ROTATIONALLY INVARIANT HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN THE HEISENBERG GROUP $\mathbb{H}^n$

MANUEL RITORÉ AND CÉSAR ROSALES

ABSTRACT. In this paper we study sets in the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ which are critical points, under a volume constraint, of the sub-Riemannian perimeter associated to the distribution of horizontal vector fields in $\mathbb{H}^n$. We define a notion of mean curvature for hypersurfaces and we show that the boundary of a stationary set is a constant mean curvature (CMC) hypersurface. Our definition coincides with previous ones.

Our main result describes which are the CMC hypersurfaces of revolution in $\mathbb{H}^n$. The fact that such a hypersurface is invariant under a compact group of rotations allows us to reduce the CMC partial differential equation to a system of ordinary differential equations. The analysis of the solutions leads us to establish a counterpart in the Heisenberg group of the Delaunay classification of constant mean curvature hypersurfaces of revolution in the Euclidean space. Hence we classify the rotationally invariant isoperimetric sets in $\mathbb{H}^n$.

1. INTRODUCTION

In the last years the study of variational questions in sub-Riemannian geometry has experimented an increasing interest. In particular, the recent development of a theory of *minimal surfaces* in this context has contributed to achieve a better understanding of the geometry of the *Heisenberg group* $\mathbb{H}^n$ endowed with its *Carnot-Carathéodory distance*.

It is well-known that minimal surfaces arise as extremal points of the area for variations preserving the boundary of the surface. In this paper, we are interested in sets contained in the Heisenberg group which are critical points of the *sub-Riemannian perimeter* under a *volume constraint*. To describe the situation in precise terms we need to recall some facts about the Heisenberg group, that will be treated in more detail in Section 2.

We denote by $\mathbb{H}^n$ the $(2n+1)$*-dimensional Heisenberg group*, which we identify with the Lie group $\mathbb{C}^n \times \mathbb{R}$ where the product is given by

$$[z,t] * [z',t'] = [z+z', t+t' + \operatorname{Im}\big(\sum_{i=1}^{n} z_i \overline{z}'_i\big)],$$

with $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$. The Lie algebra of $\mathbb{H}^n$ is generated by $(2n+1)$ left-invariant vector fields $\{X_k, Y_k, T : k = 1, \ldots, n\}$ with one non-trivial bracket relation given by $[X_k, Y_k] = -2T$. The $(2n)$-dimensional distribution generated by $\{X_k, Y_k : k = 1, \ldots, n\}$ is called the *horizontal distribution* in $\mathbb{H}^n$. Usually $\mathbb{H}^n$ is endowed with a structure of sub-Riemannian manifold by considering the Riemannian metric on the horizontal distribution so that the basis $\{X_k, Y_k : k = 1, \ldots, n\}$ is orthonormal. This metric allows us to measure the length of horizontal curves and define the *Carnot-Carathéodory distance* between two points as the infimum of length of horizontal curves joining both points, see **[Gr2]**. Since $\mathbb{H}^n$ is a group one can consider its Haar measure, which turns out to coincide with the Lebesgue measure in $\mathbb{R}^{2n+1}$. From the notions of

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distance and volume one can also define the Minkowski content in the group, and the spherical Hausdorff measure, so that different surface measures are also given on $\mathbb{H}^n$. As it is shown in [**MSC**] and [**FSSC**], both notions of "perimeter" coincide for a set with C^2 boundary.

In this paper we will follow a slightly different approach to introduce the notions of volume and perimeter in the Heisenberg group. We consider the left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ so that $\{X_k, Y_k, T : k = 1, \ldots, n\}$ is an orthonormal basis at every point. We define the volume $\text{vol}(\Omega)$ of a Borel set $\Omega \subseteq \mathbb{H}^n$ as the Riemannian measure of the set. The perimeter $\mathcal{P}(\Omega)$ is defined in the sense of De Giorgi by using horizontal vector fields on $\mathbb{H}^n$, see (2.1) for a precise definition. This notion of perimeter coincides with the one introduced in [**CDG**] and [**FSSC**], and it is a sub-Riemannian analogous of the De Giorgi perimeter in the Riemannian manifold $(\mathbb{H}^n, g)$.

With the notions of volume and perimeter above, we study in Section 3 *stationary sets* of $\mathbb{H}^n$ which are critical points of the perimeter functional for volume preserving variations. As in the Riemannian case, one may expect that for such a set some geometric quantity defined on the boundary remains constant. In Lemma 3.2 we compute the first variation of perimeter for arbitrary variations of a set and we show, as a consequence, that the boundary of a C^2 stationary set must have *constant mean curvature* (CMC). The mean curvature of a C^2 hypersurface Σ is defined in (3.4) as the Riemannian divergence relative to Σ of the *horizontal normal vector* ν_H to Σ. We remark that a notion of mean curvature in $\mathbb{H}^1$ for graphs over the xy-plane was previously introduced by S. Pauls [**Pa**]. A more general definition has been proposed by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang [**CHMY**], and by N. Garofalo and S. Pauls [**GP**]. In Section 4 we expose a method to compute the mean curvature of a C^2 hypersurface in $\mathbb{H}^n$ which in particular shows that our definition coincides with the previous ones.

The recent study of CMC hypersurfaces in $\mathbb{H}^n$ has mainly focused on minimal surfaces in $\mathbb{H}^1$. In fact, many classical questions of the theory of minimal surfaces in $\mathbb{R}^3$, such as the Plateau problem, the Bernstein problem, or the global behaviour of properly embedded surfaces, have been treated in $\mathbb{H}^1$, see [**Pa**], [**CHMY**], [**GP**], [**CH**] and [**CHY**]. These works also provide a rich variety of examples of minimal surfaces in $\mathbb{H}^1$. However, in spite of the last advances, not much is known about non-zero CMC surfaces in $\mathbb{H}^n$. In [**CHMY**] some very interesting facts about the CMC equation in $\mathbb{H}^1$, such as the uniqueness of solutions for the Dirichlet problem or the structure of the singular set, are proved. In [**BoC**], M. Bonk and L. Capogna have studied an analogous in $\mathbb{H}^n$ of the Riemannian mean curvature flow giving, as an application, another proof of the fact that C^2 stationary sets in $\mathbb{H}^n$ are bounded by CMC hypersurfaces.

The aim of this paper is to study *invariant* CMC hypersurfaces in $\mathbb{H}^n$. Some examples of invariant minimal surfaces in $\mathbb{H}^1$ were previously obtained by S. Pauls [**Pa**]. He considered the solutions of the minimal surface equation for radial graphs over the xy-plane, and discovered a family of complete surfaces of revolution which are similar to catenoids of $\mathbb{R}^3$, see Example 4.1. With the same idea he also introduced some examples of translationally invariant and helicoidal minimal surfaces.

In Section 5 of the paper we use *intrinsic* arguments to describe CMC hypersurfaces of revolution about the t-axis in $\mathbb{H}^n$. In this case we can reduce the CMC partial differential equation to a system of ordinary differential equations (Lemma 5.1). Then, a detailed analysis of the solutions leads us to our main result (Theorem 5.4) where we prove a Heisenberg analogous of the classification by C. Delaunay [**D**] of constant mean curvature hypersurfaces of revolution in $\mathbb{R}^3$, later extended by W.-Y. Hsiang [**H**] to $\mathbb{R}^n$. As a consequence, we deduce that the only compact, CMC hypersurfaces of revolution about the t-axis in $\mathbb{H}^n$ are the spherical ones in Example 4.2. Also that complete minimal hypersurfaces of revolution are hyperplanes orthogonal to the t-axis and the catenoidal type hypersurfaces in Example 4.1. Right cylinders, unduloid type hypersurfaces and non-embedded nodoids complete the classification.

It is known that the Carnot-Carathédory distance in $\mathbb{H}^n$ can be approximated in the Gromov-Hausdorff sense by a sequence of dilated Riemannian metrics g_λ associated to g, see [**Gr**] and [**P3**]. This motivates the method, already employed in [**Pa**], of obtaining CMC hypersurfaces in $\mathbb{H}^n$ as limits of constant mean curvature hypersurfaces for the approximating metrics g_λ. The classification of constant mean curvature hypersurfaces of revolution in $(\mathbb{H}^1, g)$ was established by P. Tomter [**T**]. The case of $(\mathbb{H}^n, g)$ was treated by C. Figueroa, F. Mercuri and R. Pedrosa [**FMP**]. These papers are based on the so-called *reduction technique*, which consists of considering how the constant mean curvature equation descends to the Riemannian quotient of $(\mathbb{H}^n, g)$ by a closed subgroup of isometries, see [**FMP**, §2] and the references therein. Y. Ni [**Ni**] used the same technique to obtain similar classification results for $(\mathbb{H}^1, g_\lambda)$. By letting $\lambda \to 0$, Y. Ni defined a notion of sub-Riemannian mean curvature and found the rotationally invariant CMC surfaces in $\mathbb{H}^1$ which appear in Theorem 5.4. We remark that in [**Ni**] it is not proved that any rotationally invariant CMC surface in $\mathbb{H}^1$ is obtained as such a limit. This follows from Theorem 5.4 in this paper. Our proof of Theorem 5.4 is based on intrinsic arguments and it is not a generalization to $\mathbb{H}^n$ of the above mentioned reduction technique.

In addition to the geometric interest of this work, we believe that our results could contribute to study the *isoperimetric problem* in $\mathbb{H}^n$, which consists of finding sets enclosing a given volume with the least possible perimeter. P. Pansu [**P**] proved a non-optimal isoperimetric inequality in $\mathbb{H}^1$ and pointed out [**P2**] that the isoperimetric solutions, if smooth, must have constant mean curvature. Moreover, Pansu also conjectured in [**P2**] that the minimizers must be congruent to the spherical surfaces S_H in Example 4.2. Though some advances have been achieved in this direction the conjecture remains open. It was proved by G. P. Leonardi and S. Rigot [**LR**] that isoperimetric solutions exist: they are bounded, connected, and satisfy a nice geometric separation property ([**LR**, Theorem 2.11]). In [**LM**] it is shown that any sphere S_H in $\mathbb{H}^n$ has constant mean curvature and solves the isoperimetric problem restricted to a particular class of sets with cylindrical symmetry, see also [**M**]. Recently, D. Danielli, N. Garofalo and D.-M. Nhieu [**DGN**] have proved that the spheres S_H are also the isoperimetric solutions within a wider class of sets bounded by C^1 graphs over a xy-ball. Another approximation to Pansu's conjecture may consists of proving that the isoperimetric sets in $\mathbb{H}^n$ are of class C^2 and rotationally invariant about the t-axis, up to a left translation. In case this was true, then Theorem 5.4 would show that the spherical hypersurfaces S_H are the solutions to the isoperimetric problem.

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2. Preliminaries

In order to introduce the Heisenberg group we will follow the notation used by P. Tomter [**T**]. The $(2n+1)$-*dimensional Heisenberg group* $\mathbb{H}^n$ is the Lie group $(\mathbb{R}^{2n+1}, *)$, where we consider the usual differentiable structure in $\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$, and the product

$$[z,t] * [z',t'] = [z+z', t+t' + \operatorname{Im}\big(\sum_{i=1}^{n} z_i\, \overline{z}'_i\big)].$$

For $p = [z,t] \in \mathbb{H}^n$, the *left translation* associated to p is the diffeomorphism $L_p(q) = p * q$. A basis of left-invariant vector fields is given by

$$X_k = \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial t}, \quad Y_k = \frac{\partial}{\partial y_k} - x_k \frac{\partial}{\partial t}, \quad k = 1, \ldots, n; \qquad T = \frac{\partial}{\partial t},$$

where (x_k, y_k, t) are coordinates in $\mathbb{R}^{2n+1}$.

The *horizontal distribution* in $\mathbb{H}^n$ is the $(2n)$-dimensional smooth distribution generated by $\{X_k, Y_k : k = 1, \ldots, n\}$. The notation U_H will represent the projection of a vector U to the horizontal distribution. A vector field U is called *horizontal* if $U = U_H$. Note that $[X_k, T] = [Y_k, T] = [X_k, X_j] = [Y_k, Y_j] = 0$, while $[X_k, Y_j] = -2\delta_{kj}\, T$, where δ_{kj} is the Kronecker delta. The last equality and Frobenius Theorem imply that the horizontal distribution is nonintegrable.

For a C^1 hypersurface $\Sigma \subset \mathbb{H}^n$ the *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent hyperplane $T_p\Sigma$ coincides with the horizontal distribution. The set Σ_0 is closed and has empty interior in Σ. Hence, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. For any point $p \in \Sigma - \Sigma_0$, the tangent hyperplane meets transversally the horizontal distribution, and so the intersection is $(2n-1)$-dimensional.

Consider the Riemannian metric $g = \langle \cdot\,, \cdot \rangle$ on $\mathbb{H}^n$ so that $\{X_k, Y_k, T : k = 1 \ldots n\}$ is an orthonormal basis of $\mathbb{R}^{2n+1}$ at every point. We denote by $|U|$ the modulus of a vector field U. The volume $\text{vol}(\Omega)$ of a Borel set $\Omega \subseteq \mathbb{H}^n$ is the Riemannian volume of the metric, which in this case coincides with the Lebesgue measure in $\mathbb{R}^{2n+1}$. The *perimeter* of a Borel set $\Omega \subseteq \mathbb{H}^n$ is defined as

$$\mathcal{P}(\Omega) = \sup \left\{ \int_\Omega \text{div}(U)\, dv : |U| \leqslant 1 \right\}, \tag{2.1}$$

where the supremum is taken over C^1 *horizontal vector fields* with compact support on $\mathbb{H}^n$. In the definition above, dv and $\text{div}(\cdot)$ are the Riemannian volume and divergence of the metric, respectively. This notion of perimeter coincides with the sub-Riemannian perimeter in $\mathbb{H}^n$ given in **[CDG]** and **[FSSC]**. A set Ω is said to be of *finite perimeter* if $\text{vol}(\Omega)$ and $\mathcal{P}(\Omega)$ are finite. We refer to the reader to **[FSSC]** for a detailed development about perimeter and sets of finite perimeter in $\mathbb{H}^n$.

Let Ω be an open set in $\mathbb{H}^n$ bounded by a C^2 embedded hypersurface $\Sigma = \partial\Omega$. We denote by N the unit normal vector to Σ in $(\mathbb{H}^n, g)$ pointing into Ω. By using the Riemannian divergence theorem we obtain

$$\mathcal{P}(\Omega) = \int_\Sigma |N_H|\, da, \tag{2.2}$$

where N_H is the horizontal projection of N, and da is the Riemannian measure on Σ.

We shall denote by D the Levi-Civitá connection on $(\mathbb{H}^n, g)$. The following derivatives can be easily computed

$$\begin{aligned} &D_{X_k}X_j = D_{Y_k}Y_j = D_T T = 0, \\ &D_{X_k}Y_j = -\delta_{kj}\, T, \qquad D_{X_k}T = Y_k, \qquad D_{Y_k}T = -X_k, \\ &D_{Y_k}X_j = \delta_{kj}\, T, \qquad D_T X_k = Y_k, \qquad D_T Y_k = -X_k. \end{aligned} \tag{2.3}$$

For any vector field U on $\mathbb{H}^n$ we define $G(U) = D_U T$. It follows that $G(X_k) = Y_k$, $G(Y_k) = -X_k$ and $G(T) = 0$, so that G defines a linear isometry when restricted to horizontal vector fields. Note also that

$$\langle G(U), V \rangle + \langle U, G(V) \rangle = 0, \tag{2.4}$$

for any pair of vector fields U and V.

Let Σ be a C^2 hypersurface in $\mathbb{H}^n$, and N a unit normal vector to Σ. We can describe the singular set $\Sigma_0 \subset \Sigma$ in terms of N_H, as the set $\{p \in \Sigma : N_H(p) = 0\}$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal unit normal vector* ν_H (**[DGN]**) by

$$\nu_H = \frac{N_H}{|N_H|}. \tag{2.5}$$

Consider the unit vector field Z on $\Sigma - \Sigma_0$ given by $Z = G(\nu_H)$. As Z is horizontal and orthogonal to ν_H, we conclude that Z is tangent to Σ.

Any isometry of $(\mathbb{H}^n, g)$ leaving invariant the horizontal distribution preserves the perimeter of sets in $\mathbb{H}^n$. Examples of such isometries are left translations, which act transitively on $\mathbb{H}^n$.

In $\mathbb{H}^1$ the Euclidean rotation of angle θ about the t-axis given by

$$r_\theta(x, y, t) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, t),$$

is also such a kind of isometry since it transforms the orthonormal basis $\{X, Y, T\}$ at the point p into the orthonormal basis $\{(\cos\theta)\, X + (\sin\theta)\, Y, (-\sin\theta)\, X + (\cos\theta)\, Y, T\}$ at the point $r_\theta(p)$. It was pointed out in **[FMP]** that not all the rotations about the t axis are isometries of $(\mathbb{H}^n, g)$ for $n \geqslant 2$.

3. Stationary sets and constant mean curvature hypersurfaces in $\mathbb{H}^n$

In this section we study sets of $\mathbb{H}^n$ which are critical points under a volume constraint of the perimeter functional defined in (2.1).

Let Ω be a set of finite perimeter in $\mathbb{H}^n$. Consider a C^1 vector field U with compact support on $\mathbb{H}^n$, and denote by $\{\varphi_t\}_{t\in\mathbb{R}}$ the associated group of diffeomorphisms. Let $\Omega_t = \varphi_t(\Omega)$. The family $\{\Omega_t\}$, for t small, is the variation of Ω induced by U. Let $V(t) = \operatorname{vol}(\Omega_t)$ and $\mathcal{P}(t) = \mathcal{P}(\Omega_t)$. We say that the variation *preserves volume* if $V(t)$ is constant for t small enough. We say that Ω is *stationary* if $\mathcal{P}'(0) = 0$ for any volume preserving variation. In order to describe analytically stationary sets we shall compute the first variation formula for volume and perimeter.

Suppose that Ω is bounded by a C^2 embedded hypersurface $\Sigma = \partial\Omega$. As the volume considered is the Riemannian one, it is well-known (**[BdCE]**) that

$$V'(0) = \int_\Omega \operatorname{div}(U)\, dv = -\int_\Sigma u\, da,$$

where $u = \langle U, N\rangle$ is the component of U with respect to the unit normal vector N to Σ pointing into Ω.

Now we compute the first variation of perimeter. We need a previous lemma.

Lemma 3.1. *Let $\Sigma \subset \mathbb{H}^n$ be a C^2 hypersurface and N a unit normal vector to Σ. Consider a point $p \in \Sigma - \Sigma_0$, the horizontal normal ν_H at p defined in* (2.5)*, and $Z = G(\nu_H)$. Let $\{Z_1, \ldots, Z_{2n-1}\}$ be an orthonormal family of horizontal, tangent vectors to Σ at p with $Z_1 = Z$. Then, for any $u \in T_p\mathbb{H}^n$ we have*

$$D_u N_H = (D_u N)_H - \langle N, T\rangle\, G(u) - \langle N, G(u)\rangle\, T, \tag{3.1}$$

$$u\,(|N_H|) = \langle D_u N, \nu_H\rangle - \langle N, T\rangle\, \langle G(u), \nu_H\rangle\,, \tag{3.2}$$

$$D_u \nu_H = |N_H|^{-1} \sum_{i=1}^{2n-1} (\langle D_u N, Z_i\rangle - \langle N, T\rangle\, \langle G(u), Z_i\rangle)\, Z_i + \langle Z, u\rangle\, T. \tag{3.3}$$

Proof. Equalities (3.1) and (3.2) are easily obtained by using that $N_H = N - \langle N, T\rangle\, T$. Let us prove (3.3). As $|\nu_H| = 1$ and $\{\nu_H, Z_i, T : i = 1, \ldots, 2n-1\}$ is an orthonormal basis of $T_p\mathbb{H}^n$, we get

$$D_u \nu_H = \sum_{i=1}^{2n-1} \langle D_u \nu_H, Z_i\rangle\, Z_i + \langle D_u \nu_H, T\rangle\, T.$$

Note that $\langle D_u \nu_H, T\rangle = -\langle \nu_H, G(u)\rangle = \langle Z, u\rangle$ by (2.4). On the other hand, by using (3.1) and the fact that Z_i is tangent and horizontal, we deduce

$$\langle D_u \nu_H, Z_i\rangle = |N_H|^{-1}\, \langle D_u N_H, Z_i\rangle = |N_H|^{-1}(\langle D_u N, Z_i\rangle - \langle N, T\rangle\, \langle G(u), Z_i\rangle).$$

□

For a C^1 vector field U on a hypersurface Σ, we denote by $\operatorname{div}_\Sigma U$ the Riemannian divergence of U relative to Σ, which is given by $\operatorname{div}_\Sigma U(p) := \sum_{1=1}^{2n} \langle D_{e_i}U, e_i\rangle$ for any orthonormal basis $\{e_i : i = 1, \dots, 2n\}$ of $T_p\Sigma$. Now, we can prove

Lemma 3.2. *Let Ω be a set of finite perimeter in $\mathbb{H}^n$ such that $\Sigma = \partial\Omega$ is a C^2 hypersurface. Denote by N the unit normal vector to Σ pointing into Ω and by ν_H the horizontal normal vector defined in* (2.5). *Let U be a C^1 vector field with compact support on $\mathbb{H}^n$ and normal component $u = \langle U, N\rangle$. Then the first derivative of the perimeter functional $\mathcal{P}(t)$ associated to U is given by*

$$\mathcal{P}'(0) = -\int_\Sigma \operatorname{div}_\Sigma \big(u\,(\nu_H)^\top\big)\,da + \int_\Sigma u\,(\operatorname{div}_\Sigma \nu_H)\,da,$$

provided $\operatorname{div}_\Sigma \nu_H \in L^1(\Sigma)$.

Moreover, if the support of U is disjoint from the singular set Σ_0, then

$$\mathcal{P}'(0) = \int_\Sigma u\,(\operatorname{div}_\Sigma \nu_H)\,da = \int_\Sigma |N_H|^{-1} u\,(\operatorname{div}_\Sigma \nu_H)\,dP,$$

where dP is the perimeter measure on Σ.

Proof. First we remark that the Riemannian area of the singular set Σ_0 of Σ vanishes, as was proved in **[De1]**, **[De2]** and **[Ba]**. Thus we can integrate over Σ any function u defined on the regular set $\Sigma - \Sigma_0$ provided $u \in L^1(\Sigma)$.

Call $\{\varphi_t\}_{t\in\mathbb{R}}$ to the group of diffeomorphisms associated to U and denote $\Sigma_t = \varphi_t(\Sigma)$. Let da_t be the Riemannian measure on Σ_t. Consider a C^1 vector field N whose restriction to Σ_t coincides with the unit normal vector pointing into $\Omega_t = \varphi_t(\Omega)$. Denote by $U^\top$ and $U^\perp$ the tangent and the normal part of U, respectively. By using (2.2) and the coarea formula, we have

$$\mathcal{P}(t) = \int_{\Sigma_t} |N_H|\,da_t = \int_\Sigma (|N_H| \circ \varphi_t)\,|\operatorname{Jac}\varphi_t|\,da = \int_{\Sigma-\Sigma_0} (|N_H| \circ \varphi_t)\,|\operatorname{Jac}\varphi_t|\,da,$$

where $\operatorname{Jac}\varphi_t$ is the Jacobian determinant of the map $\varphi_t : \Sigma \to \Sigma_t$. The last equality holds since Σ_0 has Riemannian area zero. Now, we differentiate with respect to t, and we use the known fact that $(d/dt)|_{t=0}\,|\operatorname{Jac}\varphi_t| = \operatorname{div}_\Sigma U$, to get

$$\begin{aligned}
\mathcal{P}'(0) &= \int_{\Sigma-\Sigma_0} \{U(|N_H|) + |N_H|\ \operatorname{div}_\Sigma U\}\,da \\
&= \int_{\Sigma-\Sigma_0} \{U^\perp(|N_H|) + \operatorname{div}_\Sigma(|N_H|\,U)\}\,da \\
&= \int_{\Sigma-\Sigma_0} \{\operatorname{div}_\Sigma(|N_H|\,U^\top) + U^\perp(|N_H|) + |N_H|\ \operatorname{div}_\Sigma U^\perp\}\,da \\
&= \int_{\Sigma-\Sigma_0} \{U^\perp(|N_H|) + |N_H|\ \operatorname{div}_\Sigma U^\perp\}\,da.
\end{aligned}$$

To obtain the last equality we have used the Riemannian divergence theorem to get that the integral of the divergence of the Lipschitzian vector field $|N_H|\,U^\top$ over Σ vanishes (the modulus of a C^1 vector field in a Riemannian manifold is a Lipschitz function). We observe that the function $U^\perp(|N_H|) + |N_H|\ \operatorname{div}_\Sigma U^\perp$ is bounded in $\Sigma - \Sigma_0$, and so it lies in $L^1(\Sigma)$.

On the other hand, we can use (3.2) to obtain

$$U^\perp(|N_H|) = \langle D_{U^\perp}N, \nu_H\rangle - \langle N, T\rangle\ \langle G(U^\perp), \nu_H\rangle = -\langle \nabla_\Sigma u, \nu_H\rangle\,,$$

since $G(U^\perp)$ is orthogonal to ν_H and $D_{U^\perp}N = -\nabla_\Sigma u$. Here $\nabla_\Sigma u$ represents the gradient of u relative to Σ. Then, we get in $\Sigma - \Sigma_0$

$$\begin{aligned} U^\perp(|N_H|) + |N_H| \operatorname{div}_\Sigma U^\perp &= -(\nu_H)^\top(u) + u\,|N_H| \operatorname{div}_\Sigma N \\ &= -\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) + u \operatorname{div}_\Sigma\big((\nu_H)^\top\big) + u \operatorname{div}_\Sigma(|N_H|\,N) \\ &= -\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) + u \operatorname{div}_\Sigma \nu_H. \end{aligned}$$

Since we are assuming $\operatorname{div}_\Sigma \nu_H \in L^1(\Sigma)$, we conclude that $\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) \in L^1(\Sigma)$ and so we have

$$\int_\Sigma \{U^\perp(|N_H|) + |N_H| \operatorname{div}_\Sigma U^\perp\}\,da = \int_\Sigma \{-\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) + u\,(\operatorname{div}_\Sigma \nu_H)\}\,da.$$

Hence

$$\mathcal{P}'(0) = -\int_\Sigma \operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,da + \int_\Sigma u\,(\operatorname{div}_\Sigma \nu_H)\,da.$$

Finally, the first integral above vanishes by virtue of the Riemannian divergence theorem whenever u has compact support disjoint from the singular set Σ_0. □

Let Σ be a C^2 hypersurface in $\mathbb{H}^n$, and N a unit normal vector field to Σ. We define the *mean curvature* of Σ with respect to N by equality

$$-2n\,H(p) = (\operatorname{div}_\Sigma \nu_H)(p), \qquad p \in \Sigma - \Sigma_0. \tag{3.4}$$

We say that Σ is of *constant mean curvature* (CMC) if H is constant on $\Sigma - \Sigma_0$. In this case we extend H by its constant value to the whole Σ. A *minimal hypersurface* in $\mathbb{H}^n$ is one for which $H = 0$. These definitions have sense even for immersed hypersurfaces.

The first variation of perimeter allows us to prove the following variational property of stationary sets

Corollary 3.3. *Let Ω be a set of finite perimeter in $\mathbb{H}^n$ bounded by a C^2 hypersurface Σ. If Ω is stationary, then Σ has constant mean curvature.*

Remark 3.4. The first variation of perimeter and the notion of mean curvature were first given by S. Pauls [**Pa**] for graphs $\Sigma = \{t = f(x,y)\}$ in $\mathbb{H}^1$, and later extended by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang [**CHMY**], and by N. Garofalo and S. Pauls [**GP**] for variations supported on the regular set of any C^2 surface in $\mathbb{H}^1$. The case of $\mathbb{H}^n$ has been treated in [**DGN**]. In the recent paper [**CHY**] the variation of perimeter has been computed for some more general variations moving the singular set. R. K. Hladky and S. Pauls [**HP**] have generalized the notion of mean curvature and Corollary 3.3 to vertically rigid sub-Riemannian manifolds. In Section 4 we will show that our definition of mean curvature agrees with the previous ones.

Remark 3.5. The mean curvature can be defined, off of the singular set, for a C^2 immersed hypersurface Σ in $\mathbb{H}^n$. The area of Σ may be defined as $\int_\Sigma |N_H|\,da$, and it can be easily shown that its first derivative equals $-\int_\Sigma 2nH\,u\,da$ for any variation of Σ with initial velocity vector field uN with compact support. On the other hand, even if Σ does not enclose some given volume, the variation of volume for a displacement of Σ in the direction of a normal vector field uN with compact support can be computed and equals $-\int_\Sigma u\,da$. The interested readers are referred to the paper by J. L. Barbosa, M. do Carmo and J. Eschenburg [**BdCE**] for details in the Riemannian setting. Hence one can conclude that immersed hypersurfaces with constant mean curvature are critical points of the sub-Riemannian boundary area under volume-preserving variations.

4. Computation of the mean curvature and examples

In this section we describe a method to compute the mean curvature defined in (3.4) of a C^2 hypersurface Σ in $\mathbb{H}^n$. Then, we will apply it to give an explicit expression for the mean curvature of a graph $t = f(x,y)$ in $\mathbb{H}^1$. This will allow us to recall two families of well-known constant mean curvature hypersurfaces in $\mathbb{H}^1$. Finally, we will compute the mean curvature of a hypersurface of revolution in $\mathbb{H}^n$ about the t-axis.

Consider a C^2 immersion $\phi : B \to \mathbb{H}^n$ defined on a $(2n)$-dimensional Riemannian manifold. Suppose that N is a unit normal vector field to $\Sigma = \phi(B)$ in $(\mathbb{H}^n, g)$. Fix a point $p \in B$ such that $\phi(p) \in \Sigma - \Sigma_0$, and consider an orthonormal basis $\{e_1, \dots, e_{2n}\}$ of T_pB. Denote $\partial_j = e_j(\phi)$. The vectors $\{\partial_j : j = 1, \dots, 2n\}$ form a basis of $T_{\phi(p)}\Sigma$. Denote by ν_H the horizontal normal vector defined in (2.5). A unit, horizontal tangent vector to the non-singular part $\Sigma - \Sigma_0$ of Σ is given by $Z = G(\nu_H)$. Take an orthonormal basis $\{Z_1, \dots, Z_{2n-1}\}$ of horizontal, tangent vectors to Σ at $\phi(p)$ with $Z_1 = Z$. If we call $S = \langle N, T\rangle\, \nu_H - |N_H|\, T$, then it is clear that $\{Z_i, S\}$ is an orthonormal basis of $T_{\phi(p)}\Sigma$, and so the mean curvature (3.4) of Σ can be computed as

$$-2nH = \sum_{i=1}^{2n-1} \left\langle D_{Z_i}\nu_H, Z_i \right\rangle + \langle D_S \nu_H, S\rangle .$$

By using the expression for $D_u\nu_H$ given in Lemma 3.1, we obtain

$$2nH = |N_H|^{-1} \sum_{i=1}^{2n-1} \mathrm{II}(Z_i, Z_i), \tag{4.1}$$

where II is the second fundamental form of Σ in $(\mathbb{H}^n, g)$ with respect to the normal N.

From the expression above it is easy to see that for the case $n = 1$

$$D_Z Z = 2H\nu_H,$$

and we deduce that our definition of mean curvature coincides with the one given in [**CHMY**].

Denote by $\mathrm{II}_{ij} = \mathrm{II}(\partial_i, \partial_j) = -\left\langle D_{\partial_i} N, \partial_j \right\rangle$. It is clear that

$$\mathrm{II}_{ij} = \left\langle N, D_{e_i}\partial_j \right\rangle . \tag{4.2}$$

On the other hand, if the coordinates of ∂_j with respect to the left-invariant basis $\{X_k, Y_k, T\}$ are given by (x_{kj}, y_{kj}, t_j), then a straightforward computation by using (2.3) shows that the coordinates of $D_{e_i}\partial_j$ with respect to $\{X_k, Y_k, T\}$ are

$$\left(e_i(x_{kj}) - t_i\, y_{kj} - t_j\, y_{ki}, e_i(y_{kj}) + t_i\, x_{kj} + t_j\, x_{ki}, e_i(t_j) + \sum_{k=1}^{n} (x_{kj}\, y_{ki} - x_{ki}\, y_{kj}) \right) . \tag{4.3}$$

The calculation of the coefficients II_{ij} allows us to compute $\mathrm{II}(Z_i, Z_i)$ so that we can obtain from (4.1) an explicit expression for the mean curvature of Σ in any particular case.

Example. Let Σ be the graph of a function $f \in C^2(B)$ over an open set $B \subseteq \mathbb{R}^2$. Then, by using (4.3), (4.2) and (4.1) for the immersion $\phi(z) = (z, f(z))$, $z \in B$, we get

$$2H = -\frac{(f_y + x)^2\, f_{xx} + (f_x - y)^2\, f_{yy} - 2\,(f_x - y)\,(f_y + x)\, f_{xy}}{\{(f_x - y)^2 + (f_y + x)^2\}^{3/2}}. \tag{4.4}$$

The expression above coincides with the divergence in $\mathbb{R}^2$ of the horizontal vector field ν_H projected to $\mathbb{R}^2$. It follows that our definition of mean curvature extends the one given by S. Pauls [**Pa**].

Now, we will show some known examples of CMC surfaces in $\mathbb{H}^n$. Of course, hyperplanes in $\mathbb{R}^{2n+1}$ are minimal hypersurfaces in $\mathbb{H}^n$. The construction of examples has been mainly focused on minimal surfaces in $\mathbb{H}^1$, see **[Pa]**, **[CH]** and **[GP]**. We are interested in the following ones

Example 4.1 (Catenoidal surfaces in $\mathbb{H}^1$)**.** For any $E > 0$, let us consider the hyperboloid of revolution Σ in $\mathbb{R}^3$ defined in coordinates $[z, t] \in \mathbb{H}^1$, by equality

$$t = \pm\sqrt{E^2\,(|z|^2 - E^2)}, \qquad |z| \geqslant E.$$

By using the expression (4.4) for the mean curvature of a graph, it is easy to check that Σ is a smooth, rotationally invariant, minimal surface in $\mathbb{H}^1$. These surfaces were characterized by S. Pauls [**Pa**, Section 4] as the unique minimal surfaces given by the unions of two radial graphs over the xy-plane.

There are no many examples of complete surfaces in $\mathbb{H}^n$ with non-zero constant mean curvature. The best known are the next ones

Example 4.2 (Spherical hypersurfaces in $\mathbb{H}^n$)**.** For any $H > 0$ consider the hypersurface S_H in $\mathbb{H}^n$ defined in coordinates $[z, t]$, for $z \in \mathbb{C}^n$, by

$$t = \pm\frac{1}{2H^2}\,\{H|z|\,\sqrt{1 - H^2\,|z|^2} + \arccos{(H|z|)}\}, \qquad |z| \leqslant \frac{1}{H}.$$

The hypersurface S_H is compact and homeomorphic to a $(2n)$-dimensional sphere. It has two singular points on the t-axis. In **[DGN]** it is shown that S_H is C^2 but not C^3 around the singular points. It was proved in **[LM]** that S_H has constant mean curvature H. These hypersurfaces were conjectured to be the (smooth) solutions to the isoperimetric problem in $\mathbb{H}^n$ by P. Pansu **[P2]**.

Finally, we shall compute the mean curvature of a rotationally invariant hypersurface in $\mathbb{H}^n$.

Let Σ be a C^2 hypersurface in $\mathbb{H}^n$ which is invariant under the group of rotations in $\mathbb{R}^{2n+1}$ about the t-axis. Denote by $\gamma(s) = (x(s), t(s))$, $s \in I$, the generating curve of Σ in the half-plane $\{xt\,(= x_1 t) : x \geqslant 0\}$. We parameterize the hypersurface Σ in cylindrical coordinates by the immersion $\phi : B \to \mathbb{H}^n$ given by $\phi(s, \omega) = (x(s)\,\omega, t(s))$, where B is the manifold $I \times \mathbb{S}^{2n-1}$ endowed with the Euclidean metric of $\mathbb{R}^{2n}$. We take the unit normal vector to Σ in $(\mathbb{H}^n, g)$ whose coordinates with respect to $\{X_k, Y_k, T\}$ are

$$\frac{(xx'\omega_{n+k} - t'\omega_k, -xx'\omega_k - t'\omega_{n+k}, x')}{\sqrt{|\gamma'|^2 + x^2(x')^2}}, \quad \text{whenever } x > 0, \tag{4.5}$$

where $\omega = (\omega_k, \omega_{n+k}) \in \mathbb{R}^n \times \mathbb{R}^n$. In particular, we deduce that the singular set Σ_0 of Σ is contained on the t-axis. Now, choose a point $p = (s, \omega) \in B$ and a geodesic frame $\{u_2, \ldots, u_{2n}\}$ of $\mathbb{S}^{2n-1}$ around ω, with $u_2(\omega) = (-\omega_{n+k}, \omega_k)$. We take the orthonormal basis $e_1 = (1, 0)$ and $e_j = (0, u_j)$ of $T_pB = \mathbb{R} \times T_\omega\mathbb{S}^{2n-1}$. Note that the coordinates of $\partial_j = e_j(\phi)$ with respect to the basis $\{X_k, Y_k, T\}$ are

$$\partial_1 = (x'\omega_k, x'\omega_{n+k}, t'), \quad \partial_2 = (-x\omega_{n+k}, x\omega_k, x^2), \quad \partial_j = (x(u_j)_k, x(u_j)_{n+k}, 0), \quad j \geqslant 3.$$

Then, we can compute from (4.2), (4.3) and (4.5) the coefficients II_{ij} of the second fundamental form of Σ with respect to N, resulting

$$\mathrm{II}_{11} = \frac{x't'' - x''t' - 2x(x')^2t'}{\sqrt{|\gamma'|^2 + x^2(x')^2}}, \qquad \mathrm{II}_{22} = \frac{xt'\,(1 + 2x^2)}{\sqrt{|\gamma'|^2 + x^2(x')^2}}, \qquad \mathrm{II}_{ii} = 0, \quad i = 3, \ldots, 2n,$$

$$\mathrm{II}_{1j} = \frac{x(t')^2 - x^3(x')^2}{\sqrt{|\gamma'|^2 + x^2(x')^2}}\,\delta_{1j}, \qquad \mathrm{II}_{ij} = 0, \qquad i, j \in \{2, \ldots, 2n\},\ i \neq j.$$

On the other hand, we consider the following orthonormal basis $\{Z_i : i = 2, \dots, 2n\}$ of horizontal, tangent vectors to Σ

$$Z_2 = Z = \frac{x\,\partial_1 - (t'/x)\,\partial_2}{\sqrt{x^2(x')^2 + (t')^2}}, \qquad Z_i = \frac{\partial_i}{x}, \quad i = 3, \dots, 2n.$$

Finally, it is easy to check from (4.1) that the mean curvature of Σ with respect to N in the regular set is the following

$$2nH = \frac{x^3\,(x't'' - x''t') + (2n-1)\,(t')^3 + 2\,(n-1)\,x^2(x')^2t'}{x\,\{x^2(x')^2 + (t')^2\}^{3/2}}. \tag{4.6}$$

Example. When the generating curve γ is the graph of a function $x(t)$ defined on the t-axis, then equation (4.6) becomes

$$2nH = \frac{(2n-1) - x^3x'' + 2\,(n-1)\,x^2(x')^2}{x\,\{1 + x^2\,(x')^2\}^{3/2}}.$$

In particular, a cylinder of radius $r > 0$ about the t-axis has constant mean curvature $H = \frac{2n-1}{2nr}$.

5. Classification of rotationally invariant CMC hypersurfaces in $\mathbb{H}^n$

In this section we study in detail the solutions of the CMC equation for hypersurfaces of revolution in $\mathbb{H}^n$ about the t-axis.

Let Σ be a C^2 hypersurface in $\mathbb{H}^n$ which is invariant under the group of rotations in $\mathbb{R}^{2n+1}$ about the t-axis. Denote by γ the generating curve of Σ in the half-plane $\{xt\,(= x_1t) : x \geqslant 0\}$. We parameterize the curve $\gamma = (x, t)$ by arc-length $s \in I$. Let N be the unit normal vector to Σ given in (4.5), and H the mean curvature of Σ with respect to N. Denote by $\sigma(s)$ the angle between the tangent vector $\gamma'(s)$ and $\frac{\partial}{\partial t}$. It is clear that $x' = \sin\sigma$ and $t' = \cos\sigma$. By substituting these equalities into (4.6), we deduce the following

Lemma 5.1. *The generating curve $\gamma = (x, t)$ of a C^2 rotationally invariant hypersurface in $\mathbb{H}^n$ with constant mean curvature H, satisfies the following system of ordinary differential equations*

$$(*)_H \qquad \begin{cases} x' = \sin\sigma, \\ t' = \cos\sigma, \\ \sigma' = (2n-1)\,\dfrac{\cos^3\sigma}{x^3} + 2\,(n-1)\,\dfrac{\sin^2\sigma\,\cos\sigma}{x} - 2nH\,\dfrac{(x^2\sin^2\sigma + \cos^2\sigma)^{3/2}}{x^2}, \end{cases}$$

whenever $x > 0$. Moreover, the system above has a first integral: the function given by

$$\frac{x^{2n-1}\cos\sigma}{\sqrt{x^2\sin^2\sigma + \cos^2\sigma}} - Hx^{2n} \tag{5.1}$$

is constant along any solution (x, t, σ).

Remarks. 1. Note that the system $(*)_H$ has a singularity for $x = 0$. We will show that the possible contact between a solution (x, t, σ) and the t-axis is perpendicular. This means that the generated hypersurface Σ is of class C^1 around the t-axis.

2. The existence of a first integral follows from Noether's theorem [**GiH**, § 4 in Chap. 3] by taking into account that the translations along the t-axis preserve the solutions of $(*)_H$, since the area and volume functionals are invariant, and hence infinitesimally invariant, by these translations. It can also be obtained by using the arguments given by N. Korevaar, R. Kusner and B. Solomon, see [**KKS**, p. 480].

3. The constant value E of the function (5.1) will be called the *energy* of the solution (x, t, σ). Notice that

$$x^{2n-1}\cos\sigma = (E + Hx^{2n})\sqrt{x^2\sin^2\sigma + \cos^2\sigma}. \tag{5.2}$$

The equation above clearly implies

$$(x^{4n-2} - (E + Hx^{2n})^2)\cos^2\sigma = (E + Hx^{2n})^2 x^2 \sin^2\sigma, \tag{5.3}$$

from which we deduce the inequality

$$x^{2n-1} \geqslant |E + Hx^{2n}|. \tag{5.4}$$

In the following results we gather some elementary properties of the solutions of $(*)_H$.

Lemma 5.2. *Let $(x(s), t(s), \sigma(s))$ be a solution of $(*)_H$ with energy E. Then, we have*

(i) *The solution can be translated along the t-axis. More precisely, $(x(s), t(s) + t_0, \sigma(s))$ is a solution of $(*)_H$ with energy E for any $t_0 \in \mathbb{R}$.*
(ii) *If $x'(s_0) = 0$, then the solution is symmetric with respect to the line $\{t = t(s_0)\}$. As consequence, we can continue a solution by reflecting across the critical points of $x(s)$.*
(iii) *The curve $(x(s_0 - s), t(s_0 - s), \pi + \sigma(s_0 - s))$ is a solution of $(*)_{-H}$ with energy $-E$.*

Lemma 5.3. *Let $(x(s), t(s), \sigma(s))$ be a solution of $(*)_H$. If $\cos\sigma(s_0) \neq 0$, then the coordinate x is a function over a small t-interval around $t(s_0)$. Moreover*

$$\frac{dx}{dt} = \tan\sigma, \qquad \frac{d^2x}{dt^2} = \frac{\sigma'}{\cos^3\sigma}, \tag{5.5}$$

where σ' is the derivative of σ with respect to s.

The first integral (5.1) allows us to give the complete description of the solutions of $(*)_H$. They are of the same types as the ones obtained by C. Delaunay **[D]** when he studied constant mean curvature surfaces of revolution in $\mathbb{R}^3$.

Theorem 5.4. *Let γ be a complete solution of the system $(*)_H$ with energy E. Then, the generated Σ is a constant mean curvature hypersurface in $\mathbb{H}^n$ of one of the following types (see Figure 1)*

(i) *If $H = 0$ and $E = 0$ then γ is a straight line orthogonal to the t-axis and Σ is a Euclidean hyperplane.*
(ii) *If $H = 0$ and $E \neq 0$ we obtain an embedded Σ of catenoidal type. Moreover, the resulting hypersurface is contained inside a slab of $\mathbb{R}^{2n+1}$ when $n \geqslant 2$.*
(iii) *If $H \neq 0$ and $E = 0$ then Σ is a compact hypersurface homeomorphic to a sphere.*
(iv) *If $EH > 0$ then γ is a periodic graph over the t-axis. The generated Σ is a cylinder or an embedded hypersurface of unduloid type.*
(v) *If $EH < 0$ then γ is a locally convex curve and Σ is a nodoid type hypersurface, which has selfintersections.*

Remark 5.5. Undularies and nodaries are curves generating undoloids and nodoids, hypersurfaces of revolution with constant mean curvature in Euclidean space. Unduloids are embedded hypersurfaces invariant by a translation parallel to the axis of revolution. Nodoids are also periodic, but they have selfintersections. The curves obtained in (iv) and (v) of Theorem 5.4 produce hypersurfaces with the same behavior.

Proof. Let $\gamma = (x, t, \sigma)$ be a complete solution of $(*)_H$ with energy E. By Lemma 5.2 (i) we can suppose that γ is defined over an interval I containing the origin, and that the initial conditions of γ are $(x_0, 0, \sigma_0)$. To prove Theorem 5.4 we distinguish several cases depending on the signs of H and E. We begin by studying minimal surfaces.

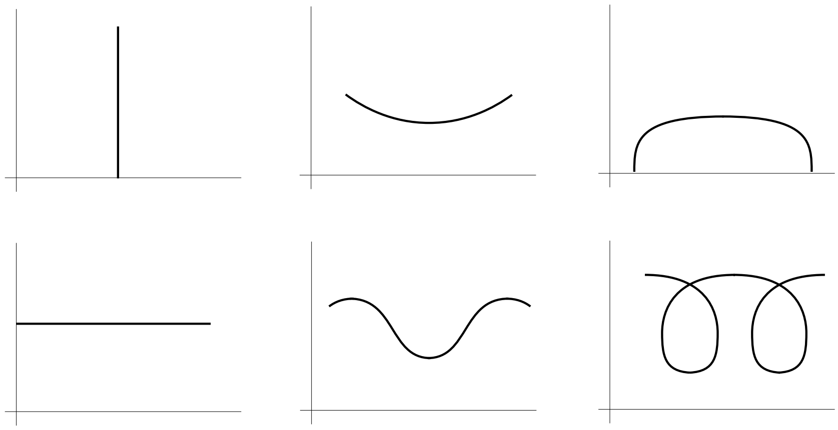

FIGURE 1. The different types for the generating curve of a rotationally invariant CMC hypersurface in $\mathbb{H}^n$: hyperplane, catenoid, sphere, cylinder, unduloid and nodoid.

• $H = 0$, $E = 0$. From (5.2) we get $\cos\sigma \equiv 0$ and so, by Lemma 5.2 (iii), we can admit that $\sigma \equiv \pi/2$. It follows that $t \equiv 0$ and $x(s) = s + x_0$. We conclude that the solution is a half-line meeting the t-axis orthogonally. The generated hypersurface Σ is a hyperplane.

• $H = 0$, $E \neq 0$. We can suppose that $E > 0$ by Lemma 5.2 (iii). By using (5.4) we see that $x^{2n-1} \geqslant E$ and so, the solution does not approach the t-axis. Moreover, the solution is defined on the whole real line since (x', t', σ') is bounded. By translating the solution along the t-axis we can admit that the initial conditions of γ are $(E^{1/(2n-1)}, 0, 0)$. From (5.2) we get $\cos\sigma > 0$. It follows from Lemma 5.3 that the x-coordinate of γ is a function of t around the origin. By (5.5) and $(*)_H$ we have

$$\frac{d^2x}{dt^2} = \frac{2n-1}{x^3} + \frac{2\,(n-1)}{x}\left(\frac{dx}{dt}\right)^2.$$

In the case $n = 1$ we can integrate this differential equation obtaining $x(t) = E^{-1}\sqrt{t^2 + E^4}$, which is a catenoidal type surface as in Example 4.1. Consider the case $n \geqslant 2$. By the symmetry of the solutions of $(*)_H$ (Lemma 5.2 (ii)) we only have to describe the curve $\gamma(s)$ for $s > 0$. From $(*)_H$ we see that $\sigma' > 0$, and so $\sigma(s) \in (0, \pi/2)$ for any $s > 0$ since $\cos\sigma > 0$. On the other hand, equation (5.5) implies that $x(t)$ is strictly increasing and strictly convex. The uniqueness of the solutions of $(*)_H$ for given initial conditions ensures that any other solution with $H = 0$ and $E > 0$ is a translation along the t-axis of the graph of the function $x(t)$ described above.

Let us see that the generated Σ is contained in a slab of $\mathbb{R}^{2n+1}$ when $n \geqslant 2$. Call $t_\infty = \lim_{s\to+\infty} t(s)$. The fact that $\sin\sigma > 0$ for $s > 0$ implies that the t-coordinate of γ is a function of $x \in (E^{1/(2n-1)}, +\infty)$. This function satisfies $dt/dx = \cot\sigma$ by (5.5). An explicit expression for $\cot\sigma$ is obtained by using (5.3). It follows that

$$t_\infty = \int_{E^{1/(2n-1)}}^{+\infty} \frac{Ex}{\sqrt{x^{4n-2} - E^2}}\,dx.$$

Finally, a comparison with the functions $x^{4n-3}\,(x^{4n-2} - E^2)^{-1/2}$ and x^{2-2n} shows that the integral above converges only if $n \geqslant 2$.

Now, we shall assume that $H \neq 0$. By Lemma 5.2 (iii) we can suppose $H > 0$. We discuss different cases attending to the sign of E.

• $H > 0$, $E = 0$. By equation (5.2) we get $\cos\sigma > 0$ on I and so, γ is a graph over the t-axis. From (5.4) we have $x \leqslant 1/H$, so that the solution could approach the t-axis. Moreover, by using (5.2), (5.3) and $(*)_H$, we obtain

$$\sigma' = \frac{\cos^3\sigma}{H^2x^5}\,(H^2x^2-2) < 0,$$

which implies that the angle is strictly decreasing along the solution γ.

As the solutions are invariant by translations along the t-axis, we can suppose that the initial conditions of γ are $(1/H, 0, 0)$. Call $\beta = \sup I$ and $t_\beta = \lim_{s\to\beta^-} t(s)$. By Lemma 5.2 (ii) we only have to study the function $x(t)$ for $t \in (0, t_\beta)$. As $\sigma' < 0$ and $\cos\sigma > 0$, we deduce that $\sigma \in (-\pi/2, 0)$ on the interval $(0, \beta)$. In particular, we get by (5.5) that $x(t)$ is strictly decreasing and strictly concave on $(0, t_\beta)$. This entails $t_\beta < +\infty$ since $x(t)$ is bounded. By using the same reasoning with $x(s)$ we prove $\beta < +\infty$. It follows that $x(s) \to 0$ when $s \to \beta^-$; otherwise, we could continue the complete solution $\gamma(s)$ for $s > \beta$, a contradiction. By (5.2) we have that γ meets the axis orthogonally. These arguments show that the generated Σ is a compact hypersurface of spherical type.

We finally see that γ coincides with the generating curve of the sphere S_H given in Example 4.2. Note that $\sin\sigma < 0$ on $(0, \beta)$. This implies that the t-coordinate of $\gamma(s)$, for $s \in (0, \beta)$, is a function of $x \in (0, 1/H)$. Moreover, by (5.5) we obtain $dt/dx = \cot\sigma < 0$. By computing $\cot\sigma$ from (5.3), we have

$$\frac{dt}{dx} = \frac{-Hx^2}{\sqrt{1-H^2x^2}}, \quad x \in (0, 1/H).$$

We can integrate the equality above to conclude that

$$t(x) = \frac{1}{2H^2}\,\{Hx\sqrt{1-H^2x^2} + \arccos(Hx)\}, \quad x \in (0, 1/H),$$

which proves the claim.

• $H > 0$, $E > 0$. By (5.2) we see that $\cos\sigma > 0$ on I and so, γ is a graph over the t-axis. Equation (5.4) gives $Hx^{2n} - x^{2n-1} + E \leqslant 0$, which implies that $x(s)$ is bounded and the solution does not meet the t-axis. By $(*)_H$ it follows that (x', t', σ') is bounded and so, the solution is defined on the whole real line. Call $x_1 = \inf x(s)$ and $x_2 = \sup x(s)$. The values x_1 and x_2 coincide with the positive zeroes of the polynomial $Hy^{2n} - y^{2n-1} + E$. We consider two cases

(i) $x_1 = x_2$. We obtain $x \equiv r > 0$, $\sigma \equiv 0$ and $t(s) = s$. The generated hypersurface is a cylinder of radius $r = (2n-1)/(2nH)$ about the t-axis.

(ii) $x_1 < x_2$. After a translation along the t-axis, we can suppose that the initial conditions of γ are $(x_1, 0, 0)$. Call $t_\infty = \lim_{s\to+\infty} t(s)$. By the symmetry of the solutions of $(*)_H$ it suffices to study the function $x(t)$ for $t \in (0, t_\infty)$. It is clear that $\sigma'(0) \geqslant 0$; in fact, as we assume $x_1 < x_2$, we have that $\sigma'(0) > 0$. In particular, by (5.5) we get that $x(t)$ is strictly increasing and strictly convex on a small interval to the right of the origin. We claim that there exists a first value $s_1 > 0$ such that $\sigma'(s_1) = 0$. Otherwise, we would deduce $\sigma(s) \in (0, \pi/2)$ for any $s > 0$, which implies by $(*)_H$ that $x(s)$ is strictly increasing and strictly convex on $(0, +\infty)$, a contradiction since $x(s)$ is bounded. Call $x_0 = x(s_1)$, $t_1 = t(s_1)$ and $\sigma_1 = \sigma(s_1)$. The definition of s_1 implies that $\sigma \in (0, \pi/2)$ and $\sigma' > 0$ on $(0, s_1)$. As a consequence

(a) The graph $x(t)$ is strictly increasing and strictly convex on $(0, t_1)$.

On the other hand, by using (5.2), (5.3) and $(*)_H$, we have

$$\sigma' = \left[\frac{\cos^3\sigma}{x^3\,(E+Hx^{2n})^3}\right]\,p(x), \tag{5.6}$$

where p is the polynomial given by $p(y) = (E + Hy^{2n})^3 - 2Hy^{6n-2} + 2(n-1)Ey^{4n-2}$. As consequence $p(x_1) > 0$ since $\sigma'(0) > 0$. Moreover, $\sigma'(s) = 0$ if and only if $p(x(s)) = 0$. It is not difficult to see that $p'(y) \neq 0$ for any $y \in [x_1, x_2]$ with $p(y) = 0$. Hence, $p'(x_0) < 0$.

It is clear that $x'(s_1) = \sin\sigma_1 > 0$. In particular, $x(s) \neq x_0$ for any $s \neq s_1$ close enough to s_1. A straightforward computation shows that

$$\sigma''(s_1) = \left[\frac{\sin\sigma_1 \, \cos^3\sigma_1}{x_0^3\,(E + Hx_0^{2n})^3} \right] p'(x_0) < 0.$$

We deduce by using (5.5) that $x(t)$ is strictly concave on a small interval to the right of t_1.

Now, we claim the existence of a first value $s_2 > s_1$ such that $\sigma(s_2) = 0$. Otherwise, $\sigma(s) \in (0, \pi/2)$ for any $s > s_1$, and so $x(s)$ would be strictly increasing and strictly concave on $(s_1, +\infty)$. Therefore, we would have $x(s) \to x_2$, $x'(s) \to 0$ and $\sigma(s) \to 0$ when $s \to +\infty$. By taking into account (5.6) and the fact that $p(x_2) \neq 0$ we would obtain $\lim_{s\to+\infty} \sigma'(s) \neq 0$, a contradiction. Clearly $x(s_2) = x_2$ by equation (5.2); moreover, $\sigma'(s_2) < 0$ and $p(x_2) < 0$. Now, we can prove that x_0 is the unique zero of $p(y)$ in $[x_1, x_2]$. By using Descartes' criterion to count the number of real roots of a polynomial, we get that $p(y)$ has at most two positive real roots (there are only two changes of sign in the sequence of coefficients of $p(y)$). But only one positive root is possible since $p(x_1) > 0$, $p(x_2) < 0$, and $p'(y) \neq 0$ for any $y \in [x_1, x_2]$ with $p(y) = 0$. Call $t_2 = t(s_2)$. By definition of s_2 and the arguments above it follows that $\sigma \in (0, \pi/2)$ and $\sigma' < 0$ on (s_1, s_2). In particular

(b) The graph $x(t)$ is strictly increasing and strictly concave on (t_1, t_2).

As $x'(s_2) = 0$ we deduce that the full solution is obtained by successive reflections across the lines $\{t = k\,t_2\}$ for k an entire. Conclusions (a) and (b) above show that the generated surface Σ is similar to an unduloid of $\mathbb{R}^{2n+1}$.

• $H > 0$, $E < 0$. In this case we deduce from (5.2) that the sign of $\cos\sigma$ and $E + Hx^{2n}$ is the same. In particular, $\cos\sigma = 0$ if and only if $x = x_0 = (-E/H)^{1/2n}$. By equation (5.4) we have $|E + Hx^{2n}| - x^{2n-1} \leqslant 0$ on I. It follows that $x(s)$ is bounded and the solution does not approach the t-axis. Call $x_1 = \inf x(s)$ and $x_2 = \sup x(s)$. These values are positive zeroes of the function $|E + Hy^{2n}| - y^{2n-1}$. The case $x_1 = x_2$ is not possible; otherwise, the solution would coincide with a cylinder. By using Descartes' criterion we deduce that the polynomials $-Hy^{2n} - y^{2n-1} - E$ and $Hy^{2n} - y^{2n-1} + E$ have at most a positive real root. This implies that $x_0 \in (x_1, x_2)$.

On the other hand, by taking into account (5.2) and $(*)_H$, we obtain

$$\sigma' = \frac{\psi}{x^3}\, p(x), \tag{5.7}$$

where $p(y) = (E + Hy^{2n})^3 - 2Hy^{6n-2} + 2(n-1)Ey^{4n-2}$, and ψ is the continuous, positive function defined by

$$\psi(s) = \begin{cases} \dfrac{\cos^3\sigma}{(E + Hx^{2n})^3} & \text{if } x(s) \neq x_0, \\ x_0^{6-6n} & \text{if } x(s) = x_0. \end{cases}$$

It is easy to check that $p(x_1) < 0$ and $p'(y) \leqslant 0$ for $y \in [x_1, x_2]$. We deduce from (5.7) that $\sigma' < 0$ along the solution. After a translation along the t-axis we can suppose that the initial conditions of γ are $(x_2, 0, 0)$. By the symmetry of the solutions we only have to study the behaviour of $\gamma(s)$ for $s > 0$. As $\sigma' < 0$, we get $\sigma < 0$ on $(0, +\infty)$. The fact that $\cos\sigma = 1$ implies that the x-coordinate of γ is a function of t around the origin. By $(*)_H$ and (5.5) it follows that $x(s)$ and $x(t)$ are strictly decreasing and strictly concave on small intervals to the right of the origin.

Now, the same arguments we used in the case $H > 0$, $E > 0$ ensure the existence of a first $s_1 > 0$ and a first $s_2 > s_1$ such that $\sigma(s_1) = -\pi/2$ and $\sigma(s_2) = -\pi$. By (5.2) we see that $x(s_1) = x_0$ and $x(s_2) = x_1$. Call $t_1 = t(s_1)$ and $t_2 = t(s_2)$. By the definition of s_1 and s_2 we have that $\sigma \in (-\pi/2, 0)$ on $(0, s_1)$ and $\sigma \in (-\pi, -\pi/2)$ on (s_1, s_2). As a consequence, the restriction of γ to $[0, s_2]$ consists of two graphs of the function $x(t)$ meeting at $t = t_1$. Moreover, by (5.5) we deduce that $x(t)$ is strictly decreasing and strictly concave on $(0, t_1)$, while it is strictly decreasing and strictly convex on (t_2, t_1). As $\{t = 0\}$ and $\{t = t_2\}$ are lines of symmetry for γ, we can reflect successively to obtain the complete solution, which is periodic. The resulting curve is embedded if and only if $t_2 = 0$; in this case, the generated Σ would be compact and homeomorphic to a torus.

Let us see that Σ has self-intersections. The fact that $\sin\sigma < 0$ on $(0, s_2)$ implies that we can see the t-coordinate of γ as a function of $x \in (x_1, x_2)$. This function satisfies $dt/dx = \cot\sigma$ by (5.5). An explicit expression for $\cot\sigma$ is obtained by using (5.3). It follows that

$$t_2 = \int_{x_1}^{x_2} \frac{(E + Hx^{2n})\, x}{\sqrt{x^{4n-2} - (E + Hx^{2n})^2}}\, dx.$$

Consider the Riemann surface R associated to the polynomial $w^2 = x^{4n-2} - (E + Hx^{2n})^2$. We may consider a lift α to R of a Jordan curve $\tilde{\alpha}$ in the xt-plane around the interval $[x_1, x_2]$ so that the only zeroes of the polynomial $x^{4n-2} - (E + Hx^{2n})^2$ in the interior of $\tilde{\alpha}$ are x_1, x_2. Hence we have

$$t_2 = \frac{1}{2} \int_\alpha \frac{(E + Hx^{2n})\, x}{w}\, dx.$$

The function $x^{-2n+2} w$ is holomorphic in a neighbourhood of α. A direct computation yields

$$d(x^{-2n+2} w) = \Big(2(n-1)\, x^{-2n+1} (E + Hx^{2n})^2 + x^{2n-1} - 2nH\, (E + Hx^{2n})\, x \Big)\, \frac{dx}{w},$$

so that

$$t_2 = \frac{1}{4nH} \int_\alpha \Big(2(n-1)\, x^{-2n+1} (E + Hx^{2n})^2 + x^{2n-1} \Big)\, \frac{dx}{w}.$$

Since the last integrand is an holomorphic one-form in a neighbourhood of α, we finally get

$$t_2 = \frac{1}{2nH} \int_{x_1}^{x_2} \frac{2(n-1)\, x^{-2n+1}\, (E + Hx^{2n})^2 + x^{2n-1}}{\sqrt{x^{4n-2} - (E + Hx^{2n})^2}}\, dx,$$

which is strictly positive. This allows us to conclude that the generated hypersurface Σ is similar to a Euclidean nodoid. □

Y. Ni [**Ni**] obtained the surfaces in Theorem 5.4, in the particular case of $\mathbb{H}^1$, by an approximation argument. Theorem 5.4 provides not only new examples of complete, embedded hypersurfaces in $\mathbb{H}^n$, $n \neq 1$, with non-zero constant mean curvature, but also the following consequences

Corollary 5.6. *The only C^2 minimal hypersurfaces of revolution in $\mathbb{H}^n$ are Euclidean hyperplanes orthogonal to the t-axis and catenoidal type hypersurfaces.*

Corollary 5.7. *The only C^2 compact, embedded, rotationally hypersurfaces of constant mean curvature in $\mathbb{H}^n$ are the spheres $\{S_H\}_{H>0}$ described in Example 4.2.*

Remark 5.8. Consider the isoperimetric problem in $\mathbb{H}^n$ which consists of finding a minimum for the perimeter functional $\mathcal{P}(\cdot)$ in the class of sets in $\mathbb{H}^n$ enclosing a fixed volume. P. Pansu conjectured in [**P2**] that the minimizers must be congruent to the spherical surfaces S_H in Example 4.2. It was proved by G. Leonardi and S. Rigot [**LR**] that the solutions to this problem exist. Moreover, isoperimetric sets are bounded, connected and satisfy a certain separation property, see [**LR**, Theorem 2.11]. The surfaces S_H also appeared in [**M**] and [**LM**] as the solutions of the restricted isoperimetric problem in $\mathbb{H}^n$ consisting of finding, for fixed volume, a minimum of the perimeter functional $\mathcal{P}(\cdot)$ in the class of sets in $\mathbb{H}^n$ bounded by two symmetric radial graphs over the hyperplane $\{t = 0\}$. Recently, D. Danielli, N. Garofalo and D. Nhieu [**DGN**, Thm. 14.6] have proved that they are also solutions of the isoperimetric problem restricted to a wider class of sets with cylindrical symmetry. We may expect that the solutions were C^2 and rotationally invariant about the t-axis, up to a left translation. In case this was proved, then Corollary 5.7 would show that the solutions to the isoperimetric problem are congruent to the family of spheres $\{S_H\}_{H>0}$ given in Example 4.2.

References

[Ba] Zoltán M. Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math. **564** (2003), 63–83. MR 2021034

[BdCE] J. Lucas Barbosa, Manfredo do Carmo and Jost Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), no. 1, 123–138. MR **88m:**53109

[BoC] Mario Bonk and Luca Capogna, *Horizontal mean curvature flow in the Heisenberg group*, in preparation.

[CDG] Luca Capogna, Donatella Danielli and Nicola Garofalo, *An isoperimetric inequality and the geometric Sobolev embedding for vector fields*, Math. Res. Lett. **1** (1994), no. 2, 203–215. MR **95a:**46048

[CH] Jih-Hsin Cheng, Jenn-Fang Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Austral. Math. Soc. **70** (2004), no 3, 507–520, MR 2103983

[CHMY] Jih-Hsin Cheng, Jenn-Fang Hwang, Andrea Malchiodi and Paul Yang *Minimal surfaces in pseudohermitian geometry*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 1, 129–177. MR 2165405

[CHY] Jih-Hsin Cheng, Jenn-Fang Hwang and Paul Yang, *Existence and uniqueness for p-area minimizers in the Heisenberg group*, arXiv:math.DG/0601208.

[DGN] Donatella Danielli, Nicola Garofalo and Duy-Minh Nhieu, *Minimal surfaces, surfaces of constant mean curvature and isoperimetry in sub-Riemannian groups*, preprint 2004.

[D] C. Delaunay, *Sur la surface of revolution dont la courbure moyenne est constante*, J. Math. Pure Appl., **16** (1841), 309–321.

[De1] M. Derridj, *Un probléme aux limites pour une classe d'opérateurs du second ordre hypoelliptiques*, Ann. Inst. Fourier, Grenoble, 21, **4** (1971), 99–148. MR 0601055

[De2] ———, *Sur un thórème de traces*, Ann. Inst. Fourier, Grenoble, 22, **2** (1972), 73–83. MR 0343011

[FMP] Christiam B. Figueroa, Francesco Mercuri and Renato N. L. Pedrosa, *Invariant surfaces of the Heisenberg groups*, Ann. Mat. Pura Appl. (4) **177** (1999), 173–194, MR 1747630

[FSSC] Bruno Franchi, Raul Serapioni and Francesco Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), no. 3, 479–531. MR **2003g:**49062

[GP] Nicola Garofalo and Scott Pauls, *The Bernstein problem in the Heisenberg group*, preprint 2004.

[GiH] Mariano Giaquinta and Stefan Hildebrandt, *Calculus of Variations I, II*, Grundlehren der Mathematischen Wissenschaften, no. 310, 311, Springer-Verlag, Berlin, 1996. MR **98b:**49002a, MR **98b:**49002b

[Gr] Michael Gromov, *Structures métriques pour les variétés riemanniennes*, volume 1 of Textes Mathématiques, CEDIC, Paris (1981). MR **2000d:**53065

[Gr2] Michael Gromov, *Carnot-Carathéodory spaces seen from within*, Sub-riemannian geometry, Prog. Math., vol 144, Birkhäuser, Basel, 1996, 79–323. MR **2000f:**53034

[HP] Robert K. Hladky and Scott Pauls, *Constant mean curvature surfaces in sub-Riemannian geometry*, arXiv:math.DG/059636.

[H] Wu-Yi Hsiang, *On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*, Duke Math. J. **49** (1982), no. 3, 485–496. MR 672494

[KKS] Nicholas J. Korevaar, Rob Kusner and Bruce Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), no. 2, 465–503. MR 1010168

[LM] Gian Paolo Leonardi and Simon Masnou, *On the isoperimetric problem in the Heisenberg group* $\mathbb{H}^n$, Ann. Mat. Pura Appl. (4) **184** (2005), no. 4, 533–553. MR 2177813

[LR] Gian Paolo Leonardi and Séverine Rigot, *Isoperimetric sets on Carnot groups*. Houston J. Math. **29** (2003), no. 3, pp. 609–637 (electronic). MR **2004d:**28008

[M] Roberto Monti, *Brunn-Minkowski and isoperimetric inequality in the Heisenberg group*, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 99–109. MR **2004c:**28021

[MSC] Roberto Monti and Francesco Serra Cassano, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. **13** (2001), 339–376. MR **2002j:**49052

[Ni] Yilong Ni, *Sub-Riemannian constant mean curvature surfaces in the Heisenberg group as limits*, Ann. Mat. Pura Appl. (4) **183** (2004), no. 4, 555–570. MR 2140530

[P] Pierre Pansu, *Une inégalité isopérimétrique sur le groupe de Heisenberg*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 2, 127–130. MR 676380.

[P2] ———, *An isoperimetric inequality on the Heisenberg group*, Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159–174 (1984). Conference on differential geometry on homogeneous spaces (Turin, 1983).

[P3] ———, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60. MR **90e:**53058

[Pa] Scott D. Pauls, *Minimal surfaces in the Heisenberg group*, Geom. Dedicata **104** (2004), 201–231. MR 2043961

[T] Per Tomter, *Constant mean curvature surfaces in the Heisenberg group*, Proc. Sympos. Pure Math. **54** (1993), 485–495. MR **94a:**53098

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE GRANADA, E–18071 GRANADA, ESPAÑA
E-mail address: ritore@ugr.es

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE GRANADA, E–18071 GRANADA, ESPAÑA
E-mail address: crosales@ugr.es