ISOPERIMETRIC REGIONS IN ROTATIONALLY SYMMETRIC CONVEX BODIES

CÉSAR ROSALES

Abstract. We consider the isoperimetric problem of minimizing perimeter for given volume in a strictly convex domain $\Omega \subset \mathbb{R}^{n+1}$ and prove that, if Ω is rotationally symmetric about some line, then any solution to this problem must be convex.

1. Introduction

In this paper we consider the problem of minimizing perimeter for given volume in a convex set. The precise situation is the following. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded convex open set. By an isoperimetric region -or simply a minimizer- in Ω we mean a set $E \subseteq \Omega$ satisfying the condition

$$P(E) \leqslant P(E'),$$

among all sets $E' \subseteq \Omega$ with $|E'| = |E|$. Here $P(\cdot)$ refers to perimeter in $\mathbb{R}^{n+1}$ (see Section 2) and $|\cdot|$ denotes the $(n+1)$-dimensional Lebesgue measure.

It should be remarked that this problem is considerably different from the free boundary one, in which the functional to minimize for given volume is the perimeter $P(\cdot, \Omega)$ relative to Ω. This was treated for instance in [**Gr**], [**RV**] and [**SZ**].

The classical isoperimetric inequality in $\mathbb{R}^{n+1}$ implies that an isoperimetric region with volume $0 < |E| \leqslant v_0$ must be a ball, where v_0 is the volume of a largest ball in Ω. The existence of isoperimetric regions for any given volume is solved in the context of sets of finite perimeter, see [**G**, Chapter 1]. Regularity questions have been studied by Gonzalez, Massari and Tamanini [**GMT**] and by Stredulinsky and Ziemer [**SZi**]. They have proved that a minimizer E satisfies that $\partial E \cap \Omega$ is a smooth hypersurface with constant mean curvature H_0 off of a singular closed set of small Hausdorff dimension. Moreover, if $\partial\Omega$ is C^2 it was shown in [**SZi**] that ∂E is $C^{1,1}$ in some neighborhood of each point in $\partial E \cap \partial\Omega$, and the mean curvature H of ∂E with respect to the inner normal satisfies $H \leqslant H_0$, $\mathcal{H}^n$-almost everywhere on ∂E. Here $\mathcal{H}^n(\cdot)$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

1991 *Mathematics Subject Classification.* Primary: 53C42. Secondary: 49Q20.

Key words and phrases. Isoperimetric problem, isoperimetric regions, constant mean curvature hypersurfaces, stability.

Work partially supported by MCYT research project BFM2001-3489.

Other important questions about isoperimetric regions are related to geometric and topological properties. We shall deal with the problem of determine whether an isoperimetric region in Ω is convex.

If $\Omega \subset \mathbb{R}^2$ this fact is true, and can be easily proved by using that the convex hull of a connected, non convex set, decreases boundary length while increases volume. For this reason, if a connected minimizer were non convex, one would be able to contract its convex hull to obtain a set in Ω with the same volume and less perimeter, a contradiction. However, this behaviour of the convex hull does not hold in higher dimensions; we can see this on a standard embedded torus in $\mathbb{R}^3$, where the circle in the xy plane has radius 1, and the rotated circle perpendicular to the xy plane has radius $r \ll 1$ ([**Ch**, p. 13]).

Assuming only that Ω is a bounded convex domain, the convexity of any minimizer is an open question in $\mathbb{R}^{n+1}$, $n \geqslant 2$.

In [**SZi**], under the further hypothesis that Ω satisfies a "great circle condition", it is proved that any isoperimetric region is convex. Nestedness, uniqueness and regularity results when Ω is not C^2 are also established. The great circle condition means that a largest open ball B in Ω has a great circle contained in $\partial\Omega$. A great circle of B is the intersection of the boundary of B with a hyperplane passing through the center of B.

In Section 3 we prove:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a strictly convex bounded domain with C^2 boundary. Assume that Ω is also rotationally symmetric about some line R. Then isoperimetric regions in Ω are convex.*

The above theorem is *independent* of the results in [**SZi**]. This is in part due to the independence between the further hypothesis assumed about Ω, see Figure 1.

Connectedness of minimizers is shown in a similar way by moving components into Ω until we produce a minimizer with non embedded boundary. Nevertheless, the techniques employed in [**SZi**] to establish the convexity are different from ours. A summary of the arguments in [**SZi**] is the following. The great circle condition implies that any minimizer E with $|E| \geqslant v_0$ (v_0 is the volume of any largest ball inside Ω) contains a largest open ball in Ω. In particular, E and its convex hull $\mathcal{C}(E)$ have the same intersection with the equatorial hyperplane P. On the other hand, $\partial\mathcal{C}(E)$ is a $C^{1,1}$ hypersurface with mean curvature H^* such that $H^* \leqslant H_0$ is satisfied $\mathcal{H}^n$-almost everywhere [**SZi**, Theorem 3.7]. Suppose that E were non convex; then, by the maximum principle the previous inequality cannot be an equality. Let X be a normal vector to P. By using the Gauss-Green theorem and the first variation formula for perimeter applied to the field X over the open set enclosed by E and $\mathcal{C}(E)$ it is shown that the piece of $\partial\mathcal{C}(E)$ over the equatorial hyperplane cannot be a graph over the equatorial disk, a contradiction. This argument does not hold in general for rotationally symmetric domains, see Figure 1.

Our proof of Theorem 1.1 goes as follows. First, by using Steiner symmetrization we show that any isoperimetric region E is rotationally symmetric about the line R and has connected intersection with any straight line orthogonal to R. We shall see that this gives the absence of singularities in $\partial E \cap \Omega$, which becomes a union of pieces of certain Delaunay hypersurfaces (Proposition 3.2). As a topological consequence, we obtain in Proposition 3.3 that a minimizer must be connected. Symmetrization is not sufficient, in general, to establish the convexity of a minimizer. Convexity will be definitively proved by a stability argument which uses Lemma 2.2.

Under the further assumption that Ω satisfies a great circle condition or that Ω is contained in $\mathbb{R}^2$, it is proved in [**SZi**, Theorems 3.31 and 3.32] that isoperimetric regions with volume exceeding $|B|$, B the union of all largest balls in Ω, are unique and nested as a function of the enclosed volume. As pointed out by the referee, the settings of this paper allow us to construct an example to illustrate that isoperimetric regions are not always nested. Such an example is given in Section 4.

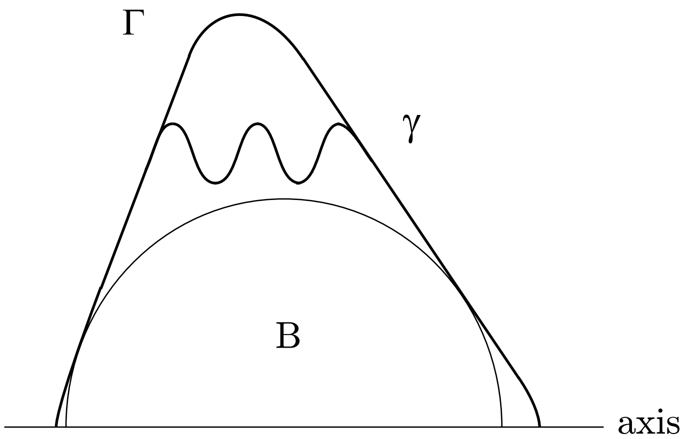

FIGURE 1. The curve Γ generates a rotationally symmetric convex set Ω in $\mathbb{R}^3$. B is the largest ball in Ω. The great circle condition is not satisfied. The curve γ consisting in a piece of Γ together with an open arc of an unduloid generates a set E which contains B. E and its convex hull do not have the same intersection with any plane passing through the center of B.

Acknowledgements. This is part of the author's Ph.D. Thesis. I would like to thank my advisor Manuel Ritoré for his kind help and advise during the elaboration of this notes. I am also grateful to the referee for his helpful suggestions for the revision of the paper.

2. Preliminaries

We denote by $|E|$ the $(n+1)$-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^{n+1}$ and by $\mathcal{H}^k(E)$, k-dimensional Hausdorff measure. The *perimeter* $P(E)$ of any Borel set $E \subseteq \mathbb{R}^{n+1}$ is given by:

$$P(E) = \sup \left\{ \int_E \operatorname{div} \phi : \phi \in C_0^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}),\ |\phi| \leqslant 1 \right\}.$$

E is said to be of *finite perimeter* if $P(E) < \infty$. The definition implies that perimeter is not changed by sets of measure zero; in other words, each set E determines an equivalence class of sets of finite perimeter. To avoid this ambiguity we always assume that:

$$0 < |E \cap B(x,r)| < |B(x,r)| \text{ for every open ball } B(x,r) \text{ such that } x \in \partial E.$$

We refer to [**G**] for background about perimeter.

By a *convex body* we mean a bounded convex open set $\Omega \subset \mathbb{R}^{n+1}$ with C^2 boundary. An *isoperimetric region in* Ω is a set $E \subseteq \overline{\Omega}$ of finite perimeter satisfying:

$$P(E) \leqslant P(E'), \text{ whenever } E' \subseteq \overline{\Omega} \text{ and } |E| = |E'|.$$

The following result summarizes what we can say about existence and regularity of isoperimetric regions in Ω.

Theorem 2.1 ([**G**], [**GMT**], [**SZi**])**.** *For every volume $0 < v < |\Omega|$ there is an isoperimetric region $E \subseteq \overline{\Omega}$ with $|E| = v$. The boundary $\Sigma = \partial E$ satisfies:*

(i) *There is a closed singular set $\Sigma_s \subset \Sigma_0 = \Sigma \cap \Omega$ of Hausdorff dimension less than or equal to $n-7$ such that $\Sigma_r = \Sigma_0 - \Sigma_s$ is a smooth embedded hypersurface.*
(ii) *Σ_r has constant mean curvature $H_0 \geqslant 0$ with respect to the inner normal.*
(iii) *Σ is $C^{1,1}$ in some neighborhood of each point in $\Sigma_1 = \Sigma \cap \partial\Omega$. Moreover, the mean curvature H of Σ is defined $\mathcal{H}^n$-almost everywhere and we have $H \leqslant H_0$.*

The above theorem follows from general regularity results in [**GMT**] and a suitable application of a result by Brézis and Kinderlehrer [**BK**] together with the formula for the first derivative of perimeter, see [**SZi**, Theorem 3.6]. By using the formula for the second derivative of perimeter for constant mean curvature hypersurfaces we obtain more information about the boundary of an isoperimetric region, namely, the stability of Σ_r.

A constant mean curvature hypersurface $M \subset \mathbb{R}^{n+1}$ is said to be *stable* if the second derivative of perimeter for variations preserving volume is nonnegative. This is analytically equivalent to the following inequality for the *index form* Q:

$$Q(u,u) = -\int_M u\,\{\Delta u + |\sigma|^2\,u\}\,dM = \int_M \{|\nabla u|^2 - |\sigma|^2\,u^2\}\,dM \geqslant 0, \tag{2.1}$$

for any smooth function u with mean zero and compact support in M ([**BdC**]). In the last formula, Δ and ∇ denote the relative Laplacian and the relative gradient on M, respectively ([**Ch2**, pp. 2–3]), dM is the Riemannian measure on M ([**Ch2**, p. 5]), and $|\sigma|^2$ is the squared sum of the principal curvatures $k_1, \ldots, k_n$. Inequality (2.1) is also valid for mean zero functions with compact support in the Sobolev space $H^1(M)$ (of functions in $L^2(M)$ whose gradient is squared integrable).

The stability condition implies that the eigenvalues of the twisted Dirichlet problem for the *Jacobi operator* $\Delta + |\sigma|^2$ on M are nonnegative. A reference for basic properties of twisted eigenvalues is [**BB**].

For the proof of Theorem 1.1 we need an instability result. Let $M \subset \mathbb{R}^{n+1}$ be a constant mean curvature hypersurface. A *Jacobi function* on M is a nontrivial smooth function u such that $\Delta u + |\sigma|^2\, u = 0$. Each component of $M - u^{-1}(0)$ is called a *nodal region* for u. If U is a nodal region compactly included in M ($\partial U \subset \operatorname{int}(M)$), then the first eigenvalue for the (non twisted) Dirichlet problem for the Jacobi operator in U is zero and the associated eigenfunction is simple and signed [**BB**].

In the above situation we have:

Lemma 2.2. *If there are two different nodal regions U_1, U_2 compactly included in M, then M is unstable.*

Proof. Consider signed functions u_i in $C^\infty(U_i)$ such that $u_i = 0$ in ∂U_i, $\int_{U_1} u_1 = -\int_{U_2} u_2$ and $\Delta u_i + |\sigma|^2\, u_i = 0$. Extending by zero, we can see u_i as a function in $H^1(M)$. It is clear that $u = u_1 + u_2$ is a $H^1(M)$-function with mean zero and compact support contained in $\overline{U}$, where $U = U_1 \cup U_2$. Inserting u in the index form (2.1) we obtain $Q(u,u) = 0$. This implies that the first eigenvalue $\lambda_1(U)$ of the twisted Dirichlet problem for the Jacobi operator in U is nonpositive. By using the monotonicity property of twisted eigenvalues we conclude that $\lambda_1(M) < 0$. Thus, M is unstable. □

Finally, we review some facts about hypersurfaces of revolution with constant mean curvature in $\mathbb{R}^{n+1}$, known as Delaunay hypersurfaces, see [**HMRR**, Lemma 4.2, Proposition 4.3] and the references therein.

Let $S \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and assume that S is invariant under the action of the group $O(n)$ of isometries of $\mathbb{R}^{n+1}$ fixing the x_1-axis. The hypersurface S is generated by a curve γ contained in the $xy\,(= x_1x_2)$-plane. We parameterize the curve $\gamma(s) = (x(s), y(s))$ by arc-length. Let σ be the angle between the tangent to γ and the positive x_1-direction. We shall consider the normal to S given by $N = (\sin\sigma, -\cos\sigma)$. The Gauss-Kronecker and the mean curvature on S are given by:

$$GK = -\,\frac{y''\,(\cos\sigma)^{n-2}}{y^{n-1}}, \quad H = \frac{1}{n}\left\{-\sigma' + (n-1)\,\frac{\cos\sigma}{y}\right\}. \tag{2.2}$$

In particular, we have:

Lemma 2.3. *The generating curve γ of an $O(n)$-invariant hypersurface $S \subset \mathbb{R}^{n+1}$ with mean curvature H with respect to the normal vector $N = (\sin\sigma, -\cos\sigma)$ satisfies the following system of ordinary differential equations:*

$$\begin{cases} x' = \cos\sigma, \\ y' = \sin\sigma, \\ \sigma' = -nH + (n-1)\,\dfrac{\cos\sigma}{y}. \end{cases} \tag{2.3}$$

Moreover, if H is constant then the function

$$y^{n-1}\cos\sigma - Hy^n \tag{2.4}$$

is constant over any solution of (2.3).

The function (2.4) is usually called a *first integral* of the system (2.3). Existence of the first integral is standard in the context of the Calculus of Variations. The constant value T of (2.4) is the *force* of the curve γ.

From a straightforward analysis of (2.3) using the first integral (2.4) we can obtain the following known properties.

Proposition 2.4. *Any local solution of the system* (2.3) *is a part of a complete solution γ, which generates a hypersurface S with constant mean curvature of several types* (*see Figure 2*).

(i) *If $TH > 0$ then $\cos\sigma > 0$ and γ is a periodic graph over the x_1-axis. It generates a periodic embedded unduloid, or a cylinder.*
(ii) *If $TH < 0$ then γ is a locally convex curve and S is a nodoid, which has selfintersections. The normal vector to S rotates monotonically.*
(iii) *If $T = 0$ and $H \neq 0$ then S is a sphere.*
(iv) *If $H = 0$ and $T \neq 0$ we obtain a catenary which generates an embedded catenoid S.*
(v) *If $H = 0$ and $T = 0$ then γ is a straight line orthogonal to the x_1-axis which generates a hyperplane.*

When speaking about unduloids or nodoids we identify the curves and the corresponding generated hypersurfaces.

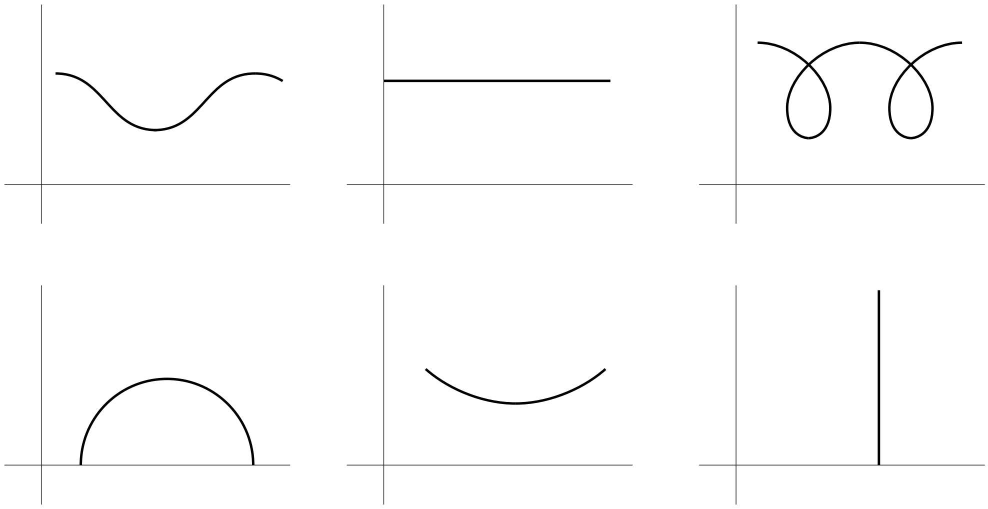

FIGURE 2. Generating curves of Delaunay hypersurfaces: unduloid, cylinder, nodoid, sphere, catenoid and hyperplane.

3. Proof of the main result

The proof of Theorem 1.1 will be broken in several previous results. We begin by introducing some notation.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a strictly convex body which is also rotationally symmetric about some line R. We shall identify R with the x_1-axis. Hence, Ω is invariant under the action of the group $O(n)$ of isometries of $\mathbb{R}^{n+1}$ fixing the x_1-axis. We denote by Γ the generating curve of $\partial\Omega$ in $\{x_1x_2 : x_2 \geqslant 0\}$. Clearly, Γ is the graph of a continuous function $G : [a, b] \to \mathbb{R}$ defined on a x_1-interval. The function G is C^2 on (a, b) and satisfies:

$$\begin{cases} G(x) > 0, & x \in (a,b), \quad G(a) = G(b) = 0, \\ G''(x) < 0, & x \in (a,b), \quad \text{that is } G \text{ is a strictly concave function.} \end{cases} \tag{3.1}$$

Consider an isoperimetric region E in Ω with boundary $\Sigma = \partial E$. The convexity of Ω allow us to apply Steiner symmetrization so that the symmetrized set $\mathcal{S}(E)$ still lies in Ω. The set $\mathcal{S}(E)$ consists in replacing the intersection between E and any hyperplane P orthogonal to R by the n-dimensional ball in P centered at $P \cap R$ and with the same $\mathcal{H}^n$ measure as $E \cap P$. We establish the following:

> (*) E can be assumed to be rotationally symmetric about the x_1-axis and such that its intersection with any straight line orthogonal to the x_1-axis is an interval.

The above statement follows from basic properties of Steiner symmetrization; a complete treatment of this topic can be found in [**T**, Section 3.8].

As a first consequence of (*) we prove:

Lemma 3.1. *$\Sigma_0 = \Sigma \cap \Omega$ is a smooth embedded hypersurface.*

Proof. Suppose that there is a singular point $p \in \Sigma_0 - R$; in this case, the set $O \subset \Sigma_0$ resulting under the action of $O(n)$ over $\{p\}$ would have Hausdorff dimension equal to $n-1$, giving us a contradiction by Theorem 2.1 (i). This proves the regularity of $\Sigma_0 - R$. Moreover, $\Sigma_0 - R$ is a constant mean curvature hypersurface (Theorem 2.1 (ii)) and, therefore, each of its components must be an open piece of one of the Delaunay hypersurfaces introduced in Proposition 2.4. The $C^{1,1}$ regularity of Σ in the points of $\Sigma \cap \partial\Omega$ (Theorem 2.1 (iii)) implies that, if one of these pieces meet the axis, then it must be part of a sphere.

Now suppose that there is a singular point $p \in \Sigma_0 \cap R$. By the above discussion, Σ_0 must contain open pieces of two different n-spheres $\mathbb{S}_i$ centered on R, with the same radius r_0, and which only meet at p. For the following reasoning we can admit that $r_0 = 1$. For any $r \in (0,1), r \approx 1$, let $\mathbb{S}_i(r) \subset \mathbb{S}_i$ be the spherical cap centered at p and such that $\operatorname{dist}(p, \partial\mathbb{S}_i(r)) = 1 - r$. Denote $\mathbb{S}(r) = \mathbb{S}_1(r) \cup \mathbb{S}_2(r)$. Let also $C(r)$ be the compact cylinder with boundary $\partial C(r) = \partial\mathbb{S}(r)$. A standard calculation using the co-area formula (see [**Ch2**, p. 86]) and a suitable change of variables show that:

$$\mathcal{H}^n[\mathbb{S}(r)] - \mathcal{H}^n[C(r)] = 2\, c_{n-1} \left[\int_\theta^{\pi/2} \cos^{n-1} u\, du - (1 - \sin\theta)\, \cos^{n-1}\theta \right],$$

where c_{n-1} is the $\mathcal{H}^{n-1}$ measure of the unit sphere of dimension $n-1$, and $\theta \in (0, \pi/2)$ is given by $\sin\theta = r$.

Define $\psi(\theta) = 1/(2c_{n-1})\,[\mathcal{H}^n[\mathbb{S}(r)] - \mathcal{H}^n[C(r)]]$ and $\phi(\theta) = \int_\theta^{\pi/2} \cos^{n-1} u\, du$, for $\theta \in (0, \pi/2)$. Using that ϕ is positive and $\lim_{\theta \to \pi/2} (\psi(\theta)/\phi(\theta)) = 1$, we deduce the existence of a value r, close enough to 1, for which $\mathcal{H}^n[\mathbb{S}(r)] > \mathcal{H}^n[C(r)]$. This inequality implies that $P(E) > P(E')$, where E' is the set resulting when we replace in E the set $\mathbb{S}(r)$ by $C(r)$. As $|E'| > |E|$ we conclude, after a contraction, that E cannot be an isoperimetric region, a contradiction. ◻

We summarize and show new properties of minimizers in the following proposition:

Proposition 3.2. *Let E be an isoperimetric region in Ω with boundary Σ. Then:*

(i) *E is rotationally symmetric about the axis R of Ω.*
(ii) *The intersection between E and any straight line orthogonal to R is an interval.*
(iii) *$\Sigma_0 = \Sigma \cap \Omega$ is a smooth embedded hypersurface. Moreover, each component of Σ_0 is an open piece of a sphere, an unduloid or a nodoid.*
(iv) *Σ is a $C^{1,1}$ hypersurface of revolution generated by as many connected graphs as components of E. Each one of these graphs touches the x_1-axis orthogonally.*

Proof. As indicated before, (i) and (ii) follow from basic properties of Steiner symmetrization. Due to the concavity of the function G in (3.1) and the $C^{1,1}$ regularity of Σ in the points of $\Sigma \cap \partial\Omega$ (Theorem 2.1 (iii)), we deduce that vertical and horizontal lines, and catenaries contained in Ω cannot meet Γ in a C^1 way, so statement (iii) is proved. Now we prove (iv). Note that Σ is a compact $C^{1,1}$ hypersurface of revolution and, therefore, it must have finitely many components. Each one of these components is generated by a finite collection of connected curves.

Let F be a component of E. For any given plane π containing the x_1-axis we have, by (ii), that if $x \in \pi \cap F$ and $\overline{x} \in \pi$ is the reflection of x with respect to the axis, then the segment $[x, \overline{x}]$ is entirely contained in F. This proves that ∂F is connected. The remaining properties follows easily from the above argument and the C^1 regularity of Σ. $\square$

Assertion (ii) in the above proposition has allowed us to obtain topological information about Σ, as we have shown in (iv). Another topological consequence of Steiner symmetrization is the following result which states the connectedness of a minimizer.

Proposition 3.3. *Any isoperimetric region E in Ω is connected.*

Proof. Note that the compactness and the regularity of Σ imply that there are only a finite number of components of E. Assume that F_1 and F_2 are two different components of E. By Proposition 3.2 (iv), each F_i is generated by a graph over a closed x_1-interval J_i contained in the segment $[a, b]$ on which the function G in (3.1) is defined. These graphs touch the axis in the extreme points of J_i and enclose portions of E; thus, the intervals J_i are disjoint. We suppose that there are no components of E with generating graph over the piece of axis between J_1 and J_2. Let $x_0 \in (a, b)$ be the unique point on which the function G reaches its maximum.

If $x_0 \in J_1$, we can move F_2 towards F_1 along the axis, without touching any other component of E, until we produce a first contact. Let F_2' the resulting set. The set defined as the union of F_2' with $F_1 \cup [E - (F_1 \cup F_2)]$ is a new minimizer in Ω with non embedded boundary, which gives us a contradiction.

If $x_0 \notin J_i,\ i = 1, 2$, then one, or both sets F_i, can be moved along the axis until the two sets touch, and we conclude in the same way. $\square$

To prove the convexity of a connected minimizer E we need to show that the Gauss-Kronecker curvature on Σ_0 is nonnegative. We know that Σ_0 is a union of open pieces of spheres, unduloids or nodoids. At first, Σ_0 could contain an unduloid piece $S \subseteq \Sigma_0$ with points of negative GK curvature, see Figure 1. We cannot use symmetrization to discard this situation unless Ω is symmetric with respect to a hyperplane orthogonal to the axis. The non existence of S will be established by invoking the instability result proved in Section 2.

Lemma 3.4. *There are no pieces of unduloids in Σ_0 containing points of negative Gauss-Kronecker curvature.*

Proof. Suppose that $S \subseteq \Sigma_0$ is an open piece of unduloid with negative curved points. The generating curve of S is the graph of a positive function $g : (c, d) \subseteq [a, b] \to \mathbb{R}$ which extends C^∞ to the whole x_1-axis (Proposition 2.4 (i)). The inclusion $\Sigma_0 \subseteq \Omega$ implies $g < G$ in (c, d). Moreover, the C^1 regularity of Σ gives $g(x) = G(x)$ and $g'(x) = G'(x)$, for $x = c, d$. It follows that $g''(x) \leqslant G''(x) < 0$ for $x = c, d$. Thus, by equation (2.2) we have proved that the contact points between ∂S and $\partial\Omega$ have positive GK curvature.

On the other hand, let x_0 be the unique point in (a, b) on which G reaches its maximum. G is strictly monotone on the intervals (a, x_0) and (x_0, b). An easy reasoning by contradiction using the existence of negative curved points in S and the periodicity of the unduloid, allow us to conclude that $c < x_0 < d$. Hence, $g'(c) = G'(c) > 0$ and $g'(d) = G'(d) < 0$.

The above properties and the behaviour of unduloids (see Figure 2) imply that S contains the closed piece of unduloid between two consecutive maxima of g. Denote by X the vector field $\frac{\partial}{\partial x_1}$ and by N the inner normal to S. Then, $u = \langle X, N \rangle$ is a Jacobi function on Σ_0 ([**BdC**]) with two different nodal regions compactly included in S (each one of these regions is given by the set between a maximum and the consecutive minimum). By Lemma 2.2 we conclude that Σ_0 is unstable, the desired contradiction. $\square$

Proof of Theorem 1.1. Let E be an isoperimetric region in Ω with boundary $\Sigma = \partial E$. By Propositions 3.3 and 3.2, E is connected and Σ is a connected $C^{1,1}$ hypersurface consisting in closed subsets of $\partial\Omega$ and open embedded pieces of spheres, unduloids or nodoids. Nodoids are locally convex by Proposition 2.4 (ii), and spheres have positive GK curvature. By Lemma 3.4 each piece of unduloid contained in Σ has nonnegative curvature. This proves that E is locally convex in the points of Σ_0. Since Ω is convex, E is locally convex in $\Sigma \cap \partial\Omega$ too. We conclude by [**M**, Theorem 1.3] that E is convex. $\square$

Remark 3.5. Steiner symmetrization does not require the C^2 smoothness or the strict convexity of Ω. Thus, every consequence obtained only by using regularity results for Σ_0 (Theorem 2.1 (i)) and symmetrization is also valid in the general case in which Ω is a rotationally symmetric convex domain. In particular, Proposition 3.2 is true up to the $C^{1,1}$ regularity of Σ, the orthogonality condition in Proposition 3.2 (iv), and the possible Delaunay hypersurfaces contained in Σ_0. Hence, Proposition 3.3 can be proved in a similar way to obtain the connectedness of any minimizer.

Remark 3.6. If Ω is not strictly convex but has C^2 boundary then Proposition 3.2 is valid. However, the proof of Lemma 3.4 does not hold in this case; in fact, it is possible to find stable pieces of unduloids with points of negative curvature contained in a C^∞ convex domain.

Denote by S_1 the closed piece of unduloid between a maximum and the consecutive minimum of its generating graph. Pedrosa and Ritoré [**PR**, Proposition 5.3] have shown that for $n \geqslant 9$ there exist *stable free boundary* pieces S_1 connecting two parallel hyperplanes in $\mathbb{R}^{n+1}$. Let S be the union of such a piece together with its reflection with respect to the hyperplane P orthogonal to the axis and containing the minimum. Note that S is the union of two nodal regions as in Lemma 3.4 but we cannot apply Lemma 2.2 to conclude that S is unstable because these regions are not compactly included in S.

To prove that S is stable let λ be a twisted Dirichlet eigenvalue for the Jacobi operator in S and u an associated eigenfunction. By using the symmetry of S with respect to P we can decompose u as the sum of a symmetric and an antisymmetric function (with respect to P). For this reason, λ is an eigenvalue for a (non twisted) Dirichlet problem or for a Neumann-Dirichlet mixed problem in S_1. In the first case $\lambda \geqslant 0$ since S_1 is a nodal region for the Jacobi function given by $u = \big\langle \frac{\partial}{\partial x_1}, N \big\rangle$, where N is the inner normal to S_1. In the second one, the mean zero condition for u and the known inequality between mixed and Neumann eigenvalues ([**Ch2**]) show that $\lambda \geqslant \lambda_k^N$ for some $k \geqslant 2$, where $\{\lambda_i^N\}$ is the collection of Neumann eigenvalues for the Jacobi operator in S_1. The free boundary stability of S_1 implies that $\lambda_k^N \geqslant 0$ and so $\lambda \geqslant 0$.

Finally, to find a convex set Ω containing S it suffices to consider the cylinder in the convex hull of S and attach to it two topological n-balls in a C^∞ way.

Remark 3.7. Let Ω be a rotationally symmetric convex domain. We can approximate the generating function G of $\partial\Omega$ by a sequence $\{G_n\}$ of C^2 strictly concave functions whose graphs are contained in a ball and lie over the graph of G. Thus, we obtain a sequence Ω_n of domains containing Ω in the hypothesis of Theorem 1.1. Denote by v_0 the volume of a largest ball in Ω. For given volume $v_0 < v < |\Omega|$ let E_n be a convex minimizer in Ω_n with $|E_n| = v$. We can suppose that the sequence E_n converges as in [**SZi**, Theorem 3.31] to a set $E \subseteq \overline{\Omega}$. Since the above convergence preserves convexity and volume we have that E is convex, $|E| = v$ and, moreover:

$$P(E) \leqslant \liminf P(E_n).$$

Finally, if E' is a set in Ω with $|E'| = v$, then the inclusion $\Omega \subseteq \Omega_n$ implies that $P(E_n) \leqslant P(E')$. Passing to the limit and using the last equation, we obtain $P(E) \leqslant P(E')$. We have just proved the existence of a convex isoperimetric region in Ω.

The above argument does not prove that any minimizer in Ω is convex unless there is a unique minimizer for the corresponding volume. Uniqueness results assuming that Ω satisfies a great circle condition or that Ω is contained in $\mathbb{R}^2$ are proved in [**SZi**, Theorems 3.31 and 3.32].

4. An example of non nested minimizers

In [**SZi**, Theorem 3.31] it is proved that, if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded convex domain that satisfies a great circle condition, then isoperimetric regions in Ω for large volume are unique and nested. In precise terms we have $E_1 \subset E_2$ whenever $|B| \leqslant |E_1| < |E_2|$, B the union of all largest balls in Ω. As indicated by examples in [**GMT2**] we cannot expect nestedness when Ω is non convex. In this section we construct an example which shows that nestedness of minimizers does not hold even if Ω is convex.

First of all note that we can approximate a solution with very large height of the Delaunay equation (2.3) by an appropriate circle. Let $\gamma = \gamma_{H,R} = (x, y, \sigma)$ be the solution of the system (2.3) for $H > 0$ constant, with initial conditions given by $\gamma(0) = (0, R, 0)$. Consider the system (2.3)* resulting by suppressing the non linear term in the third equation of (2.3). The solution $\tau = \tau_{H,R}$ of (2.3)* with $\tau(0) = (0, R, 0)$ is a circle of radius $1/(nH)$. By using the continuous dependence of the solutions of a differential equation with respect to small variations of the equation ([**RM**, Théorème 6.2 and Remarque 6.3]) we obtain:

(**) Let δ and ε be positive. Then there is a number $R_0 = R_0(\delta, \varepsilon) > 0$ such that:

$$|\gamma_{H,R}(s) - \tau_{H,R}(s)| \leqslant \varepsilon, \qquad H > 0,\ \ R \geqslant R_0,\ \ |s| \leqslant \delta.$$

Now, we will describe an example of non nested minimizers. Take a number $R_0 > 0$ corresponding to a large δ and a sufficiently small ε in (**). Choose an isosceles triangle T in the xy-plane with base on the interval $[-1, 1]$ and very large height so that the angles between the equal sides and the base are close enough to $\pi/2$ and $-\pi/2$ respectively, and the x-coordinate of a point $(x, y) \in \partial T$ with $0 < y \leqslant R_0 + 1$ is close enough to 1 or -1. Call $\Omega \subset \mathbb{R}^3$ to the convex set obtained by rotating T around the x-axis. It is clear that the largest ball $B \subset \Omega$ has radius $r < 1$ and does not have a great circle contained in $\partial\Omega$.

Select an isoperimetric region E in Ω with volume close enough to $2\pi R_0^2$. By the results in Section 3, E is convex and each component of $\Sigma_0 = \partial E \cap \Omega$ is an open piece of a sphere, a nodoid or an ($GK \geqslant 0$) unduloid with constant mean curvature $H_0 > 0$. The mean curvature of ∂E in a neighborhood of the $C^{1,1}$ points satisfies $H \leqslant H_0$ by Theorem 2.1 (iii). As $H(x, y) \to +\infty$ when $(x, y) \in \partial T$ and $y \to 0$, we deduce that ∂E cannot contain the lower vertices of T. In fact, an approximation argument such as that in Remark 3.7 tell us that there exist a minimizer E with $C^{1,1}$ boundary. By using Steiner symmetrization we also obtain that E is symmetric with respect to the yz-plane.

By the above properties, the generating curve of Σ_0 in $\{xy : y \geqslant 0\}$ consists in an upper symmetric cap of an unduloid or a nodoid -we can parameterize it as the curve $\gamma_{H_0,R}$-, and two symmetric pieces of circles with the same radius r_0 which meet the x-axis orthogonally. Denote by m the height of the points in $\gamma_{H_0,R} \cap \partial T$. As the volume enclosed by E is very close to $2\pi R_0^2$, it is clear that $R > R_0$ and we can also assume that $m < R_0 + 1$ (in the other case we immediately obtain that $B \not\subset E$). Hence, by the approximation (**) and the construction of Ω we conclude that the radius of the circle $\tau_{H_0,R}$ is very close to 1, that is, $1/(2H_0) \approx 1$. In particular, we can suppose that $H_0 \leqslant 1$, which gives $r_0 \geqslant 1 > r$. This last inequality easily implies that $B \not\subset E$ and the example is finished.

Remark 4.1. The approximation argument in Remark 3.7 allow us to show an example of a strictly convex body with non nested minimizers.

References

[**BB**] J. L. Barbosa and P. Bérard, *Eigenvalue and "twisted" eigenvalue problems, applications to CMC surfaces*, J. Math. Pures Appl. **79** (2000), no. 5, 427–450.

[**BdC**] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z., **185** (1984), 339–353.

[**BK**] H. Brézis and D. Kinderlehrer, *The smoothness of solutions to nonlinear variational inequalities*, Indiana Univ. Math. J., **23** (1974), 831–844.

[**Ch**] I. Chavel, *Isoperimetric Inequalities. Differential Geometric and Analytic Perspectives*, Cambridge Tracts in Mathematics, no. 145, Cambridge University Press, Cambridge, 2001.

[**Ch2**] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.

[**G**] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, no. 80, Birkhäuser Verlag, Basel-Boston, 1984.

[**GMT**] E. Gonzalez, U. Massari and I. Tamanini, *On the regularity of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J., **32** (1983), 25–37.

[**GMT2**] E. Gonzalez, U. Massari and I. Tamanini, *Minimal boundaries enclosing a given volume*, Manuscripta Math., **34** (1981), 381–395.

[**Gr**] M. Grüter, *Boundary regularity for solutions of a partitioning problem*, Arch. Rat. Mech. Anal., **97** (1987), 261–270.

[**HMRR**] M. Hutchings, F. Morgan, M. Ritoré and A. Ros, *Proof of the double bubble conjecture*, Ann. Math., **155** (2002), no. 2, 459–489.

[**M**] P. Mani–Levitska, *Characterizations of convex sets*, Handbook of convex geometry, Volume A, Elsevier Science Publishers B.V., Amsterdam, 1993, 19–41.

[**PR**] Renato H. L. Pedrosa and Manuel Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J., **48** (1999), no. 4, 1357–1394.

[**RV**] A. Ros and E. Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geom. Dedicata, **56** (1995), no. 1, 19–33.

[**RM**] N. Rouche and J. Mawhin, *Équations différentielles ordinaires. Tome 1: Théorie générale*, Masson et Cie, Éditeurs, Paris, 1973.

[**SZ**] P. Sternberg and K. Zumbrun, *On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint*, Comm. Anal. Geom, **7** (1999), no. 1, 199–220.

[**SZi**] E. Stredulinsky and William. P. Ziemer, *Area minimizing sets subject to a volume constraint in a convex set*, J. Geom. Anal., **7** (1997), no. 4, 653–677.

[**T**] G. Talenti, *The Standard Isoperimetric Theorem*, Handbook of convex geometry, Volume A, Elsevier Science Publishers B.V., Amsterdam, 1993, 73–123.

Granada, October 1, 2002.

Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, España

E-mail address: crosales@ugr.es