + s\, Z_{\Gamma(\varepsilon)}$, where $\Gamma(\varepsilon)$ is a piece of the integral curve through p of the vector field S defined in (2.10). Let $\gamma_\varepsilon(s) := F(\varepsilon, s)$. By Lemma 4.5 we know that $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(\varepsilon, s)$ is a non-vanishing Jacobi field orthogonal to $\gamma_\varepsilon(s)$. Moreover, the function $\langle V_\varepsilon(s), T\rangle$ is strictly negative since Σ has empty singular set and $V_\varepsilon(0) = S_{\Gamma(\varepsilon)}$. We consider the function $v(\varepsilon, s) := |\langle V_\varepsilon(s), T\rangle|^{1/2} = (|N_h|\,|V_\varepsilon(s)|)^{1/2}$, which is continuous and C^∞ along any $\gamma_\varepsilon(s)$.

Now we use the coarea formula to compute the index form (4.3) in terms of the coordinates (ε, s). The Riemannian area element can be expressed as

$$d\Sigma = |V_\varepsilon|\, d\varepsilon\, ds.$$

Hence by using the definition of v together with Lemma 4.5 (iii), equation (4.3) reads

$$\mathcal{I}(uv^{-1}|N_h|, uv^{-1}|N_h|) = \int_{I\times\mathbb{R}} \left(\frac{\partial u}{\partial s}\right)^2 d\varepsilon\, ds - \frac{3}{4}\int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u^2\, d\varepsilon\, ds, \tag{4.12}$$

for any $u \in C_0(I \times \mathbb{R})$ which is also C^1 with respect to s.

Take a non-negative C^∞ function $\phi : I \to \mathbb{R}$ with $\phi(0) > 0$ and compact support contained inside a bounded interval $I' \subseteq I$. Denote $\ell := \text{length}(I')$. Let M be a positive constant so that $|\phi'(\varepsilon)| \leqslant M$, $\varepsilon \in I$. For any $k \in \mathbb{N}$ we define the function

$$u_k(\varepsilon, s) := \phi(\varepsilon)\, \phi(s/k).$$

It is clear that $u_k \in C_0(I' \times kI')$, and that u_k is C^∞ with respect to s. By Fubini's theorem

$$\int_{I\times\mathbb{R}} \left(\frac{\partial u_k}{\partial s}\right)^2 d\varepsilon\, ds = \frac{1}{k^2}\left(\int_{I'} \phi(\varepsilon)^2\, d\varepsilon\right)\left(\int_{kI'} \phi'(s/k)^2\, ds\right) \leqslant \frac{\ell M^2}{k}\int_{I'} \phi(\varepsilon)^2\, d\varepsilon,$$

which goes to 0 when $k \to \infty$. Note also that $\{u_k\}_{k\in\mathbb{N}}$ pointwise converges when $k \to \infty$ to $u(\varepsilon, s) = \phi(0)\, \phi(\varepsilon)$. By Proposition 4.6 we have $\mathcal{L}(|N_h|) \geqslant 0$ on Σ. Thus we can apply Fatou's lemma to obtain

$$\liminf_{k\to\infty} \int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u_k^2\, d\varepsilon\, ds \geqslant \int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u^2\, d\varepsilon\, ds.$$

We conclude from (4.12) that

$$\begin{aligned}\limsup_{k\to\infty} \mathcal{I}(u_k v^{-1}|N_h|, u_k v^{-1}|N_h|) &= -\frac{3}{4}\liminf_{k\to\infty}\int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u_k^2\, d\varepsilon\, ds \\ &\leqslant -\frac{3}{4}\int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u^2\, d\varepsilon\, ds,\end{aligned}$$

which is strictly negative by Proposition 4.6 since $\langle N, T\rangle \neq 0$ inside an open neighborhood around p. Hence Σ is unstable. □

Corollary 4.8. *Let Σ be a C^2 complete, oriented, connected, area-stationary surface immersed in $\mathbb{H}^1$ with empty singular set. Then Σ is stable if and only if Σ is a Euclidean vertical plane.*

Proof. The necessary condition follows from Theorem 4.7. Conversely, suppose that Σ is a vertical plane. We can prove that Σ is an area-minimizing surface in $\mathbb{H}^1$ by using a calibration argument similar to the one in [34, Thm. 5.3], see also [4, Ex. 2.2]. In particular, Σ is stable. □

Remark 4.9. Previous results related to Corollary 4.8 were obtained in [4] and [15]. Precisely, in [4, Thm. 5.1] it is proved that the Euclidean vertical planes are the only complete stable intrinsic graphs in $\mathbb{H}^1$ associated to a C^2 function. In [15, Thm. 1.8] vertical planes are characterized as the unique complete stable C^2 Euclidean graphs with empty singular set. As we pointed out in the introduction of the paper, Corollary 4.8 does not follow from the aforementioned results. For example, they do not apply for the family of sub-Riemannian catenoids $t^2 = \lambda^2\,(x^2 + y^2 - \lambda^2)$, $\lambda \neq 0$.

5. Complete stable surfaces with non-empty singular set

In this section we give the classification of C^2 complete stable surfaces in $\mathbb{H}^1$ with non-empty singular set. By Proposition 3.3 the singular set of a C^2 area-stationary surface consists of isolated points and curves of class C^1. Moreover, the characteristic curves in the regular set meet the singular curves orthogonally. By using these facts we were able to obtain the following result in [34, Thm. 6.15].

Proposition 5.1. *Let Σ be a C^2 complete, oriented, connected, area-stationary surface immersed in $\mathbb{H}^1$ with singular set Σ_0.*

(i) *If Σ_0 contains an isolated point then Σ coincides with a Euclidean non-vertical plane.*
(ii) *If Σ_0 contains a singular curve then Σ is either congruent to the hyperbolic paraboloid $t = xy$ or to one of the helicoidal surfaces $\mathcal{H}_R$ defined below.*

In [34, Ex. 6.14] we described the helicoid $\mathcal{H}_R$ as the union of all the horizontal straight lines orthogonal to the sub-Riemannian geodesic in $\mathbb{H}^1$ obtained by the horizontal lift of the circle in the xy-plane of radius $1/R$ centered at the origin. We can parameterize $\mathcal{H}_R$ by means of the C^∞ diffeomorphism $F : \mathbb{R}^2 \to \mathcal{H}_R$ defined by

$$F(\varepsilon, s) = (s\,\sin(R\varepsilon), s\,\cos(R\varepsilon), \varepsilon/R). \tag{5.1}$$

The singular set of $\mathcal{H}_R$ consists of the helices $s = \pm 1/R$. Note that the family $\{\mathcal{H}_R\}_{R>0}$ is invariant under the dilations δ_λ defined in (2.13). In fact, it can be checked from (5.1) that $\delta_\lambda(\mathcal{H}_R) = \mathcal{H}_{R'}$ with $R' = e^{-\lambda}R$. The surfaces $\mathcal{H}_R$ coincide with the classical left-handed minimal helicoids in $\mathbb{R}^3$. In particular, they are embedded surfaces containing the vertical axis. We remark that the classical right-handed minimal helicoids in $\mathbb{R}^3$ are complete area-stationary surfaces in $\mathbb{H}^1$ with empty singular set, and so they are unstable by Theorem 4.7.

Proposition 5.1 indicates us that the study of stable surfaces in $\mathbb{H}^1$ with non-empty singular set can be reduced to three cases: Euclidean non-vertical planes, the hyperboloid $t = xy$ and the helicoids $\mathcal{H}_R$. In [34, Thm. 5.3] we showed that any complete C^2 area-stationary graph over the xy-plane is an area-minimizing surface. This gives us the stability of any plane $t = ax + by$ and any surface congruent to $t = xy$. So it remains to analyze the stability of the helicoidal surfaces $\mathcal{H}_R$.

We first compute some geometric terms of a helicoid $\mathcal{H}_R$ with respect to the system of coordinates (ε, s) in (5.1). Note that

$$\frac{\partial F}{\partial \varepsilon} = Rs\,\cos(R\varepsilon)\,X - Rs\,\sin(R\varepsilon)\,Y + f(s)\,T,$$
$$\frac{\partial F}{\partial s} = \sin(R\varepsilon)\,X + \cos(R\varepsilon)\,Y,$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(s) = \frac{1}{R} - Rs^2.$$

As a consequence, the Riemannian area element is given by

$$d\Sigma = \sqrt{f(s)^2 + R^2 s^2}\, d\varepsilon\, ds. \tag{5.2}$$

On the other hand, the cross product of $\partial F/\partial s$ and $\partial F/\partial \varepsilon$ in $(\mathbb{H}^1, g)$ provides the following unit normal vector to $\mathcal{H}_R$

$$N = \frac{f(s)\cos(R\varepsilon)\, X - f(s)\sin(R\varepsilon)\, Y - Rs\, T}{\sqrt{f(s)^2 + R^2 s^2}}, \tag{5.3}$$

and so

$$|N_h| = \frac{|f(s)|}{\sqrt{f(s)^2 + R^2 s^2}}, \qquad \langle N, T\rangle = \frac{-Rs}{\sqrt{f(s)^2 + R^2 s^2}}. \tag{5.4}$$

It follows that the straight lines $\gamma_\varepsilon(s) = F(\varepsilon, s)$, $s \in \mathbb{R}$, satisfy

$$\dot{\gamma}_\varepsilon(s) = \operatorname{sign}(1/R - |s|)\, Z, \qquad |s| \neq 1/R. \tag{5.5}$$

By taking into account (3.29) and (5.4) we get, for $|s| \neq 1/R$, that

$$\langle B(Z), S\rangle = 1 + |N_h|^{-1}\, Z(\langle N, T\rangle) = \frac{2f(s)^2 - Rf(s)}{f(s)^2 + R^2 s^2} - 1, \tag{5.6}$$

which in particular implies

$$|B(Z) + S|^2 - 4|N_h|^2 = \frac{(R^2 - 4)\, f(s)^2}{(f(s)^2 + R^2 s^2)^2}. \tag{5.7}$$

Now we are ready to deduce from Theorem 3.7 and Proposition 3.11 a stability criterion for helicoidal surfaces that plays the same role as Proposition 3.12.

Proposition 5.2. *Let Σ be the helicoid $\mathcal{H}_R$. If Σ is stable then, for any function $u \in C_0^2(\Sigma)$ such that $Z(u) = 0$ inside a small tubular neighborhood of Σ_0, we have $\mathcal{Q}(u) \geqslant 0$, where*

$$\begin{aligned} \mathcal{Q}(u) := &\int_\Sigma |N_h|^{-1} \left\{ Z(u)^2 - \left(|B(Z) + S|^2 - 4|N_h|^2\right) u^2 \right\} d\Sigma \\ &- 4 \int_{\Sigma_0} u^2\, d\Sigma_0 + \int_{\Sigma_0} S(u)^2\, d\Sigma_0. \end{aligned} \tag{5.8}$$

Here $\{Z, S\}$ is the orthonormal basis in (2.9) *and* (2.10)*, B is the Riemannian shape operator of Σ, and $d\Sigma_0$ is the Riemannian length measure on Σ_0.*

Proof. We suppose that the unit normal N to Σ is the one in (5.3). For simplicity we denote $q = |B(Z) + S|^2 - 4|N_h|^2$. By (5.7), (5.4) and (5.2) it follows that $|N_h|^{-1} q u^2 \in L^1(\Sigma)$ provided $u \in C_0(\Sigma)$. In particular, $\mathcal{Q}(u)$ is well defined for any $u \in C_0(\Sigma)$ which is piecewise C^1 in the Z-direction, satisfies $|N_h|^{-1} Z(u)^2 \in L^1(\Sigma)$, and whose restriction to Σ_0 is C^1.

Let us show, in a first step, the following statement

$$\begin{aligned} &\mathcal{Q}(v) \geqslant 0, \text{ for any } v \in C_0^2(\Sigma) \text{ such that } Z(v/\langle N, T\rangle) = 0 \\ &\qquad \text{in a small tubular neighborhood } E \text{ of } \Sigma_0. \end{aligned} \tag{5.9}$$

Note that a function v as above satisfies $Z(v)^2 = (Z(\langle N, T\rangle)^2 / \langle N, T\rangle^2)\, v^2$ in E. It follows from (5.4) and (5.2) that $|N_h|^{-1} Z(v)^2 \in L^1(\Sigma)$, and so $\mathcal{Q}(v) < \infty$.

Let σ_0 be the radius of E and K the support of v. For any $\sigma \in (0, \sigma_0/2)$ let E_σ be the tubular neighborhood of Σ_0 of radius σ. We consider functions $h_\sigma, g_\sigma \in C_0^\infty(\Sigma)$ such that $g_\sigma = 1$ on $K \cap \overline{E}_\sigma$, $\operatorname{supp}(g_\sigma) \subset E_{2\sigma}$ and $h_\sigma + g_\sigma = 1$ on K. We define the C^2 vector field

$$U_\sigma := (h_\sigma\, v)\, N + g_\sigma\, \frac{v}{\langle N, T\rangle}\, T,$$

whose support is contained in K. Note that $\langle U_\sigma, N\rangle = v$ on K. Let $\varphi_r^\sigma(p) := \exp_p(r(U_\sigma)_p)$ be the variation associated to U_σ and $A_\sigma(r) := A(\varphi_r^\sigma(\Sigma))$ the corresponding area functional. The variation φ_r^σ is vertical when restricted to E_σ. Hence we can suppose, by applying Proposition 3.11 to $w = v/\langle N,T\rangle$, that the second derivative of $A_{1\sigma}(r) := A(\varphi_r^\sigma(E_\sigma))$ is given by

$$A''_{1\sigma}(0) = \int_{\Sigma_0} S(v)^2 \, d\Sigma_0.$$

In the previous equality we have used that $\langle N,T\rangle = \pm 1$ on Σ_0. On the other hand, the second derivative of $A_{2\sigma}(r) := A(\varphi_r^\sigma(\Sigma - E_\sigma))$ can be computed from Theorem 3.7. We obtain the following expression

$$A''_{2\sigma}(0) = \int_{\Sigma-E_\sigma} |N_h|^{-1}\left\{Z(v)^2 - qv^2\right\} d\Sigma + \int_{\Sigma-E_\sigma} \operatorname{div}_\Sigma(\xi Z)\, d\Sigma + \int_{\Sigma-E_\sigma} \operatorname{div}_\Sigma(\mu Z)\, d\Sigma,$$

where

$$\xi = \langle N,T\rangle\,(1 - \langle B(Z),S\rangle)\, v^2,$$
$$\mu = |N_h|^2 \left\{ \langle N,T\rangle\,(1-\langle B(Z),S\rangle)\, \frac{g_\sigma^2\, v^2}{\langle N,T\rangle^2} - 2\langle B(Z),S\rangle\, \frac{h_\sigma\, g_\sigma\, v^2}{\langle N,T\rangle} \right\}.$$

If Σ is stable then $A''_\sigma(0) \geqslant 0$. As $A_\sigma(r) = A_{1\sigma}(r) + A_{2\sigma}(r)$ we deduce, by using the classical Riemannian divergence theorem that, for any $\sigma \in (0, \sigma_0/2)$, we have the inequality

$$\int_{\Sigma-E_\sigma} |N_h|^{-1}\left\{Z(v)^2 - qv^2\right\} d\Sigma - \int_{\partial E_\sigma} (\xi+\mu)\,\langle Z,\eta\rangle\, dl + \int_{\Sigma_0} S(v)^2\, d\Sigma_0 \geqslant 0, \tag{5.10}$$

where η is the unit normal to ∂E_σ pointing into $\Sigma - E_\sigma$ and dl denotes the Riemannian length element.

Let us compute the boundary term above. Fix $k \in \{1,2\}$. Let Λ be one of the two components of ∂E_σ at distance σ of the singular curve where $\langle N,T\rangle = (-1)^{k+1}$. By taking into account (5.5) it follows that $\eta = (-1)^{k+1} Z$ along Λ. Moreover, the functions ξ and μ are constant along Λ. Since $g_\sigma = 1$ and $h_\sigma = 0$ on Λ we have

$$\int_\Lambda \xi\,\langle Z,\eta\rangle\, d\Lambda = (-1)^{k+1}\,\langle N,T\rangle\,(1-\langle B(Z),S\rangle) \int_\Lambda v^2\, d\Lambda, \tag{5.11}$$

$$\int_\Lambda \mu\,\langle Z,\eta\rangle\, d\Lambda = (-1)^{k+1}\,|N_h|^2\,\langle N,T\rangle^{-1}\,(1-\langle B(Z),S\rangle) \int_\Lambda v^2\, d\Lambda. \tag{5.12}$$

Now we let $\sigma \to 0$ in (5.10). From the dominated convergence theorem we get

$$\lim_{\sigma\to 0} \int_{\Sigma-E_\sigma} |N_h|^{-1}\left\{Z(v)^2 - qv^2\right\} d\Sigma = \int_\Sigma |N_h|^{-1}\left\{Z(v)^2 - qv^2\right\} d\Sigma,$$
$$\lim_{\sigma\to 0} \int_{\partial E_\sigma} v^2\, dl = 2\int_{\Sigma_0} v^2\, d\Sigma_0.$$

On the other hand, equation (5.6) yields $\langle B(Z),S\rangle \to -1$ when we approach Σ_0. Moreover, we know that $|N_h| \to 0$ and $\langle N,T\rangle \to \pm 1$ when $\sigma \to 0$. This facts, together with (5.11) and (5.12) imply that

$$\lim_{\sigma\to 0} \int_{\partial E_\sigma} (\xi+\mu)\,\langle Z,\eta\rangle\, dl = 4\int_{\Sigma_0} v^2\, d\Sigma_0.$$

Hence we obtain $\mathcal{Q}(v) \geqslant 0$ from (5.10). This proves (5.9).

Now we take $u \in C_0^2(\Sigma)$ with $Z(u) = 0$ inside a small tubular neighborhood E of Σ_0. For any $\sigma \in (0,1)$ let D_σ be the open neighborhood of Σ_0 such that $|\langle N,T\rangle| = 1-\sigma$ on ∂D_σ.

We can find $\sigma_0 > 0$ such that $D_\sigma \subset E$ for $\sigma \in (0, \sigma_0)$. For such values of σ we define the function $\phi_\sigma : \Sigma \to [0, 1]$ given by

$$\phi_\sigma = \begin{cases} |\langle N, T\rangle|, & \text{in } \overline{D}_\sigma, \\ 1-\sigma, & \text{in } \Sigma - D_\sigma. \end{cases}$$

Clearly ϕ_σ is continuous and piecewise C^1 in the Z-direction. Moreover, the sequence $\{\phi_\sigma\}_{\sigma\in(0,\sigma_0)}$ pointwise converges to 1 when $\sigma \to 0$. By using (5.4) and (5.2) we can see that $|N_h|^{-1} Z(\langle N, T\rangle)^2$ extends to a continuous function on Σ, and so

$$\lim_{\sigma\to 0} \int_\Sigma |N_h|^{-1}\, Z(\phi_\sigma)^2\, d\Sigma = 0.$$

By a standard approximation argument we can slightly modify ϕ_σ around ∂D_σ in order to construct a sequence of C^2 functions $\{\psi_\sigma\}_{\sigma\in(0,\sigma_0)}$ satisfying the same properties. Define $v_\sigma := \psi_\sigma u$. This provides a sequence of functions in $C^2_0(\Sigma)$ such that $v_\sigma = u$ in Σ_0 and $Z(v_\sigma/\langle N, T\rangle) = 0$ inside a small tubular neighborhood of Σ_0. As a consequence of (5.9) we have $\mathcal{Q}(v_\sigma) \geqslant 0$ for any $\sigma \in (0, \sigma_0)$. Finally, it is straightforward to check by using the dominated convergence theorem and the Cauchy-Schwartz inequality in $L^2(\Sigma)$ that $\{\mathcal{Q}(v_\sigma)\} \to \mathcal{Q}(u)$ when $\sigma \to 0$. The proposition is proved. □

Remark 5.3. By using Proposition 3.11, inequality $\mathcal{Q}(u) \geqslant 0$ can be generalized for any C^2 stable solution Σ of the Plateau problem whose singular curves are of class C^3 whenever $\mathrm{supp}(u)$ is contained in the interior of Σ and $Z(u) = 0$ inside a tubular neighborhood of the singular curves.

Now we are ready to prove the main result of this section.

Theorem 5.4. *The helicoidal surfaces $\mathcal{H}_R$ are all unstable.*

Proof. To prove the claim it suffices to show that $\mathcal{H}_2$ is unstable. In fact, for any $R > 0$ we have $\mathcal{H}_R = \delta_\lambda(\mathcal{H}_2)$, where δ_λ is the dilation defined in (2.13) with $\lambda = \log(2/R)$. By virtue of Lemma 3.2 we deduce that $\mathcal{H}_R$ is stable if and only if $\mathcal{H}_2$ is stable.

Let $\Sigma := \mathcal{H}_2$. Consider the diffeomorphism $F : \mathbb{R}^2 \to \Sigma$ in (5.1). We denote $\gamma_\varepsilon(s) = F(\varepsilon, s)$, $s \in \mathbb{R}$. The singular set Σ_0 consists of the singular curves $F(\varepsilon, -1/2)$ and $F(\varepsilon, 1/2)$, $\varepsilon \in \mathbb{R}$. We suppose that the normal N to Σ is the one in (5.3). By equation (5.7) we get

$$|B(Z) + S|^2 - 4|N_h|^2 = 0, \qquad \text{on } \Sigma - \Sigma_0.$$

In particular, the quadratic form $\mathcal{Q}$ in (5.8) is given by

$$\mathcal{Q}(u) = \int_\Sigma |N_h|^{-1}\, Z(u)^2\, d\Sigma - 4 \int_{\Sigma_0} u^2\, d\Sigma_0 + \int_{\Sigma_0} S(u)^2\, d\Sigma_0, \tag{5.13}$$

for any $u \in C_0(\Sigma)$ which is piecewise C^1 in the Z-direction, satisfies $|N_h|^{-1} Z(u)^2 \in L^1(\Sigma)$, and whose restriction to Σ_0 is C^1. We apply in (5.13) the coarea formula. By using (5.4), (5.5) and (5.2), we deduce that

$$\begin{aligned} \mathcal{Q}(u) = \int_{\mathbb{R}^2} \frac{f(s)^2 + 4s^2}{|f(s)|} \left(\frac{\partial u}{\partial s}\right)^2 d\varepsilon\, ds - 4\int_{\mathbb{R}} u(\varepsilon, -1/2)^2\, d\varepsilon - 4 \int_{\mathbb{R}} u(\varepsilon, 1/2)^2\, d\varepsilon \\ + \int_{\mathbb{R}} \left(\frac{d}{d\varepsilon}\, u(\varepsilon, -1/2)\right)^2 d\varepsilon + \int_{\mathbb{R}} \left(\frac{d}{d\varepsilon}\, u(\varepsilon, 1/2)\right)^2 d\varepsilon. \end{aligned} \tag{5.14}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be any C^∞ function with compact support $[-\varepsilon_0, \varepsilon_0]$. For any $k > 1/2$ and $\delta > 0$, let $\phi_{k\delta} : \mathbb{R} \to [0, 1]$ be the symmetric function with respect to the origin given, for

$s \geqslant 0$, by

$$\phi_{k\delta}(s) = \begin{cases} 1, & 0 \leqslant s \leqslant k, \\ \delta^{-1}\,(-s+\delta+k), & k \leqslant s \leqslant k+\delta, \\ 0, & s \geqslant k+\delta. \end{cases}$$

Now we define the function $u_{k\delta}$ on Σ whose expression in coordinates (ε, s) is

$$u_{k\delta}(\varepsilon, s) = \phi(\varepsilon)\,\phi_{k\delta}(s).$$

Clearly $u_{k\delta}$ is a function in $C_0(\Sigma)$ which is also C^∞ with respect to ε and piecewise C^∞ in the Z-direction. Note also that

$$u_{k\delta}(\varepsilon, -1/2) = u_{k\delta}(\varepsilon, 1/2) = \phi(\varepsilon), \qquad \varepsilon \in \mathbb{R}. \tag{5.15}$$

Moreover $(\partial u_{k\delta}/\partial s)(\varepsilon, s) = \phi(\varepsilon)\,\phi'_{k\delta}(s)$, which vanishes if $|s| < k$ or $|s| > k+\delta$, and equals $\pm\phi(\varepsilon)/\delta$ if $k < |s| < k+\delta$. This implies that $Z(u_{k\delta}) = 0$ inside a tubular neighborhood of Σ_0. By using Fubini's theorem and that $|f(s)|^{-1}(f(s)^2+4s^2)$ is symmetric with respect to the origin, we have

$$\int_{\mathbb{R}^2} \frac{f(s)^2+4s^2}{|f(s)|}\left(\frac{\partial u_{k\delta}}{\partial s}\right)^2 d\varepsilon\, ds = \left(\int_{-\varepsilon_0}^{\varepsilon_0} \phi(\varepsilon)^2\, d\varepsilon\right)\left(\frac{2}{\delta^2}\int_k^{k+\delta} \frac{f(s)^2+4s^2}{|f(s)|}\, ds\right). \tag{5.16}$$

The second integral in the right-hand side can be easily computed. We obtain

$$2\int_k^{k+\delta} \frac{f(s)^2+4s^2}{|f(s)|}\, ds = \int_k^{k+\delta} \frac{16s^4+8s^2+1}{4s^2-1}\, ds = \frac{4s^3}{3} + 3s + \log\left(\frac{2s-1}{2s+1}\right)\Bigg]_k^{k+\delta}.$$

By an elementary analysis we can find a value $k > 1/2$ and $\delta = 2k+1$ such that the integral above times $1/\delta^2$ is strictly less than 8. By substituting this information into (5.16), and using (5.14) together with (5.15), we conclude for $v := u_{k\delta}$

$$\mathcal{Q}(v) < M \int_{-\varepsilon_0}^{\varepsilon_0} \phi(\varepsilon)^2\, d\varepsilon + 2\int_{-\varepsilon_0}^{\varepsilon_0} \phi'(\varepsilon)^2\, d\varepsilon,$$

for some constant $M < 0$ which does not depend on the function ϕ. If ε_0 is large enough, then we can choose ϕ with compact support $[-\varepsilon_0, \varepsilon_0]$ such that the right-hand side of the previous equation is strictly negative. This can be done since

$$\inf\left\{\left(\int_{\mathbb{R}} \phi'(\varepsilon)^2\, d\varepsilon\right)\left(\int_{\mathbb{R}} \phi(\varepsilon)^2\, d\varepsilon\right)^{-1}; \phi \in C_0^\infty(\mathbb{R})\right\} = 0.$$

Denote $\bar{\phi} := \phi_{k\delta}$ for the particular values of k and δ found above. We mollify $\bar{\phi}$ in order to obtain a sequence of functions $v_\sigma(\varepsilon, s) = \phi(\varepsilon)\bar{\phi}_\sigma(s)$ in $C_0^\infty(\Sigma)$ with $v_\sigma = v$ on Σ_0 and

$$\lim_{\sigma\to 0} \int_\Sigma |N_h|^{-1} Z(v_\sigma)^2\, d\Sigma = \int_\Sigma |N_h|^{-1} Z(v)^2\, d\Sigma.$$

Hence we have

$$\lim_{\sigma\to 0} \mathcal{Q}(v_\sigma) = \mathcal{Q}(v) < 0.$$

By Proposition 5.2 we conclude that Σ is unstable. □

Remark 5.5. Though the helicoids $\mathcal{H}_R$ are unstable, it is possible to obtain by means of a calibration argument similar to the one used for the hyperboloid $t = xy$ in [34, Thm. 5.3] that the surface obtained by removing the vertical axis from $\mathcal{H}_R$ is area-minimizing. On the other hand, the second derivative of the area in Theorem 3.7 indicates us that any non-singular variation induced by a vector field $U = vN + wT$ such that v and w are C^1 functions whose support is contained in the regular set of $\mathcal{H}_2$ satisfies $A''(0) \geqslant 0$. This means that $\mathcal{H}_2$ is also *stable under the variations used in* Theorem 3.7. The proof of Theorem 5.4 shows that, to get that $\mathcal{H}_2$ is unstable, we need to consider a function whose support intersects a large piece of $\mathcal{H}_2$ containing the vertical axis and the singular set.

6. Main result

As a consequence of our previous stability results we can prove the following.

Theorem 6.1. *Let Σ be a C^2 complete, oriented, connected, area-stationary surface immersed in $\mathbb{H}^1$. Then Σ is stable if and only if Σ is a Euclidean plane or Σ is congruent to the hyperbolic paraboloid $t = xy$. In particular, Σ is area-minimizing.*

Proof. If Σ is stable and the singular set Σ_0 is empty then Σ must be a vertical plane by Theorem 4.7. If Σ is stable and $\Sigma_0 \neq \emptyset$ then Proposition 5.1 and Theorem 5.4 imply that Σ coincides with a non-vertical Euclidean plane, or it is congruent to the hyperbolic paraboloid $t = xy$. That Euclidean planes and surfaces congruent to $t = xy$ are area-minimizing follows from [4, Ex. 2.2] and [34, Thm. 5.3]. □

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Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, Spain

E-mail address: `ahurtado@ugr.es`

Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, Spain

E-mail address: `ritore@ugr.es`

Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, Spain

E-mail address: `crosales@ugr.es`