SOME ISOPERIMETRIC COMPARISON THEOREMS FOR CONVEX BODIES IN RIEMANNIAN MANIFOLDS

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Abstract. We prove that the isoperimetric profile of a convex domain Ω with compact closure in a Riemannian manifold (M^{n+1}, g) satisfies a second order differential inequality that only depends on the dimension of the manifold and on a lower bound on the Ricci curvature of Ω. Regularity properties of the profile and topological consequences on isoperimetric regions arise naturally from this differential point of view.

Moreover, by integrating the differential inequality, we obtain sharp comparison theorems: not only can we derive an inequality that should be compared with Lévy-Gromov Inequality but we also show that if $\mathrm{Ric} \geqslant n\delta$ on Ω, then the profile of Ω is bounded from above by the profile of the half-space $\mathbb{H}^{n+1}_\delta$ in the simply connected space form with constant sectional curvature δ. As consequence of isoperimetric comparisons we obtain geometric estimations for the volume and the diameter of Ω, and for the first non-zero Neumann eigenvalue for the Laplace operator on Ω.

1. Introduction

Let Ω be a domain (connected open set) with non-empty boundary of a Riemannian manifold (M^{n+1}, g). The so-called *partitioning problem* in Ω consists on finding, for a given $V < \mathrm{vol}(\Omega)$, a minimum of the perimeter functional $\mathcal{P}(\cdot\,, \Omega)$ in the class of sets in Ω that enclose volume V. Here $\mathrm{vol}(E)$ is the $(n+1)$-dimensional Hausdorff measure of a set $E \subseteq M$ and $\mathcal{P}(E, \Omega)$ denotes the *perimeter* of E *relative to* Ω, which essentially measures the area of $\partial E \cap \Omega$ (see Section 2 for a precise definition). Solutions to the partitioning problem are called *isoperimetric regions* or *minimizers* in Ω of volume V.

The partitioning problem is object of an intensive study. The first questions taken into consideration were related to the existence and regularity of minimizers. In the light of standard results in Geometric Measure Theory [**M1**], inside a smooth domain Ω with compact closure, minimizers do exist for any given volume and their boundaries are smooth, up to a closed set of singularities with high Hausdorff codimension, (see Proposition 2.3 for a precise statement). Recently, geometric and topological properties of minimizers have been studied by A. Ros and E. Vergasta [**RV**] and P. Sternberg and K. Zumbrun [**SZ2**] inside a Euclidean convex body, and by M. Ritoré and C. Rosales [**RR**] inside Euclidean cones. However, in spite of the last advances, the complete description of isoperimetric regions has been achieved only for certain convex domains such as half-spaces in the simply connected space forms, Euclidean balls, Euclidean slabs, and Euclidean convex cones, among others. A beautiful survey containing most of the results above, including recent progress and open questions is the one by A. Ros [**Ro**].

Date: November 7, 2003.

2000 *Mathematics Subject Classification.* 53C20, 49Q20.

Key words and phrases. Isoperimetric profile, isoperimetric regions, differential inequality, comparison theorems.

Much of the information concerning the partitioning problem is contained in the *isoperimetric profile of* Ω: the function $I_\Omega(V)$ which assigns to V the least-perimeter separation of volume V in Ω. In this paper, assuming that Ω is a convex domain with compact closure in (M^{n+1},g), we prove regularity properties of the profile, connectivity results for minimizers and for their boundaries, and above all, we obtain sharp lower and upper bounds for the isoperimetric profile involving the infimum of the Ricci curvature of Ω.

We begin this work with a preliminary section where we introduce the notation and give some basic results. For example, Proposition 2.1 is an adaptation to the partitioning problem of a result by P. Bérard and D. Meyer [**BM**] in which it is shown that the isoperimetric profile I_Ω approaches asymptotically the profile of the half-space in $\mathbb{R}^{n+1}$ for small volumes. We also summarize existence and regularity results for isoperimetric regions in Proposition 2.3, and state an analytic comparison result for the solutions of a differential inequality (Theorem 2.5) that will be useful in Section 4.

In Section 3, inspired by previous results by C. Bavard and P. Pansu [**BP**], P. Sternberg and K. Zumbrun [**SZ2**], F. Morgan and D. Johnson [**MJ**], and V. Bayle [**Ba2**], we prove (Theorem 3.2) that the renormalized isoperimetric profile $Y_\Omega = I_\Omega^{(n+1)/n}$ of a smooth convex domain Ω with compact closure satisfies a second order differential inequality of the type

$$Y_\Omega'' \leqslant C\, Y_\Omega^{(1-n)/(1+n)}, \tag{1.1}$$

where C is a constant depending on the dimension of the ambient manifold and on a lower bound on the Ricci curvature over Ω.

The idea of the proof of (1.1) relies on a local comparison of Y_Ω with the renormalized profile $P(V)^{(n+1)/n}$ associated to the deformation of a minimizer E -which exists by the compactness of $\overline{\Omega}$- given by equidistant hypersurfaces to $\partial E \cap \Omega$. Some technical difficulties arise due to the possible presence in high dimensions of singularities in $\partial E \cap \Omega$. These difficulties are solved by an approximation argument consisting in the construction of "almost parallel variations" (Lemma 3.1). This scheme of proof was previously used in [**MJ**] and [**Ba2**] to get a differential inequality for the isoperimetric profile of a closed Riemannian manifold, and in [**RR**] to characterize isoperimetric regions in smooth convex cones. As in [**RR**], our proof differs from those of [**MJ**] and [**Ba2**] in the presence of a boundary term involving the second fundamental form of $\partial\Omega$ which can be controlled by using the convexity of Ω.

From the differential inequality (1.1), that yields concavity of the profile under the assumption of non-negative Ricci curvature on Ω (Theorem 3.5), we derive regularity properties of the profile (Proposition 3.7) and topological consequences related to the connectivity of minimizers and isoperimetric hypersurfaces (Propositions 3.11 and 3.9). Similar previous results for closed Riemannian manifolds and for convex bodies in the Euclidean setting were established in [**BP**], [**MJ**], [**SZ2**], [**K**] and [**Ba2**].

In Section 4 we use analytic arguments to obtain isoperimetric comparison theorems. As a matter of fact, integration of the differential inequality (1.1) makes possible to compare the isoperimetric profile of a smooth convex domain Ω with compact closure and $\mathrm{Ric} \geqslant n\delta$, with an exact solution of the differential equation associated to (1.1) that satisfies either the same initial conditions or the same boundary conditions. On the one hand we prove in Theorem 4.1 the isoperimetric inequality

$$I_\Omega(V) \leqslant I_{\mathbb{H}^{n+1}_\delta}(V), \qquad V \in [0, \mathrm{vol}(\Omega)], \tag{1.2}$$

where $\mathbb{H}^{n+1}_\delta$ is a half-space in the simply connected space form with constant sectional curvature δ. In Remark 4.5 we indicate that the geometric arguments employed by F. Morgan and D. Johnson in [**MJ**, Theorem 3.5] can be adapted to prove that (1.2) is also valid for unbounded convex domains. In Theorem 4.6 we show that equality in (1.2) for some $V_0 \in (0, \mathrm{vol}(\Omega)]$ implies that $\partial\Omega$ is a totally geodesic hypersurface of the ambient manifold, and Ω has constant sectional curvature δ in a neighborhood of $\partial\Omega$.

On the other hand, in Corollary 4.9 we deduce a lower bound for the profile that should be compared with Lévy–Gromov inequality [**Gr**]. In precise terms, we prove that any Borel set E contained in a smooth convex body Ω with $\mathrm{Ric} \geqslant n\delta > 0$, satisfies

$$\frac{\mathcal{P}(E,\Omega)}{\mathrm{vol}(\Omega)} \geqslant \frac{\mathcal{P}(E^*,\mathbb{H}^{n+1}_\delta)}{\mathrm{vol}(\mathbb{H}^{n+1}_\delta)}, \tag{1.3}$$

where $E^* \subseteq \mathbb{H}^{n+1}_\delta$ is a half-ball centered at $\partial\mathbb{H}^{n+1}_\delta$ with

$$\frac{\mathrm{vol}(E)}{\mathrm{vol}(\Omega)} = \frac{\mathrm{vol}(E^*)}{\mathrm{vol}(\mathbb{H}^{n+1}_\delta)}.$$

Moreover, inequality (1.3) is sharp since equality for a proper set $E \subset \Omega$ implies that Ω is isometric to $\mathbb{H}^{n+1}_\delta$.

Our isoperimetric inequalities in Section 4 can be used, as in [**Ga**] and [**Ba2**], to derive comparison theorems for convex bodies involving geometric quantities such as the volume or the diameter, see Theorem 2.7, Remark 4.2 and Theorem 4.13. Furthermore, by reproducing the symmetrization arguments in [**BM**, Théorème 5] we prove in Theorem 4.15 that if $\mathrm{Ric} \geqslant n\delta > 0$ on Ω, then the lowest non-zero eigenvalue for the Laplace operator in Ω with Neumann boundary condition is bounded from below by the one of the half-sphere $\mathbb{H}^{n+1}_\delta$ of radius $1/\sqrt{\delta}$, with equality if and only if Ω is isometric to $\mathbb{H}^{n+1}_\delta$.

Finally, we have added in a last section as an appendix a geometric proof of inequality (1.2) for the case of a smooth convex body Ω in $\mathbb{R}^{n+1}$.

As mentioned in [**SZ2**], in addition to the geometric interest of this work, we remark that the partitioning problem can be linked with a well-studied variational question related to phase transitions (see also [**SZ1**]).

Acknowledgements. The idea of this work was conceived while V. Bayle was visiting the University of Granada in the spring of 2003. The paper was finished during a short stay of C. Rosales at the Institut Fourier of Grenoble in the spring of 2004. The first author was supported by the Marie Curie Research Training Networks "EDGE", HPRN-CT-2000-00101. The second author was supported by MCyT-Feder research project BFM2001-3489. Both authors express their deep thanks to Manuel Ritoré for his encouragement and helpful comments during the preparation of these notes. We also thank the referee for helpful comments for the revision of the paper.

2. Preliminaries

2.1. **The isoperimetric profile.** Let Ω be a smooth domain (connected open set) with compact closure $\overline{\Omega}$ contained in a Riemannian manifold (M^{n+1}, g). The $(n+1)$-dimensional and the k-dimensional Hausdorff measures of a Borel set $E \subseteq M$ will be denoted by $\mathrm{vol}(E)$ and $\mathcal{H}_k(E)$ respectively. For any measurable set $E \subseteq M$, let $\mathcal{P}(E,\Omega)$ be the De Giorgi *perimeter* of E *relative to* Ω, defined as

$$\mathcal{P}(E,\Omega) = \sup\left\{\int_E \operatorname{div} Y\, d\mathcal{H}_{n+1} : g(Y,Y) \leqslant 1\right\},$$

where Y is a smooth vector field over M with compact support contained in Ω, and $\operatorname{div} Y$ is the divergence of Y [**Ch2**, p. 140]. If, for instance, E has C^2 boundary, then $\mathcal{P}(E,\Omega) = \mathcal{H}_n(\partial E \cap \Omega)$ by the Gauss–Green theorem.

A set $E \subseteq M$ is said to be of *finite perimeter in* Ω if $\mathcal{P}(E,\Omega) < \infty$. We refer to the reader to [**Gi**], [**Z**] and [**Ch3**] for background about perimeter, sets of finite perimeter, and their use in the context of Geometric Measure Theory.

The *isoperimetric profile* of Ω is the function $I_\Omega : [0, \operatorname{vol}(\Omega)] \to \mathbb{R}^+ \cup \{0\}$ given by

$$I_\Omega(V) = \inf\{\mathcal{P}(E,\Omega) :\ E \subseteq \Omega,\ \operatorname{vol}(E) = V\},$$

where the infimum is taken over sets of finite perimeter in Ω. We define the *renormalized isoperimetric profile* of Ω as the function

$$Y_\Omega = I_\Omega^{(n+1)/n}.$$

Through this paper we shall use the following basic properties of the isoperimetric profile

- I_Ω is a non-negative function which only vanishes at $V = 0$ and $V = \operatorname{vol}(\Omega)$.
- $I_\Omega(V) = I_\Omega(\operatorname{vol}(\Omega) - V), \quad V \in [0, \operatorname{vol}(\Omega)]$.
- I_Ω is a lower semicontinuous function [**Gi**, Theorems 1.9 and 1.19].

The following proposition is an adaptation of a result by P. Bérard and D. Meyer [**BM**, App. C], in which the cited authors show that the isoperimetric profile of a closed manifold (M^{n+1}, g) (i.e., a compact Riemannian manifold without boundary) asymptotically approaches the profile of $\mathbb{R}^{n+1}$ for small volumes.

Proposition 2.1. *Let Ω be a smooth domain with compact closure and non-empty boundary in a Riemannian manifold (M^{n+1}, g). Denote by $\mathbb{H}^{n+1}$ the half-space $\{x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$. Then, the asymptotic behaviour of the isoperimetric profile of Ω at the origin is*

$$I_\Omega(V) \underset{\substack{V\to 0\\ V>0}}{\sim} I_{\mathbb{H}^{n+1}}(V) = 2^{-1/(n+1)}\,\gamma_{n+1}\,V^{n/(n+1)},$$

where $\gamma_{n+1} = \mathcal{H}_n(\mathbb{S}^n)/\mathcal{H}_{n+1}(B(1))^{n/(n+1)}$ stands for the $(n+1)$-dimensional Euclidean isoperimetric constant.

As a consequence, the right derivative of the renormalized profile at the origin is given by

$$(Y_\Omega)'_r(0) = 2^{-1/n}\,\gamma_{n+1}^{(n+1)/n}.$$

Proof. The only change with respect to the proof by P. Bérard and D. Meyer that must be taken into account consists in proving a localization lemma as in [**BM**, p. 531] for any small geodesic ball B centered at $\partial\Omega$ and intersected with Ω. In precise terms, we need to show that, inside $B \cap \Omega$, the isoperimetric inequality for the relative perimeter infinitesimally behaves as in $\mathbb{H}^{n+1}$. This property comes from the fact that $B \cap \Omega$ is diffeomorphic to a half-ball in $\mathbb{H}^{n+1}$ centered at $\partial\mathbb{H}^{n+1}$ with Lipschitz constants arbitrarily close to 1. Finally, a compactness argument as in [**BM**] allows us to pass from the localization lemmae to a global isoperimetric inequality. $\square$

Remark 2.2. The asymptotic behaviour in the proposition above provides upper and lower bounds on the profile for small volumes. In fact, for any $\varepsilon > 0$, there exists $V(\Omega, \varepsilon) > 0$ such that

$$(1-\varepsilon)\, I_{\mathbb{H}^{n+1}}(V) \leqslant I_\Omega(V) \leqslant (1+\varepsilon)\, I_{\mathbb{H}^{n+1}}(V), \qquad \text{whenever } V \leqslant (\Omega, \varepsilon).$$

The last inequalities and the ones given in [**BM**, App. C] imply that a set E in Ω such that $\mathrm{vol}(E) = V$ and $\mathcal{P}(E, \Omega) = I_\Omega(V)$ for a small volume V, must meet the boundary of Ω.

Now, we introduce another notion of isoperimetric profile (see [**Gr**], [**Ga**] and [**Ba2**]), which is sometimes more relevant in order to obtain comparison theorems. It is given by the function $h_\Omega : [0,1] \to \mathbb{R}^+ \cup \{0\}$, defined for all β in $[0,1]$ by

$$h_\Omega(\beta) = \frac{I_\Omega(\beta\, \mathrm{vol}(\Omega))}{\mathrm{vol}(\Omega)}. \tag{2.1}$$

This point of view, which somehow corresponds to the choice of a probability measure on Ω, will be considered in the proof of a Lévy-Gromov type inequality (Theorem 4.8).

2.2. **Isoperimetric regions: existence and regularity.** Let Ω be a smooth domain of a Riemannian manifold (M^{n+1}, g). An *isoperimetric region* -or simply a *minimizer*- in Ω for volume $V \in (0, \mathrm{vol}(\Omega))$ is a set $E \subseteq \Omega$ such that $\mathrm{vol}(E) = V$ and $\mathcal{P}(E, \Omega) = I_\Omega(V)$.

In the following proposition we summarize some results from Geometric Measure Theory concerning the existence and regularity of isoperimetric regions in Ω.

Proposition 2.3 ([**Gi**], [**GMT**], [**G1**], [**M2**], [**Bo**])**.** *Let Ω be a smooth domain with compact closure in a Riemannian manifold (M^{n+1}, g). For any $V \in (0, \mathrm{vol}(\Omega))$ there is an open set $E \subset \Omega$ which minimizes the perimeter relative to Ω for volume V. The boundary $\Lambda = \overline{\partial E \cap \Omega}$ can be written as a disjoint union $\Sigma \cup \Sigma_0$, where Σ is the regular part of Λ and $\Sigma_0 = \Lambda - \Sigma$ is the set of singularities. Precisely, we have*

(i) *$\Sigma \cap \Omega$ is a smooth, embedded hypersurface with constant mean curvature.*
(ii) *If $p \in \Sigma \cap \partial\Omega$, then Σ is a smooth, embedded hypersurface with boundary contained in $\partial\Omega$ in a neighborhood of p; in this neighborhood Σ has constant mean curvature and meets $\partial\Omega$ orthogonally.*
(iii) *Σ_0 is a closed set of Hausdorff dimension less than or equal to $n-7$.*
(iv) *At every point $q \in \Sigma_0$ there is a tangent minimal cone $C_q \subset T_qM$ different from a hyperplane. The square sum $|\sigma|^2 = k_1^2 + \ldots + k_n^2$ of the principal curvatures of Σ tends to ∞ when we approach q from Σ.*

In the preceding proposition the regular set Σ is defined as follows: for $p \in \Sigma$ there is a neighborhood W of p in Σ such that W is a smooth, embedded hypersurface without boundary or with boundary contained in $\partial\Omega$. Note that a consequence of the proposition above is the absence of interior points in Σ meeting $\partial\Omega$ tangentially, see [**G2**].

Remark 2.4. The regular hypersurface Σ associated to a minimizer in Ω with large volume need not meet the boundary of Ω. An example illustrating this situation can be found at the end of Section 2 in [**RR**].

2.3. **An analytic comparison result.** Let $f : I \to \mathbb{R}$ be a function defined on an open interval. For any $x_0 \in I$ we denote by $\overline{D^2 f}(x_0)$ the *upper second derivative* of f at x_0, defined by

$$\overline{D^2 f}(x_0) = \limsup_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2}. \tag{2.2}$$

The main tool that we shall employ in Section 4 to derive comparison theorems from differential inequalities is the following technical result. A detailed proof is included in [**Ba1**, App. C].

Theorem 2.5. *Let $f, g : [0,a] \to \mathbb{R}$ be continuous functions with positive values on $(0,a)$. Let $H : \mathbb{R}^+ \to \mathbb{R}$ be the function $H(x) = -\alpha\delta x^{(2-\alpha)/\alpha}$, where $\delta \in \mathbb{R}$ and $\alpha \geqslant 2$. Suppose that f satisfies the second order differential inequality*

$$\overline{D^2} f(x) \leqslant H[f(x)], \qquad x \in (0,a),$$

while g is a C^2-function that satisfies the differential equation

$$g''(x) = H[g(x)], \qquad x \in (0,a).$$

Then, we have

(i) *If $f(0) = g(0)$ and $f(a) = g(a)$, then $f \geqslant g$ on $[0,a]$. Moreover, if $f(x_0) = g(x_0)$ for some $x_0 \in (0,a)$, then $f = g$ on $[0,a]$.*
(ii) *If $f(0) = g(0)$ and the right derivatives at the origin verify $f'_r(0) \leqslant g'_r(0) < +\infty$, then $f \leqslant g$ on $[0,a]$. Moreover, if $f(x_0) = g(x_0)$ for some $x_0 \in (0,a]$, then $f = g$ on $[0,x_0]$.*

The theorem above can be seen as a generalization of the fact that a concave function f on $[0,a]$ is pinched between any tangent line and the secant line passing through $(0, f(0))$ and $(a, f(a))$. Indeed, the philosophy of these comparisons is to pinch the solution of the differential inequality between two exact solutions of the corresponding differential equation with the same initial or boundary conditions.

2.4. **Convex domains in Riemannian manifolds.** The term “convex domain” is used in different non-equivalent ways in the literature. We adopt the following definition:

Let Ω be a domain of a Riemannian manifold (M^{n+1}, g). We say that Ω is *convex* if any two points $p, q \in \Omega$ can be joined by a minimizing geodesic of M which is contained in Ω. A convex domain Ω with compact closure in M will be called a *convex body*.

The convexity of a smooth domain Ω implies the *local convexity* of $\partial\Omega$, which means that all the geodesics in M tangent to $\partial\Omega$ are locally outside of Ω. As R. Bishop proved ([**Bi**]), the local convexity of $\partial\Omega$ is equivalent to an analytic condition (the so-called *infinitesimal convexity*) involving the second fundamental form of $\partial\Omega$. As a consequence, for a smooth convex domain Ω of a Riemannian manifold, the second fundamental form II_p of $\partial\Omega$ with respect to the inner normal vector is positive semidefinite at any $p \in \partial\Omega$.

Remark 2.6. Most of the results of the paper in which the convexity of Ω is assumed are also valid under the weaker condition that II_p is positive semidefinite at any $p \in \partial\Omega$.

The following result is a standard application to the setting of convex bodies of two well-known comparison theorems in Riemannian Geometry. It will be useful in order to show that our isoperimetric inequalities in Section 4 are sharp.

Theorem 2.7. *Let Ω be a smooth convex domain of a complete Riemannian manifold (M^{n+1}, g). For $\delta > 0$, denote by $\mathbb{H}^{n+1}_\delta$ the $(n+1)$-dimensional half-sphere of radius $1/\sqrt{\delta}$. If the Ricci curvature of M satisfies $\mathrm{Ric} \geqslant n\delta > 0$ on Ω, then*

(i) *$\overline{\Omega}$ is compact and $\mathrm{diam}(\Omega) \leqslant \mathrm{diam}(\mathbb{H}^{n+1}_\delta) = \pi/\sqrt{\delta}$ (Bonnet–Myers Theorem).*
(ii) *If $\partial\Omega \neq \emptyset$, then $\mathrm{vol}(\Omega) \leqslant \mathrm{vol}(\mathbb{H}^{n+1}_\delta)$ and equality implies that $\partial\Omega$ is totally geodesic in M and Ω is isometric to $\mathbb{H}^{n+1}_\delta$ (Bishop's Theorem).*

Remark 2.8. In the theorem above we assume $\mathrm{Ric} \geqslant n\delta > 0$ only in Ω. In Section 4 we shall see that the two geometric inequalities in Theorem 2.7 can be obtained by using isoperimetric comparisons. In Theorem 4.13 we characterize the half-spheres as the only convex domains for which equality in Theorem 2.7 (i) holds.

3. The differential inequality

Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1}, g). Our main goal in this section is to prove that the renormalized isoperimetric profile Y_Ω satisfies a differential inequality as that as in (1.1). We shall then derive some immediate consequences related to the regularity of the profile and the connectivity of isoperimetric regions in Ω.

Let us start with the proof of the differential inequality. As we pointed out in Section 1, the idea of the proof consists in a local comparison of Y_Ω with the relative profiles associated to "almost parallel variations" of a minimizer E in Ω for a fixed volume V_0. These variations will be constructed by using the following lemma

Lemma 3.1. *Let E be an isoperimetric region inside a smooth domain Ω with compact closure in a Riemannian manifold (M^{n+1}, g). Denote by Σ the regular part of $\Lambda = \overline{\partial E \cap \Omega}$. Then, there is a sequence $\{\varphi_\varepsilon : \Sigma \to \mathbb{R}\}_{\varepsilon>0}$ of smooth functions with compact support in Σ, such that*

(i) $0 \leqslant \varphi_\varepsilon \leqslant 1, \quad \varepsilon > 0.$

(ii) $\{\varphi_\varepsilon\} \to 1$ *in the Sobolev space* $H^1(\Sigma)$, *that is*

$$\lim_{\varepsilon\to 0} \int_\Sigma \varphi_\varepsilon^2 \, d\mathcal{H}_n = \mathcal{P}(E, \Omega), \qquad \lim_{\varepsilon\to 0} \int_\Sigma |\nabla \varphi_\varepsilon|^2 \, d\mathcal{H}_n = 0,$$

where $\nabla\varphi_\varepsilon$ is the gradient of φ_ε relative to Σ.

(iii) $\lim_{\varepsilon\to 0} \varphi_\varepsilon(p) = 1, \quad p \in \Sigma.$

A complete proof of the lemma above when Ω is a Euclidean domain can be found in [**SZ2**, Lemma 2.4]. The general case is treated in a similar way, see [**MR**] and [**Ba2**, Proposition 1.1] for further details. In [**MR**, Lemma 3.1] it was shown that the existence of $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is guaranteed for a bounded, constant mean curvature hypersurface Σ with a closed singular set $\Sigma_0 = \overline{\Sigma} - \Sigma$ such that $\mathcal{H}_{n-2}(\Sigma_0) = 0$ or consisting of isolated points.

Now, we can prove the main result of this section. Recall that $\overline{D^2 f}(x_0)$ denotes the upper second derivative of a function f at x_0, as defined in (2.2).

Theorem 3.2. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1}, g). Suppose that the Ricci curvature of M satisfies $\mathrm{Ric} \geqslant n\delta$ on Ω. Then, the renormalized isoperimetric profile $Y_\Omega = I_\Omega^{(n+1)/n}$ verifies*

$$\overline{D^2 Y_\Omega}(V) \leqslant -(n+1)\,\delta\, Y_\Omega(V)^{(1-n)/(1+n)}, \qquad V \in (0, \mathrm{vol}(\Omega)). \tag{3.1}$$

If equality holds for some $V_0 \in (0, \mathrm{vol}(\Omega))$ then the boundary $\Lambda = \overline{\partial E \cap \Omega}$ of any minimizer E in Ω of volume V_0 is a smooth, totally umbilical hypersurface such that

$$\mathrm{Ric}(N, N) \equiv n\delta \ \text{ on } \Lambda \quad \text{ and } \quad \mathrm{II}(N, N) \equiv 0 \ \text{ on } \Lambda \cap \partial\Omega,$$

where N is the unit normal to Λ which points into E, and II is the second fundamental form of $\partial\Omega$ with respect to the inner normal.

Moreover, if Ω coincides with the half-space $\mathbb{H}^{n+1}_\delta$ in the simply connected space form with constant sectional curvature δ, then equality holds in (3.1) for any $V \in (0, \mathrm{vol}(\Omega))$.

Proof. Fix $V_0 \in (0, \mathrm{vol}(\Omega))$. By Proposition 2.3 there is a minimizer E in Ω of volume V_0. By the same result, the regular part Σ of $\Lambda = \overline{\partial E \cap \Omega}$ is a smooth, embedded hypersurface which meets $\partial\Omega$ orthogonally. The mean curvature of Σ with respect to the unit normal N pointing into E is a constant H_0. The boundary $\Sigma \cap \partial\Omega$ could be empty, see Remark 2.4. In this case, we adopt the convention that the integrals over $\Sigma \cap \partial\Omega$ are all equal to 0.

Consider a sequence of functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ as in Lemma 3.1. Fix $\varepsilon > 0$ and take a smooth vector field X_ε with compact support over M, such that $X_\varepsilon(q) \in T_q(\partial\Omega)$ whenever $q \in \partial\Omega$ and $X_\varepsilon = \varphi_\varepsilon N$ in Σ. The flow of diffeomorphisms $\{\phi_t\}_{t\in(-\gamma,\gamma)}$ of X_ε in $\overline{\Omega}$ induces a variation $\{E_t = \phi_t(E)\}_t$ of E through sets of finite perimeter contained in Ω. Call $\mathcal{P}_\varepsilon(t) = \mathcal{P}(E_t, \Omega)$ and $V_\varepsilon(t) = \mathrm{vol}(E_t)$. By the first variation for perimeter and volume

$$\mathcal{P}_\varepsilon'(0) = \int_\Sigma \mathrm{div}_\Sigma X_\varepsilon \, d\mathcal{H}_n = -\int_\Sigma nH_0 \, \varphi_\varepsilon \, d\mathcal{H}_n, \tag{3.2}$$

$$V_\varepsilon'(0) = \int_E \mathrm{div}\, X_\varepsilon \, d\mathcal{H}_{n+1} = -\int_\Sigma \varphi_\varepsilon \, d\mathcal{H}_n, \tag{3.3}$$

where div_Σ is the divergence relative to Σ. As $V_\varepsilon'(0) < 0$, we can write t as a function of the volume $V = V(t)$ for V close to V_0; hence, we can define $\mathcal{P}_\varepsilon(V) = \mathcal{P}_\varepsilon[t(V)]$.

Now, consider the function $g_\varepsilon(V) = \mathcal{P}_\varepsilon(V)^{(n+1)/n}$ defined on a neighborhood of V_0. By using the definition of isoperimetric profile and the fact that E is a minimizer, it is clear that

$$Y_\Omega(V) \leqslant g_\varepsilon(V), \qquad Y_\Omega(V_0) = g_\varepsilon(V_0),$$

from which we deduce

$$\overline{D^2 Y_\Omega}\,(V_0) \leqslant \overline{D^2 g_\varepsilon}\,(V_0) = \left(\frac{n+1}{n}\right) \mathcal{P}_\varepsilon(V_0)^{1/n} \left\{ \frac{1}{n}\, \mathcal{P}_\varepsilon'(V_0)^2\, \mathcal{P}_\varepsilon(V_0)^{-1} + \mathcal{P}_\varepsilon''(V_0) \right\}. \tag{3.4}$$

Now, we shall compute the derivatives $\mathcal{P}_\varepsilon'(V_0)$ and $\mathcal{P}_\varepsilon''(V_0)$. The first one is calculated by using (3.2) and (3.3). We get

$$\mathcal{P}_\varepsilon'(V_0) = \mathcal{P}_\varepsilon'(0)\, V_\varepsilon'(0)^{-1} = nH_0. \tag{3.5}$$

On the other hand, the calculation of $\mathcal{P}_\varepsilon''(V_0)$ requires second variation of perimeter and volume. By following the proof of [**SZ2**, Theorem 2.5] (in fact, the only change is that a new term involving the Ricci curvature appears), it is obtained

$$\begin{aligned} \mathcal{P}_\varepsilon''(V_0) = {} & \left(\int_\Sigma \varphi_\varepsilon \, d\mathcal{H}_n \right)^{-2} \\ & \times \left[\int_\Sigma (|\nabla \varphi_\varepsilon|^2 - (\mathrm{Ric}(N,N) + |\sigma|^2)\, \varphi_\varepsilon^2)\, d\mathcal{H}_n - \int_{\Sigma\cap\partial\Omega} \mathrm{II}(N,N)\, \varphi_\varepsilon^2 \, d\mathcal{H}_{n-1} \right], \end{aligned} \tag{3.6}$$

where $|\sigma|^2$ is the squared sum of the principal curvatures of Σ with respect to N, and II is the second fundamental form of $\partial\Omega$ with respect to the inner normal.

Now, if we take lim sup in the equality above when $\varepsilon \to 0$ and we use Lemma 3.1 together with Fatou's Lemma, we have

$$\begin{aligned} \limsup_{\varepsilon\to 0} \mathcal{P}_\varepsilon''(V_0) & \leqslant -\mathcal{P}(E,\Omega)^{-2} \left[\int_\Sigma (\mathrm{Ric}(N,N) + |\sigma|^2)\, d\mathcal{H}_n + \int_{\Sigma\cap\partial\Omega} \mathrm{II}(N,N)\, d\mathcal{H}_{n-1} \right] \\ & \leqslant -n\,(\delta + H_0^2)\, \mathcal{P}(E,\Omega)^{-1}, \end{aligned} \tag{3.7}$$

where in the last inequality we have used the assumption on the Ricci curvature, the well-known inequality $|\sigma|^2 \geqslant nH_0^2$, and the convexity of Ω.

Thus, if we pass to the limit in (3.4) and we use (3.5) together with (3.7), we deduce

$$\begin{aligned}\overline{D^2 Y_\Omega}(V_0) &\leqslant \left(\frac{n+1}{n}\right)\mathcal{P}(E,\Omega)^{1/n}\left\{nH_0^2\,\mathcal{P}(E,\Omega)^{-1}+\limsup_{\varepsilon\to 0}\,\mathcal{P}''_\varepsilon(V_0)\right\}\\ &\leqslant -(n+1)\,\delta\,\mathcal{P}(E,\Omega)^{(1-n)/n}=-(n+1)\,\delta\,Y_\Omega(V_0)^{(1-n)/(1+n)},\end{aligned}\tag{3.8}$$

and (3.1) is proved. Moreover, if equality holds in (3.8) then we also have equality in (3.7), and so Σ is totally umbilical, $\operatorname{Ric}(N,N)\equiv n\delta$ on Σ, and $\mathrm{II}(N,N)\equiv 0$ on $\Sigma\cap\partial\Omega$. Furthermore, the singular set $\Sigma_0=\Lambda-\Sigma$ is empty by Proposition 2.3 (iv) since $|\sigma|^2$ is bounded.

Finally, suppose that $\Omega=\mathbb{H}^{n+1}_\delta$. By reflecting with respect to $\partial\mathbb{H}^{n+1}_\delta$ we get that any minimizer in $\mathbb{H}^{n+1}_\delta$ is obtained by intersecting a geodesic ball B centered at $\partial\mathbb{H}^{n+1}_\delta$ with $\mathbb{H}^{n+1}_\delta$. As ∂B is a totally umbilical hypersurface and $\partial\mathbb{H}^{n+1}_\delta$ is totally geodesic, we have equality in (3.7). On the other hand, equality holds in the first inequality of (3.8) since $Y_{\mathbb{H}^{n+1}_\delta}$ equals the renormalized profile $\mathcal{P}(V)^{(n+1)/n}$ given by parallel hypersurfaces to $\partial B\cap\mathbb{H}^{n+1}_\delta$. □

Remark 3.3. By using the profile h_Ω defined in (2.1) we easily see that Theorem 3.2 is also valid for the renormalized profile $y_\Omega=h_\Omega^{(n+1)/n}$. In particular,

$$\overline{D^2 y_\Omega}(\beta)\leqslant -(n+1)\,\delta\,y_\Omega(\beta)^{(1-n)/(1+n)},\qquad \beta\in(0,1),\tag{3.9}$$

with equality for all $\beta\in(0,1)$ when Ω coincides with $\mathbb{H}^{n+1}_\delta$ $(\delta>0)$.

Remark 3.4. Let (M^{n+1},g) be a closed Riemannian manifold with $\operatorname{Ric}\geqslant n\delta$. Then, M can be seen as a convex body Ω with $\partial\Omega=\emptyset$, and the proof of (3.1) remains valid with the only change that the terms involving $\Sigma\cap\partial\Omega$ vanish. With a similar proof, V. Bayle [**Ba2**, Theorem 1.1] proved that (3.1) holds for the renormalized profile $y_M=h_M^{(n+1)/n}$. Another type of differential inequality for the isoperimetric profile I_M was previously established by F. Morgan and D. Johnson [**MJ**, Proposition 3.3].

The remainder of this section is devoted to deduce some immediate consequences of Theorem 3.2.

One of the easiest and most obvious applications of the differential inequality (3.1) is the following theorem. The proof only uses the fact that a lower semicontinuous function on an interval $f:I\to\mathbb{R}$ such that $\overline{D^2 f}\leqslant 0$ in the interior of I must be concave ([**Ba1**]).

Theorem 3.5. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1},g) with non-negative Ricci curvature. Then, the renormalized profile of Ω is concave. As consequence, the isoperimetric profile I_Ω is concave and, therefore, increasing on $[0,\operatorname{vol}(\Omega)/2]$.*

Remark 3.6. The concavity of the profile of a smooth convex body $\Omega\subset\mathbb{R}^{n+1}$ was obtained by P. Sternberg and K. Zumbrun [**SZ2**, Theorem 2.8]. The observation that, in fact, the renormalized profile of $\Omega\subset\mathbb{R}^{n+1}$ is concave is due to E. Kuwert [**K**].

Now, we generalize to the setting of convex bodies the regularity properties obtained for the isoperimetric profile of a closed Riemannian manifold (see [**BP**], [**MJ**] and [**Ba2**]). As an analytic outcome of Theorem 3.5 we have that, under non-negative Ricci curvature, the isoperimetric profile of Ω has the regularity properties of concave functions. In the following proposition we show that no assumption on the Ricci curvature is needed.

Proposition 3.7. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1},g). Then the renormalized isoperimetric profile Y_Ω has left and right derivatives satisfying*

$$(Y_\Omega)'_l(V)\geqslant (Y_\Omega)'_r(V),\quad V\in(0,\operatorname{vol}(\Omega)).$$

As a consequence, $(Y_\Omega)'_l(\mathrm{vol}(\Omega)/2)$ is non-negative. Moreover, Y_Ω is differentiable on the interval $(0,\mathrm{vol}(\Omega))$ except on an at most countable set.

As to the isoperimetric profile I_Ω, it has left and right derivatives for every V in $(0,\mathrm{vol}(\Omega))$, such that

$$(I_\Omega)'_l(V)\geqslant nH_E\geqslant (I_\Omega)'_r(V),$$

where H_E is the inward mean curvature associated to a minimizer E in Ω of volume V. As a consequence, $(I_\Omega)'_l(\mathrm{vol}(\Omega)/2)$ is non-negative. Furthermore, I_Ω is differentiable on $(0,\mathrm{vol}(\Omega))$ except on an at most countable set.

Proof. The regularity properties and the inequality between the side derivatives come from the differential inequality (3.1), which implies that locally around $V_0\in(0,\mathrm{vol}(\Omega))$ the renormalized profile Y_Ω is concave, up to the addition of a constant times $(V-V_0)^2$. Now fix $V_0\in(0,\mathrm{vol}(\Omega))$ and take a minimizer E in Ω of volume V_0. Let $\mathcal{P}(V)$ be the relative profile associated to an almost parallel variation of E as constructed in the proof of Theorem 3.2. It is obvious that $I_\Omega(V)\leqslant\mathcal{P}(V)$ for V close to V_0, and $I_\Omega(V_0)=\mathcal{P}(V_0)$. As $\mathcal{P}'(V_0)=nH_E$ (see (3.5)) we deduce that $(I_\Omega)'_l(V_0)\geqslant nH_E\geqslant(I_\Omega)'_r(V_0)$. □

Remark 3.8. The asymptotic behaviour of the profile given in Proposition 2.1 allows us to deduce the following consequences from Proposition 3.7

(i) I_Ω is continuous on $[0,\mathrm{vol}(\Omega)]$.
(ii) $\lim_{V\to 0}(I_\Omega)'_r(V)=+\infty$.
(iii) The inward mean curvature associated to a minimizer in Ω explodes when the enclosed volume tends to zero.

We finish this section by showing some topological restrictions related to the connectivity of minimizers inside a convex body. We derive them by using a well-known argument (see [**SZ2**] and [**MJ**]), which relies on the second variation formula of perimeter (3.6).

Proposition 3.9. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1},g) such that $\mathrm{Ric}\geqslant n\delta$ on Ω. Denote by II the second fundamental form of $\partial\Omega$ with respect to the inner normal. Let E be an isoperimetric region in Ω, Σ the regular part of $\overline{\partial E\cap\Omega}$, and N the normal to Σ pointing into E. Then*

(i) *If $\delta>0$, then Σ is connected.*
(ii) *If $\delta=0$ and Σ consists of more than one component, then Σ is totally geodesic and we have $\mathrm{Ric}(N,N)\equiv 0$ in Σ and $\mathrm{II}(N,N)\equiv 0$ in $\Sigma\cap\partial\Omega$. As a consequence, if Σ is non-connected, and Ω is strictly convex in the sense that $\mathrm{II}>0$, then $\Sigma\cap\partial\Omega=\emptyset$.*
(iii) *If $\delta\leqslant 0$, then there exists $V_1\in(0,\mathrm{vol}(\Omega))$ such that Σ is connected if $\mathrm{vol}(E)\leqslant V_1$.*

Proof. Call $V_0=\mathrm{vol}(E)$ and denote by H_0 the mean curvature of Σ with respect to N. Let Σ' be a component of Σ and $\{\varphi_\varepsilon\}\subset C_0^\infty(\Sigma')$ a sequence as in Lemma 3.1. By following the proof of Theorem 3.2 we consider almost parallel variations of E and the associated perimeter functions $\mathcal{P}_\varepsilon(V)$. Call $\alpha(V_0)=\limsup_{\varepsilon\to 0}\mathcal{P}''_\varepsilon(V_0)$. From (3.7) we know that

$$\alpha(V_0)\leqslant -n\,(\delta+H_0^2)\,\mathcal{P}(E,\Omega)^{-1}, \tag{*}$$

due to the hypothesis on the Ricci curvature, the convexity of Ω and the inequality $|\sigma|^2\geqslant nH_0^2$.

We assert that $\alpha(V_0)<0$ implies that Σ is connected. Otherwise, we would use almost parallel variations with $\varepsilon\approx 0$ to expand one component Σ_1 and shrink another one Σ_2 so that the resulting variation preserves volume while reducing perimeter, see [**SZ2**, Theorem 2.6] for details; this would give us a contradiction with the minimality of E.

Now we distinguish two cases. If $\delta \geqslant 0$, then $\alpha(V_0) \leqslant 0$ and an easy discussion of equality cases in (*) proves the claim. If $\delta \leqslant 0$, then the explosion of the mean curvature for small volumes (Remark 3.8 (iii)) yields the existence of V_1 such that $\alpha(V) < 0$ for $V \in [0, V_1]$. □

Remark 3.10. Topological restrictions on isoperimetric hypersurfaces inside a Euclidean convex body were obtained by A. Ros and E. Vergasta [**RV**] and by P. Sternberg and K. Zumbrun [**SZ2**]. On the one hand, statement (ii) in the proposition above is proved in [**SZ2**, Theorem 2.6] for a convex body $\Omega \subset \mathbb{R}^{n+1}$. Furthermore, it is shown that strict convexity of Ω implies that Σ is connected. We must point out that, in general, this cannot be achieved when Ω is not a Euclidean domain since $\Sigma \cap \partial\Omega$ could be empty, see Remark 2.4. On the other hand, in [**RV**, Theorem 5] some conditions on the genus g and the number r of boundary components of Σ are obtained when $\Omega \subset \mathbb{R}^3$. In precise terms, they proved that the only possible values for g and r are

(i) $g = 0$ and $r = 1, 2$ or 3;
(ii) $g = 2$ or 3 and $r = 1$.

It has been recently conjectured that an isoperimetric hypersurface inside a strictly convex body of $\mathbb{R}^3$ must be homeomorphic to a disk ([**Ro**]).

Let Ω be a smooth convex body of a Riemannian manifold and let $n\delta$ be a lower bound on the Ricci curvature of Ω. By Proposition 3.9 we have that a minimizer E in Ω is connected when $\delta > 0$, or when $\delta \leqslant 0$ and $\mathrm{vol}(E)$ is small enough. At first, the second variation of perimeter is not sufficient, in the case $\delta \leqslant 0$, to discard a minimizer with finitely many components bounded by totally geodesic hypersurfaces. However, by using that the profile is concave when $\delta = 0$ we can prove

Proposition 3.11. *Let Ω be a smooth convex body of a Riemannian manifold with non-negative Ricci curvature. Then, isoperimetric regions in Ω are connected.*

Proof. Suppose that E is a minimizer of volume $V_0 \in (0, \mathrm{vol}(\Omega))$ and that E_1 is a connected component of E with volume $V_1 < V_0$. By the definition of isoperimetric profile and the fact that the set of singularities in $\overline{\partial E \cap \Omega}$ does not contribute to perimeter, we get

$$I_\Omega(V_0) = \mathcal{P}(E, \Omega) = \mathcal{P}(E_1, \Omega) + \mathcal{P}(E - E_1, \Omega) \geqslant I_\Omega(V_1) + I_\Omega(V_0 - V_1).$$

On the other hand, the concavity of Y_Ω (Theorem 3.5) gives us

$$Y_\Omega(V_0) \leqslant Y_\Omega(V_1) + Y_\Omega(V_0 - V_1),$$

and so, as $I_\Omega(V_1)$ and $I_\Omega(V_0 - V_1)$ are positive, and since the function $x \longmapsto x^{\frac{n}{n+1}}$ is strictly concave, we deduce

$$I_\Omega(V_0) < I_\Omega(V_1) + I_\Omega(V_0 - V_1),$$

which leads us to a contradiction. This proves that $V_1 = V_0$, and E is therefore connected. □

4. Comparison theorems

In this section, we shall integrate the differential inequality (3.1) in order to prove comparison theorems for the isoperimetric profile of a smooth convex body Ω in a Riemannian manifold (M^{n+1}, g). The underlying philosophy of these results consists in using the analytic Theorem 2.5 to compare a profile f, which can be Y_Ω or the function y_Ω defined in Remark 3.3, with a solution g of the differential equation associated to (3.1) having the same initial conditions or the same boundary values as f. In the first case we shall obtain an upper bound for the profile I_Ω, while in the second one, we shall deduce a lower bound for h_Ω that can be interpreted as a Lévy-Gromov type inequality. We must remark that the two comparisons are quite different although they arise from the same differential inequality. A detailed analysis of equality cases will allow us to deduce global geometric consequences on Ω.

Through this section we also illustrate how to use our isoperimetric inequalities to deduce other geometric an analytic comparisons. In this way, we give alternative proofs of the inequalities in Theorem 2.7, and we characterize the half-spheres as the only convex domains for which equality in Theorem 2.7 (i) holds. Finally, we prove a comparison result for the first non-zero Neumann eigenvalue of the Laplace operator on Ω that can be seen as a generalization of the Obata–Lichnerowicz theorem [**Ch**, Theorem 9, p. 82].

4.1. Upper bounds on the isoperimetric profile.

Theorem 4.1. *Let Ω be a smooth convex body with non-empty boundary of a complete Riemannian manifold (M^{n+1},g). Suppose that the Ricci curvature of M satisfies $\mathrm{Ric} \geqslant n\delta$ on Ω. Then*

$$I_\Omega(V) \leqslant I_{\mathbb{H}^{n+1}_\delta}(V), \qquad V \in [0,\mathrm{vol}(\Omega)], \tag{4.1}$$

where $\mathbb{H}^{n+1}_\delta$ is a half-space in the $(n+1)$-dimensional simply connected space form with constant sectional curvature δ.

If equality holds in (4.1) for some $V_0 \in (0,\mathrm{vol}(\Omega)]$, then $I_\Omega = I_{\mathbb{H}^{n+1}_\delta}$ on $[0,V_0]$, and the boundary $\overline{\partial E\cap\Omega}$ of any minimizer E in Ω of volume $V\in(0,V_0)$ is a smooth, totally umbilical hypersurface. Moreover, if $V_0=\mathrm{vol}(\Omega)$ (which implies $\delta>0$), then Ω is isometric to $\mathbb{H}^{n+1}_\delta$.

Proof. The comparison arises from the fact that a continuous solution of the differential inequality (3.1) is bounded from above by a solution of the differential equation

$$f'' = -(n+1)\,\delta\, f^{(1-n)/(1+n)} \tag{4.2}$$

with the same initial conditions (Theorem 2.5). Then, by using that the renormalized profile of $\mathbb{H}^{n+1}_\delta$ satisfies (4.2) (see Theorem 3.2) and taking into account the asymptotic behaviour of the renormalized profile Y_Ω at the origin (Proposition 2.1), we obtain

$$Y_\Omega(V) \leqslant Y_{\mathbb{H}^{n+1}_\delta}(V), \qquad V\in[0,\min\{\mathrm{vol}(\Omega),\mathrm{vol}(\mathbb{H}^{n+1}_\delta)\}]. \tag{4.3}$$

From the inequality above we get (4.1) once we show that $\mathrm{vol}(\Omega)\leqslant\mathrm{vol}(\mathbb{H}^{n+1}_\delta)$. This volume comparison is trivial if $\delta\leqslant 0$ while in the case $\delta>0$, the opposite inequality would allow us to deduce from (4.3) that $Y_\Omega(\mathrm{vol}(\mathbb{H}^{n+1}_\delta))\leqslant 0$, which is a contradiction since the profile is positive in $(0,\mathrm{vol}(\Omega))$.

Finally, if both profiles coincide at $V_0\in(0,\mathrm{vol}(\Omega)]$ then they must coincide in $[0,V_0]$ by Theorem 2.5. The umbilicality of a minimizer of volume $V<V_0$ follows from the discussion, given in Theorem 3.2, of equality cases in (3.1). If $V_0=\mathrm{vol}(\Omega)$, then $\mathrm{vol}(\Omega)=\mathrm{vol}(\mathbb{H}^{n+1}_\delta)$ and Ω is isometric to $\mathbb{H}^{n+1}_\delta$ by Theorem 2.7 (ii). □

Remark 4.2. Note that we have given another proof of the volume comparison $\mathrm{vol}(\Omega)\leqslant\mathrm{vol}(\mathbb{H}^{n+1}_\delta)$ of Theorem 2.7 (ii) by using the isoperimetric inequality (4.3).

Remark 4.3. When $n=1$ the differential inequality (3.1) turns out to be linear and Theorem 4.1 follows since the function $E(V)=Y_\Omega(V)-Y_{\mathbb{H}^2_\delta}(V)$ is concave on $[0,\mathrm{vol}(\Omega)]$, and the tangent line at the origin coincides with the x-axis. After an explicit calculation of $Y_{\mathbb{H}^2_\delta}$, inequality (4.1) reads

$$I^2_\Omega(V) \leqslant V\,(2\pi - \delta V), \qquad V \in [0, \mathrm{vol}(\Omega)].$$

Remark 4.4. For a closed Riemannian manifold (M^{n+1}, g) with $\mathrm{Ric} \geqslant n\delta$, the integration of the differential inequality (3.1) would give us the comparison

$$I_M \leqslant I_{\mathbb{M}^{n+1}_\delta}, \qquad V \in [0, \mathrm{vol}(M)], \tag{4.4}$$

where $\mathbb{M}^{n+1}_\delta$ stands for the $(n+1)$-dimensional simply connected space form with constant sectional curvature δ. This result was previously proved by F. Morgan and D. Johnson [**MJ**, Theorem 3.4].

Remark 4.5. Inequality (4.1) is also valid for a smooth, unbounded, convex domain Ω with non-empty boundary and $\mathrm{Ric} \geqslant n\delta$. This can be proved by showing, as was done for closed Riemannian manifolds in [**MJ**, Theorem 3.5], that the perimeter in Ω of a "half-ball" $B = \Omega \cap B(p, r)$ centered at a point $p \in \partial\Omega$ is less than or equal to the area of the geodesic half-ball $\widetilde{B}$ in $\mathbb{H}^{n+1}_\delta$ of the same volume, with equality only if B is isometric to $\widetilde{B}$ and $\partial\Omega$ is geodesic at p. The arguments in the proof by F. Morgan and D. Johnson rely on comparison theorems involving the volume of metric balls ([**Ch2**, Theorem 3.9]) and the area of metric spheres ([**Ch2**, Proposition 3.3]). These theorems do not use the compactness of the ambient manifold and can be easily generalized to our setting.

This alternative proof of (4.1) also allows us to deduce geometric consequences on Ω when we have equality in (4.1). We summarize them in the next result

Theorem 4.6. *Let Ω be a smooth convex domain with* $\mathrm{Ric} \geqslant n\delta$ *in a complete Riemannian manifold (M^{n+1}, g). Then*

(i) *If Ω has non-empty boundary then* (4.1) *holds, and equality for some $V_0 \in (0, \mathrm{vol}(\Omega)]$ implies that $\partial\Omega$ is totally geodesic in M and Ω has constant sectional curvature δ in a neighborhood of $\partial\Omega$.*

(ii) *If $\partial\Omega = \emptyset$, then* (4.4) *holds, and equality for some $V_0 \in (0, \mathrm{vol}(\Omega)]$ implies that M is isometric to a quotient of the simply connected space form $\mathbb{M}^{n+1}_\delta$ with constant sectional curvature δ.*

Remark 4.7. In general, we cannot improve statement (i) in the theorem above to the stronger conclusion that equality in (4.1) for some V_0 implies that Ω has constant sectional curvature δ. For example, denote by Ω the domain obtained from attaching the half-sphere of $\mathbb{S}^2$ centered at the north pole to the compact cylinder $\mathbb{S}^1 \times [-1, 0]$ through the circle $\mathbb{S}^1 \times \{0\}$. It is clear that $I_\Omega = I_{\mathbb{H}^2_0}$ for small values; however, Ω is not a flat domain.

4.2. **A Lévy-Gromov type inequality for convex bodies.** Let (M^{n+1}, g) be a closed Riemannian manifold with $\mathrm{Ric} \geqslant n\delta > 0$. Denote by h_M the profile of M as defined in (2.1). The Lévy-Gromov inequality [**Gr**] states that

$$h_M(\beta) \geqslant h_{\mathbb{M}^{n+1}_\delta}(\beta), \qquad \beta \in [0, 1], \tag{4.5}$$

where $\mathbb{M}^{n+1}_\delta$ is an $(n+1)$-dimensional sphere of radius $1/\sqrt{\delta}$. Moreover, if equality holds in (4.5) for some $\beta \in (0, 1)$, then M is isometric to $\mathbb{M}^{n+1}_\delta$.

Inequality (4.5) can be obtained by integrating a differential inequality similar to (3.1), see [**Ba1**]. With the same technique, we generalize (4.5) to the setting of convex bodies.

Theorem 4.8. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1}, g). Suppose that the Ricci curvature of M over Ω satisfies* $\mathrm{Ric} \geqslant n\delta > 0$*. Then,*

$$h_\Omega(\beta) \geqslant h_{\mathbb{H}_\delta^{n+1}}(\beta), \qquad \beta \in [0,1], \tag{4.6}$$

where $\mathbb{H}_\delta^{n+1}$ is an $(n+1)$-dimensional half-sphere of radius $1/\sqrt{\delta}$.

Moreover, if Ω has non-empty boundary then equality holds in (4.6) *for some $\beta_0 \in (0,1)$ if and only if Ω is isometric to $\mathbb{H}_\delta^{n+1}$.*

Proof. The inequality follows from the fact, given in Theorem 2.5 (i), that any function satisfying the differential inequality (3.9) is bounded from below by an exact solution of the corresponding differential equation with the same boundary values. Furthermore, if we have equality for some $\beta_0 \in (0,1)$, then $h_\Omega = h_{\mathbb{H}_\delta^{n+1}}$ on $[0,1]$, and by the asymptotic behaviour of h_Ω at the origin (Proposition 2.1), we deduce that $\mathrm{vol}(\Omega) = \mathrm{vol}(\mathbb{H}_\delta^{n+1})$. From statement (ii) in Theorem 2.7 we conclude that Ω is isometric to $\mathbb{H}_\delta^{n+1}$. □

The preceding result can be given in the following alternative form

Corollary 4.9. *Let Ω be a smooth convex body of a Riemannian manifold (M^{n+1}, g). Suppose that the Ricci curvature of M over Ω satisfies* $\mathrm{Ric} \geqslant n\delta > 0$*. Then, for any Borel set $E \subseteq \Omega$, we have*

$$\frac{\mathcal{P}(E,\Omega)}{\mathrm{vol}(\Omega)} \geqslant \frac{\mathcal{P}(E^*, \mathbb{H}_\delta^{n+1})}{\mathrm{vol}(\mathbb{H}_\delta^{n+1})},$$

where $E^* \subseteq \mathbb{H}_\delta^{n+1}$ is a geodesic half-ball centered at $\partial\mathbb{H}_\delta^{n+1}$ such that

$$\frac{\mathrm{vol}(E)}{\mathrm{vol}(\Omega)} = \frac{\mathrm{vol}(E^*)}{\mathrm{vol}(\mathbb{H}_\delta^{n+1})}.$$

Moreover, if Ω has non-empty boundary and equality holds for some set $E \subseteq \Omega$ with $\mathrm{vol}(E) \in (0, \mathrm{vol}(\Omega))$*, then Ω is isometric to an $(n+1)$-dimensional half-sphere of radius $1/\sqrt{\delta}$.*

Remark 4.10. Let $h_C(\Omega)$ be the Cheeger isoperimetric constant of a smooth convex body Ω of a Riemannian manifold (M^{n+1}, g), defined by

$$h_C(\Omega) = \inf\left\{\frac{\mathcal{P}(E,\Omega)}{\min\{\mathrm{vol}(E), \mathrm{vol}(\Omega\setminus E)\}} : \mathrm{vol}(E) \in (0, \mathrm{vol}(\Omega))\right\}.$$

Note that

$$h_C(\Omega) = \inf\left\{\frac{h_\Omega(\beta)}{\min\{\beta, 1-\beta\}} : \beta \in (0,1)\right\},$$

and so, if the Ricci curvature of M is non-negative on Ω, we deduce by the concavity of the profile (Theorem 3.5)

$$h_C(\Omega) = 2\, h_\Omega(1/2),$$

which yields $h_C(\Omega) \geqslant h_C(\mathbb{H}_\delta^{n+1})$ when $\mathrm{Ric} \geqslant n\delta > 0$ in Ω by (4.6).

Now, by reproducing the arguments in [**Ba2**] (see also [**Ba1**]), we can refine Theorem 4.8, so as to get, under the same assumption on the Ricci curvature,

$$h_\Omega(\beta) \geqslant \left[\frac{h_C(\Omega)}{h_C(\mathbb{H}_\delta^{n+1})}\right]^{\frac{1}{n+1}} h_{\mathbb{H}_\delta^{n+1}}(\beta), \qquad \beta \in [0,1]. \tag{4.7}$$

Moreover, if there is $\beta_0 \in (0,1)$ such that (4.7) is an equality, then Ω is isometric to $\mathbb{H}_\delta^{n+1}$.

4.3. **Some consequences of Theorem 4.8.** We first show how to use Theorem 4.8 to give a characterization of equality cases in Theorem 2.7 (i). We need a previous result (see [**Ga**] for closed Riemannian manifolds), linking the diameter of a domain Ω and the profile h_Ω.

Lemma 4.11. *The diameter of a smooth domain Ω of a complete Riemannian manifold (M^{n+1}, g) satisfies*

$$\operatorname{diam}(\Omega) \leqslant \int_0^1 \frac{d\beta}{h_\Omega(\beta)},$$

with equality when Ω coincides with an $(n+1)$-dimensional half-sphere.

Proof. Suppose that $\operatorname{vol}(\Omega) < \infty$ (in other case $h_\Omega \equiv 0$). If Ω is unbounded, then choose any point $p_0 \in \Omega$. If Ω is bounded, fix a point $p_0 \in \overline{\Omega}$ such that $\operatorname{dist}(p_0, p_1) = \operatorname{diam}(\Omega)$ for some $p_1 \in \overline{\Omega}$. Denote by S_t and B_t the metric sphere and the metric open ball in M centered at p_0 with radius $t > 0$. By the coarea formula [**Ch3**, Corollary I.3.1], the volume of a set $E \subseteq M$ can be computed as

$$\operatorname{vol}(E) = \int_0^{+\infty} \mathcal{H}_n(E \cap S_t)\, dt,$$

and so the function $\beta(r) = \operatorname{vol}(\Omega \cap B_r)/\operatorname{vol}(\Omega)$ is absolutely continuous on $[0, \operatorname{diam}(\Omega)]$ and satisfies

$$\beta'(r) = \frac{\mathcal{H}_n(\Omega \cap S_r)}{\operatorname{vol}(\Omega)} \geqslant \frac{\mathcal{P}(\Omega \cap B_r, \Omega)}{\operatorname{vol}(\Omega)} \geqslant h_\Omega(\beta(r)), \tag{4.8}$$

for almost all $r \in [0, \operatorname{diam}(\Omega)]$, with equality when Ω coincides with a half-sphere. The proof finishes by integrating in (4.8). □

Remark 4.12. The asymptotic behaviour of h_Ω at the origin (Proposition 2.1) ensures that the upper bound on the diameter given in the lemma above is finite when Ω is bounded.

As a consequence of Lemma 4.11 and Theorem 4.8 we can prove for convex bodies the analogous of the well-known Topogonov–Cheng theorem [**Ch2**, Theorem 3.11] for closed Riemannian manifolds. Note that the following result is not a direct consequence of the aforementioned one for closed manifolds since we are assuming that $\operatorname{Ric} \geqslant n\delta > 0$ only in Ω.

Theorem 4.13. *Let Ω be a smooth convex body with non-empty boundary of a Riemannian manifold (M^{n+1}, g). If the Ricci curvature of M satisfies $\operatorname{Ric} \geqslant n\delta > 0$ on Ω, then*

$$\operatorname{diam}(\Omega) \leqslant \frac{\pi}{\sqrt{\delta}},$$

and equality holds if and only if Ω is isometric to a half-sphere of radius $1/\sqrt{\delta}$.

Remark 4.14. By following the arguments in [**Ba2**, Corollary 3.7 and Theorem 4.3] we could say that, for a smooth convex body Ω with non-empty boundary and $\operatorname{Ric} \geqslant n\delta > 0$, having a diameter close to $\pi/\sqrt{\delta}$ (resp. a volume close to $\operatorname{vol}(\mathbb{H}^{n+1}_\delta)$) is equivalent to the fact that $h_\Omega - h_{\mathbb{H}^{n+1}_\delta}$ is uniformly close to 0 on $[0,1]$ (resp. $h_\Omega/h_{\mathbb{H}^{n+1}_\delta}$ is uniformly close to 1 on $(0,1)$). This means that almost maximality of the diameter or almost maximality of the volume both entail, in certain sense, almost minimality of the profile.

We finish this section with an eigenvalues comparison theorem. The application of an isoperimetric inequality to obtain eigenvalues estimates was first given by G. B. Faber and E. Krahn for smooth Euclidean domains with compact closure ([**Ch**, Theorem 2, p. 87]). In [**BM**] and [**BBG**] it is shown how the ideas of G. B. Faber and E. Krahn, together with

Lévy–Gromov inequality (4.5), lead to sharp estimates for the first eigenvalue of the Laplace operator with Dirichlet boundary condition on a smooth, bounded domain of a complete Riemannian manifold (M^{n+1}, g) with $\mathrm{Ric} \geqslant n\delta > 0$. Other estimates for Dirichlet eigenvalues obtained in a similar way can be found in [**Ga**] and [**Ba2**].

In the setting of a smooth convex domain Ω with $\partial\Omega \neq \emptyset$, the fact that isoperimetric hypersurfaces in the model $\mathbb{H}^{n+1}_\delta$ intersect the boundary orthogonally, seems to indicate that the Neumann boundary condition on $\partial\Omega$ is more appropriated if we want to derive an eigenvalues comparison from inequality (4.6). In fact, we can prove

Theorem 4.15. *Let Ω be a smooth convex body with non-empty boundary of a complete Riemannian manifold (M^{n+1}, g). If the Ricci curvature of M satisfies $\mathrm{Ric} \geqslant n\delta > 0$ on Ω, then*

$$\lambda_1^N(\Omega) \geqslant \lambda_1^N(\mathbb{H}^{n+1}_\delta) = (n+1)\,\delta, \tag{4.9}$$

where the notation $\lambda_1^N(\Omega)$ stands for the lowest non-zero eigenvalue of the Laplace operator on Ω with Neumann boundary condition on $\partial\Omega$. Moreover, if (4.9) is an equality, then Ω is isometric to a half-sphere $\mathbb{H}^{n+1}_\delta$ of radius $1/\sqrt{\delta}$.

Proof. We give a brief desciption of the proof, which follows the symmetrization argument in [**BM**, Théorème 5]. For any non-trivial function $u \in C^\infty(\Omega)$, denote by $R_\Omega(u)$ the Rayleigh quotient of u, given by

$$R_\Omega(u) = \left(\int_\Omega |\nabla u|^2 \, d\mathcal{H}_{n+1}\right)\left(\int_\Omega u^2 \, d\mathcal{H}_{n+1}\right)^{-1}.$$

Due to the variational characterization of Neumann eigenvalues there exists a smooth, mean zero function u on $\overline{\Omega}$, such that $R_\Omega(u) = \lambda_1^N(\Omega)$ and $\partial u/\partial\nu = 0$ on $\partial\Omega$, where ν is the inward normal vector to $\partial\Omega$. Suppose that u has finitely many non-degenerate critical points (condition (ND)). The symmetrization technique allows us to construct, by using a suitable family of concentric half-balls in $\mathbb{H}^{n+1}_\delta$ centered at a fix boundary point, a function u^* defined on $\mathbb{H}^{n+1}_\delta$ such that

(i) u^* is a non-trivial Sobolev function on $\mathbb{H}^{n+1}_\delta$.
(ii) u^* has mean zero over $\mathbb{H}^{n+1}_\delta$.
(iii) $R_\Omega(u) \geqslant R_{\mathbb{H}^{n+1}_\delta}(u^*)$ with equality if and only if Ω is isometric to $\mathbb{H}^{n+1}_\delta$ (here is the point where Theorem (4.8) is used).

By using statement (iii) and the variational characterization of Neumann eigenvalues, the proof of the theorem follows.

If u does not satisfy condition (ND), then we get (4.9) by approximation since $\lambda_1^N(\Omega)$ is the limit of a sequence $\{R_\Omega(u_n)\}_{n\in\mathbb{N}}$, where each u_n has mean zero and satisfies condition (ND). In this situation, the discussion of the equality case is not so obvious; we appeal to [**BM**, p. 520]. □

Remark 4.16. In [**Re**, Theorem 4], R. Reilly proved that Theorem 4.15 is valid when working with the first eigenvalue of the Laplace operator on Ω with Dirichlet boundary condition on $\partial\Omega$. In [**E**, Theorem 4.3] it is shown that, in fact, inequality (4.9) can be obtained as a consequence of Reilly's formula. We remark that inequality (4.9) can also be deduced by using Bochner's formula ([**Ch**, p. 83]).

Remark 4.17. By using inequality (4.7) instead of (4.6) in the proof of Theorem 4.15, we obtain

$$\lambda_1^N(\Omega) \geqslant \left[\frac{h_C(\Omega)}{h_C(\mathbb{H}_\delta^{n+1})}\right]^{\frac{2}{n+1}} \lambda_1^N(\mathbb{H}_\delta^{n+1}),$$

with equality if and only if Ω is isometric to $\mathbb{H}_\delta^{n+1}$.

5. Appendix: an alternative proof of inequality (4.1) in the euclidean case

Here we give a geometric proof of the fact that the isoperimetric profile of a convex body $\Omega \subset \mathbb{R}^{n+1}$ is bounded from above by the profile of the half-space $\mathbb{H}^{n+1} = \{x_{n+1} > 0\}$. The proof relies on the fact that the local convexity of a domain Ω around a boundary point implies $I_\Omega \leqslant I_{\mathbb{H}^{n+1}}$ for small volumes.

Proposition 5.1. *Let Ω be a smooth domain in $\mathbb{R}^{n+1}$. If Ω has a local supporting hyperplane at a point $x \in \partial\Omega$, then there exists $V_0 > 0$ such that $I_\Omega(V) \leqslant I_{\mathbb{H}^{n+1}}(V)$, whenever $V \in [0, V_0]$.*

Proof. We follow the proof in [**RR**, Proposition 3.6]. Denote by $\mathcal{P}(r)$ and $V(r)$ respectively the perimeter in Ω and the volume of the ball B_r of radius $r > 0$ centered at x intersected with Ω. Let $\widetilde{V}(r)$ be the volume of the cone subtended by $\partial B_r \cap \Omega$ and vertex at x. We have the relation

$$\mathcal{P}(r) = (n+1)\,\frac{\widetilde{V}(r)}{r}.$$

On the one hand, since Ω is locally convex around x, we have $V(r) \geqslant \widetilde{V}(r)$ for r small, so that

$$\mathcal{P}(r) = (n+1)\,\frac{\widetilde{V}(r)}{r} \leqslant (n+1)\,\frac{V(r)}{r}.$$

On the other hand, if $\mathcal{P}_e(r)$ and $V_e(r)$ respectively are the area and the volume of a half-ball in $\mathbb{H}^{n+1}$ of radius $r > 0$, we have

$$\frac{\mathcal{P}_e(r)}{V_e(r)} = \frac{n+1}{r},$$

and so

$$\frac{\mathcal{P}(r)}{V(r)} \leqslant \frac{\mathcal{P}_e(r)}{V_e(r)}.$$

Since $V(r) \leqslant V_e(r)$ due to the local convexity of Ω around x, we finally get

$$\frac{\mathcal{P}(r)}{V(r)^{n/(n+1)}} = \frac{\mathcal{P}(r)}{V(r)}\,V(r)^{1/(n+1)} \leqslant \frac{\mathcal{P}_e(r)}{V_e(r)}\,V_e(r)^{1/(n+1)} = \frac{\mathcal{P}_e(r)}{V_e(r)^{n/(n+1)}} = d_n,$$

where d_n is the constant that appears in the expression of the isoperimetric profile of the half-space $I_{\mathbb{H}^{n+1}}(V) = d_n V^{n/(n+1)}$.

Hence, for small r, we obtain the relation $\mathcal{P}(r) \leqslant I_{\mathbb{H}^{n+1}}(V(r))$, which proves the claim. □

Proof of inequality (4.1)*:* Let Ω be a smooth convex body in $\mathbb{R}^{n+1}$. As the renormalized profile $Y_{\mathbb{H}^{n+1}}$ is linear as function of V, and Y_Ω is concave (Theorem 3.5), the proof trivially follows from Proposition 5.1.

Remark 5.2. Though we have succeeded in comparing the profiles for small volumes with geometric arguments, the global comparison has required global analytic properties of the profile.

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