AREA-STATIONARY AND STABLE SURFACES IN THE SUB-RIEMANNIAN HEISENBERG GROUP $\mathbb{H}^1$

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Dedicated to Prof. Manfredo P. do Carmo on the occasion of his 80th birthday

One of the most celebrated papers by Manfredo is his joint work with Peng [19] on the classification of complete orientable stable minimal surfaces in $\mathbb{R}^3$. The authors, through a clever use of the second variation formula of the area together with some information on the conformal geometry of the surface, provided a beautiful and straightforward proof of the following

Theorem (do Carmo and Peng [19, Thm. 1.2]). *Let $x : M \to \mathbb{R}^3$ be a stable complete minimal immersion of a two-dimensional, orientable, connected manifold M. Then $x(M) \subset \mathbb{R}^3$ is a plane.*

The same result was independently proved by Fischer-Colbrie and Schoen as a particular case of their work on stable minimal surfaces in 3-manifolds with nonnegative scalar curvature [21]. A third proof of the result by do Carmo and Peng was given by Pogorelov [39]. Later on, another one, using harmonic vector fields, was given by Palmer [36]. Previous results on stable minimal surfaces had been obtained by do Carmo and Barbosa [3] and by do Carmo and Silveira [18].

Complete orientable stable minimal surfaces are a generalization of entire minimal graphs. These are stable by a simple variational argument involving a suitable coordinate of the unit normal vector field. In this way, the result by do Carmo and Peng can be considered an extension of the well-known Bernstein's Theorem

Theorem (Bernstein's Theorem [8]). *The only entire minimal graphs in $\mathbb{R}^3$ are planes.*

The second variation formula of the area was later used by Barbosa and do Carmo [4] and by Barbosa, do Carmo and Eschenburg [5] to characterize the geodesic spheres as the only compact orientable stable constant mean curvature surfaces in simply-connected space-forms.

All these results popularized a classical tool in Calculus of Variations, the second variation formula, for the treatment of variational problems related to the area functional in Riemannian Geometry.

In this paper we will describe recent development concerning the variational theory of the sub-Riemannian area functional in the Heisenberg group $\mathbb{H}^1$. The theory is quite different from the Riemannian one, and unexpected phenomena arise. However, we have the same problem of finding the area-minimizing surfaces and the same basic strategy of characterizing the critical points of the area functional and computing the second variation for such

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critical points. We have organized the material into two sections: in the following we gather some basic results on the geometry of the Heisenberg group $\mathbb{H}^1$, variational formulas for the sub-Riemannian area, and geometric properties and characterization of area-stationary surfaces. Most of this material can be found in [42]. In the final section, we use the second variation formula of the area to characterize the stable area-stationary surfaces in $\mathbb{H}^1$. This is the main result in [31].

1. Area-stationary surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$

Indispensable references to understand the geometry and analytical properties of the sub-Riemannian Heisenberg group $\mathbb{H}^1$ include Folland and Stein [22], Gromov [27], [26], Montgomery [32], and Capogna, Danielli, Pauls and Tyson [9]. The study of minimal surfaces in sub-Riemannian geometry was initiated by Garofalo and Nhieu [24]. The theory of finite perimeter sets in $\mathbb{H}^n$ has been developed in [23].

In these notes, all the considered surfaces will be of class C^2. While this is a reasonable hypothesis from the point of view of differential geometry, it is known that there exists critical points of the sub-Riemannian area, even global minimizers, with less regularity [12], [40].

1.1. **The geometry of $\mathbb{H}^1$.** The *Heisenberg group* $\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3,*)$, where the product $*$ is defined, for any pair of points $[z,t]$, $[z',t']\in\mathbb{R}^3\equiv\mathbb{C}\times\mathbb{R}$, by

$$[z,t]*[z',t']:=[z+z',t+t'+\operatorname{Im}(z\overline{z}')],\qquad (z=x+iy).$$

For $p\in\mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q)=p*q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X:=\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial t},\qquad Y:=\frac{\partial}{\partial y}-x\,\frac{\partial}{\partial t},\qquad T:=\frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar distribution generated by X and Y. The *horizontal projection* of a tangent vector U onto $\mathcal{H}$ will be denoted by U_h. A vector field U is *horizontal* if $U=U_h$.

We denote by $[U,V]$ the Lie bracket of two C^1 vector fields U and V on $\mathbb{H}^1$. Note that $[X,T]=[Y,T]=0$, while $[X,Y]=-2T$, so that $\mathcal{H}$ is a bracket-generating distribution. Moreover, by Frobenius theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega:=-y\,dx+x\,dy+dt$.

We shall consider on $\mathbb{H}^1$ the Riemannian metric $g=\langle\cdot,\cdot\rangle$ so that $\{X,Y,T\}$ is an orthonormal basis at every point, and denote by D the Levi-Civita connection in $(\mathbb{H}^1,g)$. The restriction of g to $\mathcal{H}$ coincides with the usual sub-Riemannian metric in $\mathbb{H}^1$. A 90-degree rotation J is defined on every plane $\mathcal{H}_p$ by taking $J(X_p)=Y_p$, $J(Y_p)=-X_p$.

1.2. **Carnot-Carathéodory distance and geodesics.** Let $\gamma:I\to\mathbb{H}^1$ be a piecewise C^1 curve defined on a compact interval $I\subset\mathbb{R}$. The *length* of γ is the usual Riemannian length $L(\gamma):=\int_I|\dot\gamma(\varepsilon)|\,d\varepsilon$, where $\dot\gamma$ is the tangent vector of γ. A *horizontal curve* γ in $\mathbb{H}^1$ is a C^1 curve whose tangent vector always lies in the horizontal distribution. For two given points in $\mathbb{H}^1$ we can find, by Chow's connectivity theorem [26, Sect. 1.2.B], a horizontal curve joining these points. The *Carnot-Carathéodory distance* d_{cc} between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points. The topology associated to d_{cc} coincides with the usual topology in $\mathbb{R}^3$, see [7, Cor. 2.6]. Two given points can be joined by a, non-necessarily unique, sub-Riemannian geodesic $\gamma : I \to \mathbb{H}^1$, which is a C^∞ curve and satisfies the equation

$$D_{\dot\gamma}\dot\gamma + 2\lambda J(\dot\gamma) = 0,$$

for some $\lambda \in \mathbb{R}$ called the *curvature* of the geodesic [33].

1.3. **Surfaces in $\mathbb{H}^1$.** Let Σ be a C^1 surface immersed in $\mathbb{H}^1$. The *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent plane $T_p\Sigma$ coincides with $\mathcal{H}_p$. As Σ_0 is closed and has empty interior in Σ, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. It was proved in [17, Lem. 1], see also [2, Thm. 1.2], that, for a C^2 surface, the Hausdorff dimension of Σ_0 with respect to the Riemannian distance on $\mathbb{H}^1$ is less than or equal to one. In particular, the Riemannian area of Σ_0 vanishes. If N is a unit vector normal to Σ in $(\mathbb{H}^1, g)$, then we can describe the singular set as $\Sigma_0 = \{p \in \Sigma : N_h(p) = 0\}$, where $N_h = N - \langle N, T\rangle T$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal Gauss map* ν_h and the *characteristic vector field* Z, by

$$\nu_h := \frac{N_h}{|N_h|}, \qquad Z = J(\nu_h). \tag{1.1}$$

As Z is horizontal and orthogonal to ν_h, we conclude that Z is tangent to Σ. Hence Z_p generates $T_p\Sigma \cap \mathcal{H}_p$. The integral curves of Z in $\Sigma - \Sigma_0$ will be called (*oriented*) *characteristic curves* of Σ. They are both tangent to Σ and horizontal. If we define

$$S := \langle N, T\rangle\, \nu_h - |N_h|\, T, \tag{1.2}$$

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$.

1.4. **Sub-Riemannian area.** Given a C^1 immersed surface Σ with a unit normal vector N, we define the (sub-Riemannian) *area* of Σ by

$$A(\Sigma) := \int_\Sigma |N_h|\, d\Sigma, \tag{1.3}$$

where $d\Sigma$ is the Riemannian area element on Σ. If Σ is a C^2 surface bounding a set Ω, then $A(\Sigma)$ coincides with all the notions of perimeter of Ω and area of Σ introduced by different authors, see [23, Prop. 2.14], [35, Thm. 5.1] and [23, Cor. 7.7].

1.5. **Mean curvature. The first variation of the sub-Riemannian area.** For a C^2 immersed surface Σ with a unit normal vector N, we denote by B the Riemannian shape operator of Σ with respect to N. It is defined for any vector W tangent to Σ by $B(W) = -D_W N$. The Riemannian mean curvature of Σ is $-2H_R = \operatorname{div}_\Sigma N$, where $\operatorname{div}_\Sigma$ denotes the Riemannian divergence relative to Σ.

Let Σ be a C^2 immersed surface in $\mathbb{H}^1$ with a unit normal vector N. We define the (sub-Riemannian) *mean curvature* of Σ as in [41] and [42], by the equality

$$-2H(p) = (\operatorname{div}_\Sigma \nu_h)(p), \qquad p \in \Sigma - \Sigma_0, \tag{1.4}$$

where ν_h is the horizontal Gauss map defined in (1.1). We say that Σ is a *minimal surface* if the mean curvature vanishes on $\Sigma - \Sigma_0$. This notion of mean curvature agrees with the ones introduced by other authors [13], [37], [11].

Like in the Riemannian case, the sub-Riemannian mean curvature appears in the computation of the first variation formula of the area (1.3), but we shall see immediately fundamental differences

Lemma 1.1 ([42, Lem. 4.3]). *Let $\Sigma \subset \mathbb{H}^1$ be an oriented C^2 immersed surface. Suppose that U is a C^2 vector field with compact support on Σ and normal component $u = \langle U, N\rangle$. Then the first derivative at $s = 0$ of the area functional $A(s)$ associated to U is given by*

$$A'(0) = \int_\Sigma u \left(\operatorname{div}_\Sigma \nu_h \right) d\Sigma - \int_\Sigma \operatorname{div}_\Sigma \left(u \, (\nu_h)^\top \right) d\Sigma, \tag{1.5}$$

provided $\operatorname{div}_\Sigma \nu_h \in L^1_{loc}(\Sigma)$.

Moreover, if Σ is area-stationary then

$$A'(0) = \int_\Sigma u \, (\operatorname{div}_\Sigma \nu_h) \, d\Sigma.$$

The second term in formula (1.5) may not vanish if the singular set Σ_0 is not empty. The structure of Σ_0 for C^2 surfaces with a certain condition on the sub-Riemannian mean curvature, including the surfaces with H constant, has been studied by Cheng, Hwang, Malchiodi and Yang [11, Sect. 3], who have proved that it consists on isolated points and singular curves, and that the characteristic curves cross the singular curves in a C^1 way. As a consequence of the first variation formula of the area (1.5) and the results in [11] we obtained

Proposition 1.2 ([42, Thm. 4.8, Thm. 4.17]). *Let $\Sigma \subset \mathbb{H}^1$ be an oriented C^2 immersed surface. Then Σ is area-stationary if and only if Σ is minimal and the characteristic curves meet orthogonally the singular curves.*

Moreover, the characteristic curves of Σ are horizontal Riemannian geodesics, i.e., they are horizontal straight lines.

Similar results also hold in other ambient spaces, see [28] and [30]. In higher dimensional Heisenberg groups, the Hausdorff codimension of the singular set is high enough to make the second term in (1.5) irrelevant. Hence area-stationary surfaces in $\mathbb{H}^n$, $n \geqslant 2$, are just minimal surfaces.

1.6. **Examples of area-stationary surfaces and classification results.** The first known examples of minimal surfaces in $\mathbb{H}^1$ were described in the family of t-graphs [37, § 4]. Given a function $u \in C^2(D)$, where $D \subset \mathbb{R}^2$, it is easy to check from (1.4) that the graph $t = u(x, y)$ is a minimal surface in $\mathbb{H}^1$ if and only if

$$(u_y + x)^2 u_{xx} - 2\,(u_y + x)\,(u_x - y)\,u_{xy} + (u_x - y)^2\,u_{yy} = 0, \tag{1.6}$$

which is a degenerate elliptic and hyperbolic PDE. The Plateau problem for t-graphs has been studied in [37], [11], [12], [38].

Interesting examples include the Euclidean planes, either the vertical ones with no singular points and the nonvertical ones with an isolated singular point, the hyperbolic paraboloid $t = xy$, the sub-Riemannian catenoids $t^2 = \lambda^2\,(x^2 + y^2 - \lambda^2)$, $\lambda \neq 0$ [37], [41], and the helicoids $\mathcal{H}_r$, $r > 0$, described in [42, Ex. 6.14] as the union of all the horizontal straight lines orthogonal to the sub-Riemannian geodesic in $\mathbb{H}^1$ obtained by the horizontal lift of the circle in the xy-plane of radius $1/r$ centered at the origin. We can parameterize $\mathcal{H}_r$ by

$$F_r(\varepsilon, s) = (s\,\sin(r\varepsilon), s\,\cos(r\varepsilon), \varepsilon/r). \tag{1.7}$$

The singular set of $\mathcal{H}_r$ consists of the helices $s = \pm 1/r$. Note that the family $\{\mathcal{H}_r\}_{r>0}$ is invariant under the sub-Riemannian dilations δ_λ defined by

$$\delta_\lambda(x, y, t) := (e^\lambda x, e^\lambda y, e^{2\lambda} t).$$

In fact $\delta_\lambda(\mathcal{H}_r) = \mathcal{H}_\mu$ with $\mu = e^{-\lambda}r$. The surfaces $\mathcal{H}_r$ coincide with the classical left-handed minimal helicoids in $\mathbb{R}^3$. In particular, they are embedded surfaces containing the vertical axis. We remark that the classical right-handed minimal helicoids in $\mathbb{R}^3$ are complete area-stationary surfaces in $\mathbb{H}^1$ with empty singular set.

By using these facts we were able to obtain the following, see [42, Thm. 6.15].

Proposition 1.3. *Let Σ be a C^2 complete, oriented, connected, area-stationary surface immersed $\mathbb{H}^1$ with singular set Σ_0.*

(i) *If Σ_0 contains an isolated point then Σ coincides with a Euclidean non-vertical plane.*
(ii) *If Σ_0 contains a singular curve then Σ is either congruent to the hyperbolic paraboloid $t = xy$ or to one of the helicoidal surfaces $\mathcal{H}_r$ defined above.*

There are many examples of complete area-stationary surfaces with empty singular set, including *intrinsic graphs* (Riemannian graphs over vertical planes), as those described by Barone, Serra-Cassano and Vittone [6]. See also the paper by Cheng and Hwang [10] for a classification of minimal surfaces in $\mathbb{H}^1$.

1.7. **The second variation of the sub-Riemannian area.** Second variation formulas of the area for particular surfaces and variations supported in the regular set have appeared in several papers. In [11], such a formula was obtained for C^3 surfaces inside a 3-dimensional pseudo-hermitian manifold. In [6], a second variation formula was proved for variations by intrinsic graphs of class C^2 in $\mathbb{H}^1$. In [13], it is computed the second derivative of the area associated to a C^2 variation by Euclidean straight lines of a C^2 surface without singular points in $\mathbb{H}^1$. We would like to stress that the variations we consider in Theorem 1.4 can move the singular set of the surface.

In the next theorem we compute the second derivative of the area functional for an arbitrary *normal variation by Riemannian geodesics* of a C^2 minimal surface in $\mathbb{H}^1$.

Theorem 1.4 ([31, Thm. 3.7])**.** *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 immersed minimal surface with singular set Σ_0. Consider the C^1 vector field $U = uN$, where N is a unit normal vector to Σ and $u \in C^1_0(\Sigma)$. Then, the second derivative of the area for the variation induced by U is given by*

$$A''(0) = \int_\Sigma |N_h|^{-1} \left\{ Z(u)^2 - \left(|B(Z)+S|^2 - 4|N_h|^2 \right) u^2 \right\} d\Sigma + \int_\Sigma \operatorname{div}_\Sigma(\xi Z)\, d\Sigma,$$

provided all the integrals above are finite. Here $\{Z, S\}$ is the orthonormal basis defined in (1.1) and (1.2), B is the Riemannian shape operator of Σ, and

$$\xi = \langle N, T\rangle \, (1 - \langle B(Z), S\rangle)\, u^2.$$

In particular, if $u \in C^1_0(\Sigma - \Sigma_0)$ then

$$A''(0) = \int_\Sigma |N_h|^{-1} \left\{ Z(u)^2 - \left(|B(Z)+S|^2 - 4|N_h|^2 \right) u^2 \right\} d\Sigma.$$

By integration by parts [31, § 3.3] we get

$$\mathcal{Q}(u) := A''(0) = - \int_\Sigma u\, \mathcal{L}(u)\, d\Sigma,$$

where $\mathcal{L}$ is the second order subelliptic operator on Σ given by

$$\begin{aligned}\mathcal{L}(u) := |N_h|^{-1} \big\{ Z(Z(u)) &+ 2\,|N_h|^{-1} \langle N, T\rangle \langle B(Z), S\rangle\, Z(u) \\ &+ (|B(Z)+S|^2 - 4|N_h|^2)\, u \big\}.\end{aligned}$$

As in the Euclidean case, we define a *stable area-stationary* surface in $\mathbb{H}^1$ as a C^2 area-stationary surface with non-negative second derivative of the area under compactly supported variations.

2. Stable surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$

Stable area-stationary surfaces in $\mathbb{H}^1$ have been considered in previous papers in connection with some Bernstein type problems. Let us describe some related works.

In [11], a classification of all the complete C^2 solutions to the minimal surface equation (1.6) for t-graphs in $\mathbb{H}^1$ is given. In [42], this classification was refined by showing that the only complete area-stationary t-graphs are Euclidean non-vertical planes or those congruent to the hyperbolic paraboloid $t = xy$. By means of a calibration argument it is also proved in [42] that they are all area-minimizing.

In [14] and [6] the Bernstein problem for *intrinsic graphs* in $\mathbb{H}^1$ was studied. A C^1 intrinsic graph has empty singular set. Examples of C^2 complete area-stationary intrinsic graphs different from vertical Euclidean planes were found in [25] and [14], and they were classified by Barone, Serra Cassano and Vittone in [6]. A remarkable difference with respect to the case of the t-graphs is the existence of complete C^2 area-stationary intrinsic graphs which are not area-minimizing, see [14]. The second variation formula of the area was computed in [6] to establish that the only complete stable C^2 intrinsic graphs are the Euclidean vertical planes. An interesting calibration argument, also given in [6], yields that the vertical planes are in fact area-minimizing surfaces in $\mathbb{H}^1$.

In the interesting paper [15], it is proven that C^2 complete stable area-stationary Euclidean graphs with empty singular set must be vertical planes. This is done by showing that such graphs contain a particular example of unstable surfaces called *strict graphical strips* unless the surface is a vertical plane. From the geometrical point of view, a graphical strip is a C^2 surface given by the union of a family of horizontal lines L_t passing through and filling a vertical segment so that the angle function of the horizontal projection of L_t is a monotone function. The graphical strip is strict if the angle function is strictly monotone. If the angle function is constant we have a piece of a vertical plane. We would like to remark that there are examples of complete area-stationary surfaces with empty singular set which do not contain a graphical strip, such as the sub-Riemannian catenoids $t^2 = \lambda^2\,(x^2+y^2-\lambda^2)$, $\lambda \neq 0$. Hence the main result in [15] does not apply to general surfaces.

The authors, in a joint work with Ana Hurtado, have proved the following

Theorem 2.1 ([31, Thm. 6.1]). *The only complete, orientable, connected, stable area-stationary surfaces in $\mathbb{H}^1$ of class C^2 are the Euclidean planes and the surfaces congruent to the hyperbolic paraboloid $t = xy$.*

After the distribution of the paper [31], we were informed of the related work [16] by Danielli, Garofalo, Nhieu and Pauls, where the authors prove that stable embedded minimal surfaces with empty singular set are vertical planes.

In particular, Theorem 2.1 provides the classification of all the complete C^2 orientable area-minimizing surfaces in $\mathbb{H}^1$. In the Heisenberg groups $\mathbb{H}^n$, with $n \geqslant 5$, there is no counterpart to Theorem 2.1, as some examples have been constructed in [6] of complete area-minimizing intrinsic graphs different from Euclidean vertical hyperplanes. For $n = 2, 3, 4$ it is still unknown if similar examples can be obtained. We would like to mention that examples of area-minimizing surfaces in $\mathbb{H}^1$ with low Euclidean regularity have been obtained in [38], [12], [40] and [34]. Hence our results are optimal in the class of C^2 area-stationary surfaces. Finally, the techniques used to prove Theorem 2.1 can be employed to prove classification results for complete stable area-stationary surfaces under a volume constraint in the first Heisenberg group [43], and inside the sub-Riemannian three-sphere [29].

The strategy of the proof is to use the second variation formula with suitable test functions. We first consider the case of empty singular set. So let $\Sigma \subset \mathbb{H}^1$ be a (complete orientable) stable area-stationary surface. Then $|N_h|$ does not vanish on Σ. Observe that the function $|N_h|$ is associated to the variational vector field induced by the surfaces equidistant to Σ in the Carnot-Carathéodory distance, see [1]. Hence, our construction of the test function v is somewhat similar to the Euclidean case, where the equivalent test function is $u \equiv 1$. Using Fischer-Colbrie's results [20], a stable minimal surface is conformally a compact Riemann surface minus a finite number of points, so that a logarithmic cut-off function v of $u \equiv 1$ has compact support and yields instability unless the surface is a plane. We remark that the function $|N_h|$ was already used as a test function in [6] and [15]. We first observe that, for any function f with compact support on Σ, we get

$$\mathcal{Q}(f|N_h|) = \int_\Sigma |N_h| \left\{ Z(f)^2 - \mathcal{L}(|N_h|)\, f^2 \right\} d\Sigma. \tag{2.1}$$

If Σ were compact then taking $f \equiv 1$ we would conclude from the following

Proposition 2.2 ([31, Prop. 4.6])**.** *Let $\Sigma \subset \mathbb{H}^1$ be a complete immersed C^2 area-stationary surface without singular points. Then $\mathcal{L}(|N_h|) \geqslant 0$. Moreover*

(1) $\mathcal{L}(|N_h|) = 0$ *if and only if* $\langle N, T \rangle = 0$, $\langle B(Z), S \rangle = 1$.
(2) $\mathcal{L}(|N_h|) \equiv 0$ *only on vertical planes.*

We prove that Σ is a vertical plane. Assume it is not, and consider a horizontal line L contained in Σ such that $\mathcal{L}(|N_h|) > 0$ on L. This line exists by Proposition 2.2. We parameterize a neighborhood in Σ of L by the map $F : I \times \mathbb{R} \to \Sigma$ given by

$$F(\varepsilon, s) := \Gamma(\varepsilon) + s\, Z_{\Gamma(\varepsilon)},$$

where $\Gamma : I \to \Sigma$ is a small portion of an integral curve of S passing through a given point of L. For fixed ε, the curve $s \mapsto F(\varepsilon, s)$ is the horizontal Riemannian geodesic with initial conditions $(\Gamma(\varepsilon), Z_{\Gamma(\varepsilon)})$. Then we get

$$d\Sigma := |V_\varepsilon|\, d\varepsilon\, ds,$$

where V_ε is the Riemannian Jacobi field

$$V_\varepsilon(s) := \frac{\partial F}{\partial \varepsilon}(\varepsilon, s).$$

Let $v := |\langle V, T \rangle|^{1/2}$. Then we get from (2.1)

$$\mathcal{Q}(uv^{-1}|N_h|) = \int_{I\times\mathbb{R}} \left(\frac{\partial u}{\partial s}\right)^2 d\varepsilon\, ds - \frac{3}{4}\int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u^2 d\varepsilon\, ds,$$

for any function u with compact support in Σ.

Take a non-negative C^∞ function $\phi : I \to \mathbb{R}$ with $\phi(0) > 0$ and compact support contained inside a bounded interval $I' \subseteq I$. For any $k \in \mathbb{N}$ define the function

$$u_k(\varepsilon, s) := \phi(\varepsilon)\, \phi(s/k).$$

Then we have

$$\begin{aligned} \limsup_{k\to\infty} \mathcal{Q}(u_k v^{-1}|N_h|) &= -\frac{3}{4} \liminf_{k\to\infty} \int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u_k^2\, d\varepsilon\, ds \\ &\leqslant -\frac{3}{4}\int_{I\times\mathbb{R}} \mathcal{L}(|N_h|)\, u^2\, d\varepsilon\, ds, \end{aligned}$$

which is strictly negative since $\mathcal{L}(|N_h|)$ is non-negative and strictly positive on L. This contradiction shows that Σ must be a vertical plane.

Vertical planes are stable area-stationary surfaces since they are area-minimizing. The proof follows by a calibration argument since the unit normal vector to a vertical plane is the restriction of a divergence-free left-invariant vector field on $\mathbb{H}^1$, see [6].

In case Σ has singular points then Σ must be either a Euclidean non-vertical plane, congruent to a hyperbolic paraboloid, or a helicoidal surface $\mathcal{H}_r$, $r > 0$. Both Euclidean non-vertical planes and surfaces congruent to the hyperbolic paraboloid are stable because they are area-minimizing. Since $\mathcal{H}_r = \delta_\lambda(\mathcal{H}_2)$, for $\lambda := \log(2/r)$, it is enough to show that $\mathcal{H}_2$ is unstable. The singular set of $\mathcal{H}_2$ is composed of two connected curves Γ_1, Γ_2 corresponding to $s = \pm 1/2$. For a compactly supported function $u : \mathcal{H}_2 \to \mathbb{R}$ we can write, using (1.7), the second variation formula in the following way

$$\begin{aligned} A''(0) &= \int_\Sigma |N_h|^{-1} Z(u)^2 \, d\Sigma - 4 \int_{\Gamma_1} u^2 d\ell - 4 \int_{\Gamma_2} u^2 d\ell \\ &= \int_{\mathbb{R}^2} \frac{f(s) + 4s^2}{|f(s)|} \left(\frac{\partial u}{\partial s} \right)^2 d\varepsilon \, ds - 4 \int_{\mathbb{R}} u(\varepsilon, +1/2)^2 \, d\varepsilon - 4 \int_{\mathbb{R}} u(\varepsilon, -1/2)^2 \, d\varepsilon, \end{aligned}$$

where $f(s) := (1/2) - 2s^2$. Observe that for a function u with compact support in the regular part of the surface we obtain $A''(0) \geqslant 0$. Hence the general second variation formula of the area, for variations moving the singular set, will be necessary to prove the instability of $\mathcal{H}_2$.

Let $\phi : \mathbb{R} \to [0, 1]$ be a C^∞ function with $\phi(\varepsilon) = 1$ if $|\varepsilon| \leqslant 1$ and $\phi(\varepsilon) = 0$ if $|\varepsilon| \geqslant 2$. For any $k > 1/2$ and $\delta > 0$, let $\phi_{k\delta} : \mathbb{R} \to [0, 1]$ be the symmetric function with respect to the origin given, for $s \geqslant 0$, by

$$\phi_{k\delta}(s) = \begin{cases} 1, & 0 \leqslant s \leqslant k, \\ \delta^{-1}(-s + \delta + k), & k \leqslant s \leqslant k + \delta, \\ 0, & s \geqslant k + \delta. \end{cases}$$

Then there is $k > 1/2$, $\delta = 2k + 1$ such that $u(\varepsilon, s) := \phi(\varepsilon)\phi_{k\delta}(s)$ satisfy

$$A''(0) < 0. \tag{2.2}$$

By mollification we can find a smooth function such that (2.2) holds. This completes the proof of Theorem 2.1.

References

1. N. Arcozzi and F. Ferrari, *Metric normal and distance function in the Heisenberg group*, Math. Z. **256** (2007), no. 3, 661–684. MR MR2299576
2. Z. M. Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math. **564** (2003), 63–83. MR MR2021034 (2005d:43007)
3. J. L. Barbosa and M. do Carmo, *On the size of a stable minimal surface in R^3*, Amer. J. Math. **98** (1976), no. 2, 515–528. MR MR0413172 (54 #1292)
4. J. L. Barbosa and M. P. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), no. 3, 339–353. MR MR731682 (85k:58021c)
5. J. L. Barbosa, M. P. do Carmo, and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), no. 1, 123–138. MR MR917854 (88m:53109)
6. V. Barone Adesi, F. Serra Cassano, and D. Vittone, *The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations*, Calc. Var. Partial Differential Equations **30** (2007), no. 1, 17–49. MR MR2333095
7. A. Bellaïche, *The tangent space in sub-Riemannian geometry*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 1–78. MR MR1421822 (98a:53108)
8. S. Bernstein, *Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique.*, Charikov, Comm. Soc. Math. (2) **15** (1915-1917), 38–45 (French).

9. L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progress in Mathematics, vol. 259, Birkhäuser Verlag, Basel, 2007. MR MR2312336
10. J.-H. Cheng and J.-F. Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Austral. Math. Soc. **70** (2004), no. 3, 507–520. MR MR2103983 (2005f:53010)
11. J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang, *Minimal surfaces in pseudohermitian geometry*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 1, 129–177. MR MR2165405 (2006f:53008)
12. J.-H. Cheng, J.-F. Hwang, and P. Yang, *Existence and uniqueness for p-area minimizers in the Heisenberg group*, Math. Ann. **337** (2007), no. 2, 253–293. MR MR2262784
13. D. Danielli, N. Garofalo, and D.-M. Nhieu, *Sub-Riemannian calculus on hypersurfaces in Carnot groups*, Adv. Math. **215** (2007), no. 1, 292–378. MR MR2354992
14. ———, *A notable family of entire intrinsic minimal graphs in the Heisenberg group which are not perimeter minimizing*, Amer. J. Math. **130** (2008), no. 2, 317–339. MR MR2405158
15. D. Danielli, N. Garofalo, D.-M. Nhieu, and S. D. Pauls, *Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group*, J. Differential Geom. **81** (2009), no. 2, 251–295.
16. ———, *Stable complete embedded minimal surfaces in $\mathbb{H}^1$ with empty characteristic locus are vertical planes*, arXiv math.DG/0903.4296, 2009.
17. M. Derridj, *Sur un théorème de traces*, Ann. Inst. Fourier (Grenoble) **22** (1972), no. 2, 73–83. MR MR0343011 (49 #7755)
18. M. do Carmo and A. M. Da Silveira, *Globally stable complete minimal surfaces in* $\mathbf{R}^3$, Proc. Amer. Math. Soc. **79** (1980), no. 2, 345–346. MR MR565370 (81b:53010)
19. M. P. do Carmo and C. K. Peng, *Stable complete minimal surfaces in* $\mathbf{R}^3$ *are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 903–906. MR MR546314 (80j:53012)
20. D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985), no. 1, 121–132. MR MR808112 (87b:53090)
21. D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211. MR MR562550 (81i:53044)
22. G. B. Folland and Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982. MR MR657581 (84h:43027)
23. B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), no. 3, 479–531. MR MR1871966 (2003g:49062)
24. N. Garofalo and D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), no. 10, 1081–1144. MR MR1404326 (97i:58032)
25. N. Garofalo and S. D. Pauls, *The Bernstein Problem in the Heisenberg Group*, arXiv math.DG/0209065 v2, 2002.
26. M. Gromov, *Carnot-Carathéodory spaces seen from within*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–323. MR MR1421823 (2000f:53034)
27. ———, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999, Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR MR1699320 (2000d:53065)
28. R. K. Hladky and S. D. Pauls, *Constant mean curvature surfaces in sub-Riemannian geometry*, J. Differential Geom. **79** (2008), no. 1, 111–139. MR MR2401420
29. A. Hurtado and C. Rosales, *Stable surfaces inside the sub-Riemannian three-sphere*, in preparation.
30. ———, *Area-stationary surfaces inside the sub-Riemannian three-sphere*, Math. Ann. **340** (2008), no. 3, 675–708. MR MR2358000 (2008i:53038)
31. Ana Hurtado, Manuel Ritoré, and César Rosales, *The classification of complete stable area-stationary surfaces in the Heisenberg group* $\mathbb{H}^1$, arXiv:0810.5249v2 [math.DG], 2008.
32. Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR MR1867362 (2002m:53045)
33. R. Monti, *Some properties of Carnot-Carathéodory balls in the Heisenberg group*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), no. 3, 155–167 (2001). MR MR1841689 (2002c:53048)
34. R. Monti, F. Serra Cassano, and D. Vittone, *A negative answer to the Bernstein problem for intrinsic graphs in the Heisenberg group*, Bollettino dell unione matematica italiana (2008), no. 3, 709–728, ISSN 1972-6724.
35. R. Monti and F. Serra Cassano, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. Partial Differential Equations **13** (2001), no. 3, 339–376. MR MR1865002 (2002j:49052)

36. Bennett Palmer, *Stability of minimal hypersurfaces*, Comment. Math. Helv. **66** (1991), no. 2, 185–188. MR MR1107838 (92m:58023)
37. Scott D. Pauls, *Minimal surfaces in the Heisenberg group*, Geom. Dedicata **104** (2004), 201–231. MR MR2043961 (2005g:35038)
38. ———, *H-minimal graphs of low regularity in $\mathbb{H}^1$*, Comment. Math. Helv. **81** (2006), no. 2, 337–381. MR MR2225631 (2007g:53032)
39. A. V. Pogorelov, *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR **260** (1981), no. 2, 293–295. MR MR630142 (83b:49043)
40. M. Ritoré, *Examples of area-minimizing surfaces in the Heisenberg group H^1 with low regularity*, Calc. Var. Partial Differential Equations **34** (2009), no. 2, 179–192.
41. M. Ritoré and C. Rosales, *Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$*, J. Geom. Anal. **16** (2006), no. 4, 703–720. MR MR2271950
42. M. Ritoré and C. Rosales, *Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$*, Adv. Math. **219** (2008), no. 2, 633–671. MR MR2435652
43. C. Rosales, *Complete noncompact stable cmc surfaces with empty singular set in the first Heisenberg group*, in preparation.

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