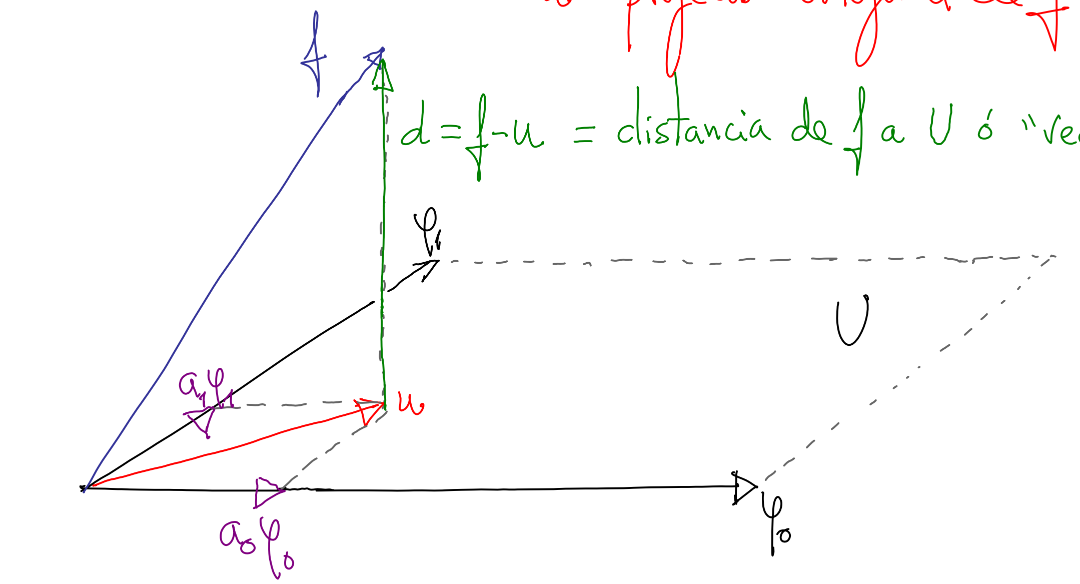
MÍNIMOS CUADRADOS

Título de la nota 21/03/2011

A) INTERPRETACIÓN GEOMÉTRICA.

u = proyección ortogonal de f sobre U

$d = f - u$ = distancia de f a U ó "vector error"



$$f = u + d = a_0\varphi_0 + a_1\varphi_1 + d \qquad d \in U^{\perp}$$

$$\begin{rcases}\langle f | \varphi_0 \rangle = a_0 \langle \varphi_0 | \varphi_0 \rangle + a_1 \langle \varphi_1 | \varphi_0 \rangle + \cancel{\langle d | \varphi_0 \rangle}^{\,0} \\ \langle f | \varphi_1 \rangle = a_0 \langle \varphi_0 | \varphi_1 \rangle + a_1 \langle \varphi_1 | \varphi_1 \rangle + \cancel{\langle d | \varphi_1 \rangle}^{\,0}\end{rcases} \text{Ecuaciones normales}$$

Forma matricial:

$$\underbrace{\begin{pmatrix} \langle f|\varphi_0\rangle \\ \langle f|\varphi_1\rangle \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} \langle \varphi_0|\varphi_0\rangle & \langle \varphi_1|\varphi_0\rangle \\ \langle \varphi_0|\varphi_1\rangle & \langle \varphi_1|\varphi_1\rangle \end{pmatrix}}_{G\ :\ \text{matriz de Gram}} \overbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}^{a}$$

$$b = G \cdot a$$

$$\boxed{a = G^{-1} b}$$

B) BASES ORTOGONALES

base ortogonal

Supongamos que $B(U)=\{\tilde{\varphi}_0,\tilde{\varphi}_1\}$ base con $\langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle=0$

$$\langle f|\tilde{\varphi}_0\rangle = a_0\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle + a_1\cancelto{0}{\langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle} + \cancelto{0}{\langle d|\tilde{\varphi}_0\rangle} \Rightarrow a_0=\frac{\langle f|\tilde{\varphi}_0\rangle}{\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle}$$

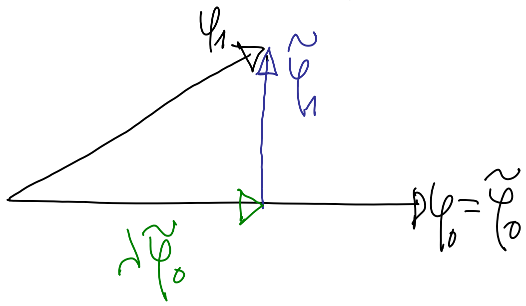
$$\langle f|\tilde{\varphi}_1\rangle = a_0\cancelto{0}{\langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle} + a_1\langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle + \cancelto{0}{\langle d|\tilde{\varphi}_1\rangle} \Rightarrow a_1=\frac{\langle f|\tilde{\varphi}_1\rangle}{\langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle}$$

$$u=a_0\tilde{\varphi}_0+a_1\tilde{\varphi}_1=\frac{\langle f|\tilde{\varphi}_0\rangle}{\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle}\tilde{\varphi}_0+\frac{\langle f|\tilde{\varphi}_1\rangle}{\langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle}\tilde{\varphi}_1$$

Coeficientes de Fourier

C) ALGORIMO DE GRAM-SCHMIDT.

Ejemplo 1: Dada $B=\{\varphi_0,\varphi_1\}$ una base de U, encontrar $\tilde{B}=\{\tilde{\varphi}_0,\tilde{\varphi}_1\}$ base de U con $\langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle=0 \Rightarrow$ ortogonal

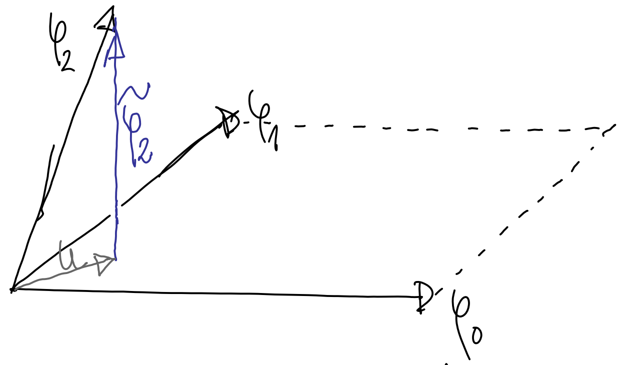


$\tilde{\varphi}_1=\varphi_1-\alpha\tilde{\varphi}_0$; multiplicando por $\tilde{\varphi}_0$:

$$0=\langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle=\langle\varphi_1|\tilde{\varphi}_0\rangle-\alpha\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle$$

$$\Rightarrow \alpha=\frac{\langle\varphi_1|\tilde{\varphi}_0\rangle}{\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle} \Rightarrow \boxed{\tilde{\varphi}_1=\varphi_1-\frac{\langle\varphi_1|\tilde{\varphi}_0\rangle}{\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle}\tilde{\varphi}_0}$$

Ejemplo 2 Dada $B = \{\varphi_0, \varphi_1, \varphi_2\}$ base de U, encontrar $\tilde{B} = \{\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2\}$ base de U con $\langle \tilde{\varphi}_i | \tilde{\varphi}_j \rangle = 0$ si $i \neq j$



$$\tilde{\varphi}_0 = \varphi_0$$

$$\tilde{\varphi}_1 = \varphi_1 - \frac{\langle \varphi_1 | \tilde{\varphi}_0 \rangle}{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle} \tilde{\varphi}_0$$

$$\tilde{\varphi}_2 = \varphi_2 - u = \varphi_2 - \frac{\langle \varphi_2 | \tilde{\varphi}_0 \rangle}{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle} \tilde{\varphi}_0 - \frac{\langle \varphi_2 | \tilde{\varphi}_1 \rangle}{\langle \tilde{\varphi}_1 | \tilde{\varphi}_1 \rangle} \tilde{\varphi}_1 \qquad (\text{"}\varphi_2 \longleftrightarrow f\text{"})$$

En general:

$$\tilde{\varphi}_n = \varphi_n - \sum_{k=0}^{n-1} \frac{\langle \varphi_n | \tilde{\varphi}_k \rangle}{\langle \tilde{\varphi}_k | \tilde{\varphi}_k \rangle} \tilde{\varphi}_k$$

MINIMOS CUADRADOS CONTINUO

Ejemplo Aproximar $f(x)=e^x$ en $U=P_2=Lin\{\varphi_0(x)=1, \varphi_1(x)=x, \varphi_2(x)=x^2\}$

Con producto escalar $\langle f|g\rangle=\int_{-1}^{1} f(x)g(x)\,dx$.

$\varphi_j(x)=x^j$

$j=0,1,2$

$$f=u+d=a_0\varphi_0+a_1\varphi_1+a_2\varphi_2+d$$

$$b_j=\langle f|\varphi_j\rangle=a_0\langle\varphi_0|\varphi_j\rangle+a_1\langle\varphi_1|\varphi_j\rangle+a_2\langle\varphi_2|\varphi_j\rangle+\cancelto{0}{\langle d|\varphi_j\rangle}$$

$$b_j=\langle f|\varphi_j\rangle=\int_{-1}^{1} f(x)\varphi_j(x)\,dx=\int_{-1}^{1} e^x x^j\,dx$$

$$b_0=\int_{-1}^{1} e^x\cdot 1\,dx=\left[e^x\right]_{-1}^{1}=e^1-e^{-1}=e-\frac{1}{e}$$

$$b_1=\int_{-1}^{1} e^x\cdot x\,dx=\left|\begin{matrix} u=x, & du=1dx \\ dv=e^x dx, & v=e^x\end{matrix}\right|=\left[xe^x\right]_{-1}^{1}-\underbrace{\int_{-1}^{1} e^x dx}_{b_0}=\cancel{e}+e^{-1}-(\cancel{e}-e^{-1})=\frac{2}{e}$$

$$b_2=\int_{-1}^{1} e^x\cdot x^2\,dx=\left|\begin{matrix} u=x^2, & du=2x\,dx \\ dv=e^x dx, & v=e^x\end{matrix}\right|=\left[x^2e^x\right]_{-1}^{1}-2\underbrace{\int x e^x dx}_{b_1}=e-\frac{1}{e}-\frac{4}{e}=e-\frac{5}{e}$$

$$\langle\varphi_i|\varphi_j\rangle=\int_{-1}^{1}\varphi_i(x)\varphi_j(x)\,dx=\int_{-1}^{1} x^i x^j\,dx=\int_{-1}^{1} x^{i+j}\,dx=\left[\frac{x^{i+j+1}}{i+j+1}\right]_{-1}^{1}$$

$$G=\begin{pmatrix}\langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle & \langle\varphi_0|\varphi_2\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_2|\varphi_0\rangle & \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle\end{pmatrix}=\begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5}\end{pmatrix}$$

$$\overbrace{\begin{pmatrix} e-\frac{1}{e} \\ \frac{2}{e} \\ e-\frac{5}{e} \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}}^{G} \overbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}}^{a} \Rightarrow \quad a = G^{-1} b \Rightarrow$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{8} & 0 & \frac{45}{8} \end{pmatrix} \begin{pmatrix} e-\frac{1}{e} \\ \frac{2}{e} \\ e-\frac{5}{e} \end{pmatrix} \simeq \begin{pmatrix} 0'9963 \\ 1'104 \\ 0'5367 \end{pmatrix}$$

$$u(x) \simeq 0'9963 \cdot 1 + 1'104 \cdot x + 0'5367\, x^2 \approx f(x) = e^x \quad \text{para } x \in [-1,1]$$

<u>Ejercicio</u> Repetir el ejercicio anterior utilizando una base ortogonal $\tilde{B}(U) = \{\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2\}$ obtenida a partir de $B(U) = \{\varphi_0, \varphi_1, \varphi_2\}$ por el procedimiento de Gram-Schmidt.

$$\tilde{\varphi}_0 = \varphi_0 \;,\; \tilde{\varphi}_1 = \varphi_1 - \frac{\cancel{\langle \varphi_1 | \tilde{\varphi}_0 \rangle}^{\,0}}{\underbrace{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle}_{2}} \tilde{\varphi}_0 = \varphi_1$$

$\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2$: <u>polinomios de Legendre</u> (salvo normalización)

$$\tilde{\varphi}_2 = \varphi_2 - \frac{\overbrace{\langle \varphi_2 | \tilde{\varphi}_0 \rangle}^{2/3}}{\underbrace{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle}_{2}} \tilde{\varphi}_0 - \frac{\cancel{\langle \varphi_2 | \tilde{\varphi}_1 \rangle}^{\,0}}{\langle \tilde{\varphi}_1 | \tilde{\varphi}_1 \rangle} \tilde{\varphi}_1 = \varphi_2 - \frac{1}{3}\varphi_0 \;,\; \boxed{\tilde{\varphi}_2(x) = x^2 - \frac{1}{3}}$$

$$u(x) = \tilde{a}_0 \tilde{\varphi}_0 + \tilde{a}_1 \tilde{\varphi}_1 + \tilde{a}_2 \tilde{\varphi}_2$$

$$\tilde{a}_0 = \frac{\langle f | \tilde{\varphi}_0 \rangle}{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle} = e - e^{-1} \simeq 1'1752 \;,\quad \tilde{a}_1 = \frac{\langle f | \tilde{\varphi}_1 \rangle}{\langle \tilde{\varphi}_1 | \tilde{\varphi}_1 \rangle} = \frac{2/e}{2/3} \simeq 1'1036$$

$$\tilde{a}_2 = \frac{\langle f|\tilde{\varphi}_2\rangle}{\langle \tilde{\varphi}_2|\tilde{\varphi}_2\rangle} = \frac{\langle f|\varphi_2 - \frac{1}{3}\varphi_0\rangle}{\langle \varphi_2 - \frac{1}{3}\varphi_0 | \varphi_2 - \frac{1}{3}\varphi_0\rangle} = \frac{\langle f|\varphi_2\rangle - \frac{1}{3}\langle f|\varphi_0\rangle}{\langle \varphi_2|\varphi_2\rangle - \frac{2}{3}\langle \varphi_2|\varphi_0\rangle + \frac{1}{9}\langle \varphi_0|\varphi_0\rangle} =$$

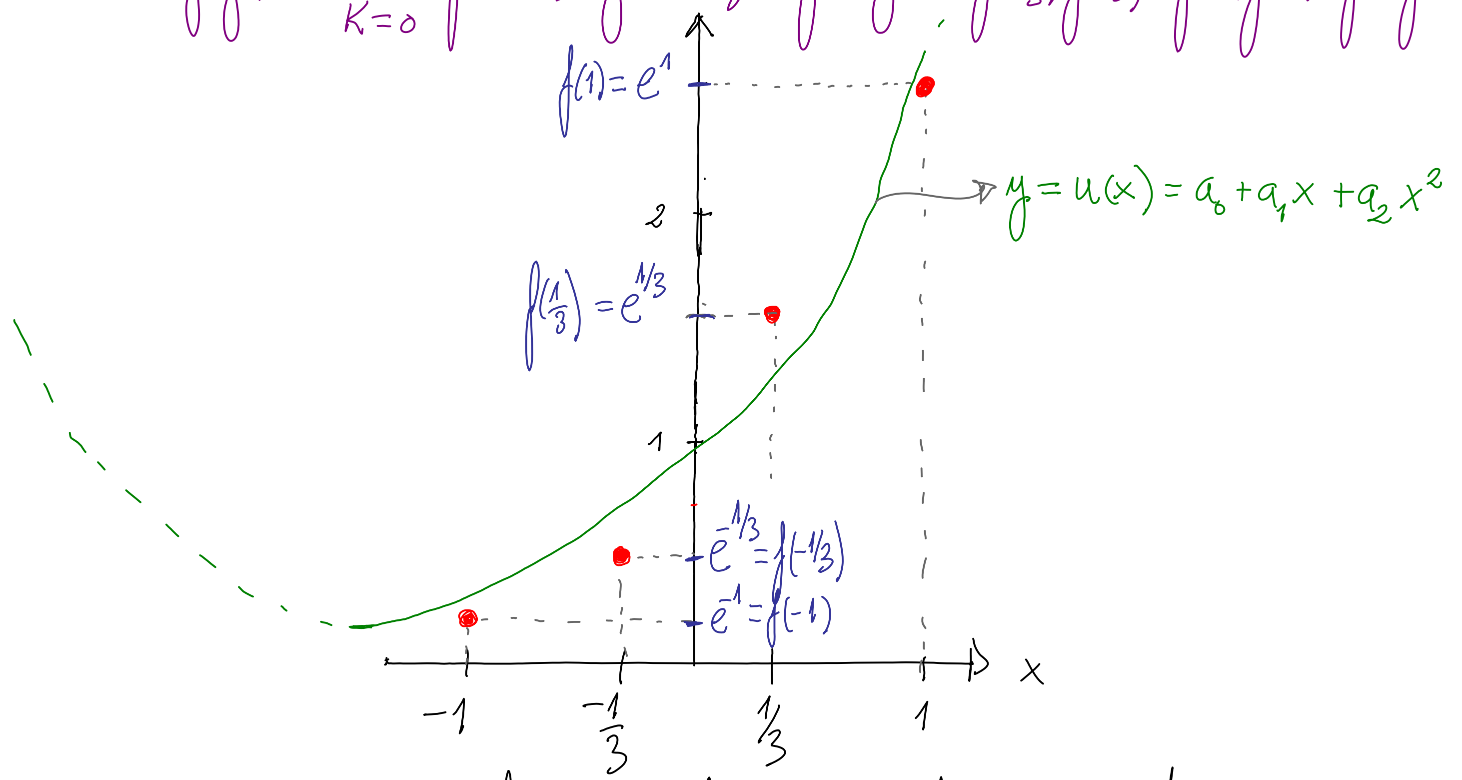
$$= \frac{e - \frac{5}{e} - \frac{1}{3}\left(e - \frac{1}{e}\right)}{\frac{2}{5} - \frac{2}{3}\frac{2}{3} + \frac{1}{9}2} \simeq 0'5367$$

$$u(x) = \tilde{a}_0\tilde{\varphi}_0(x) + \tilde{a}_1\tilde{\varphi}_1(x) + \tilde{a}_2\tilde{\varphi}_2(x) = 1'1752 \cdot 1 + 1'1036x + 0'5367\left(x^2 - \frac{1}{3}\right)$$

MÍNIMOS CUADRADOS DISCRETO

Ejemplo Aproximar e^x en P_2 con producto escalar

$$\langle f|g\rangle = \sum_{k=0}^{3} f\left(-1+\frac{2k}{3}\right) g\left(-1+\frac{2k}{3}\right) = f(-1)g(-1) + f\left(-\frac{1}{3}\right)g\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right)g\left(\frac{1}{3}\right) + f(1)g(1)$$



$$b_0 = \langle f|\varphi_0\rangle = f(-1)\varphi_0(-1) + f\left(-\frac{1}{3}\right)\varphi_0\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right)\varphi_0\left(\frac{1}{3}\right) + f(1)\varphi_0(1) =$$

$$= e^{-1} \cdot 1 + e^{-1/3} \cdot 1 + e^{1/3} \cdot 1 + e^{1} \cdot 1$$

$$b_1 = \langle f | \varphi_1 \rangle = f(-1)\varphi_1(-1) + f\left(-\tfrac{1}{3}\right)\varphi_1\left(-\tfrac{1}{3}\right) + f\left(\tfrac{1}{3}\right)\varphi_1\left(\tfrac{1}{3}\right) + f(1)\varphi_1(1) =$$

$$= e^{-1}\cdot(-1) + e^{-1/3}\cdot\left(-\tfrac{1}{3}\right) + e^{1/3}\cdot\left(\tfrac{1}{3}\right) + e^{1}\cdot(1)$$

$$b_2 = \langle f | \varphi_2 \rangle = f(-1)\varphi_2(-1) + f\left(-\tfrac{1}{3}\right)\varphi_2\left(-\tfrac{1}{3}\right) + f\left(\tfrac{1}{3}\right)\varphi_2\left(\tfrac{1}{3}\right) + f(1)\varphi_2(1) =$$

$$= e^{-1}\cdot(-1)^2 + e^{-1/3}\cdot\left(-\tfrac{1}{3}\right)^2 + e^{1/3}\cdot\left(\tfrac{1}{3}\right)^2 + e^{1}\cdot(1)^2$$

$$G = \begin{pmatrix} \overbrace{\langle \varphi_0 | \varphi_0 \rangle}^{1\cdot1+1\cdot1+1\cdot1+1\cdot1} & \langle \varphi_0 | \varphi_1 \rangle & \overbrace{\langle \varphi_0 | \varphi_2 \rangle}^{1\cdot(-1)^2+1\cdot(-\frac{1}{3})^2+1\cdot(\frac{1}{3})^2+1\cdot(1)^2} \\ \langle \varphi_1 | \varphi_0 \rangle & \langle \varphi_1 | \varphi_1 \rangle & \langle \varphi_1 | \varphi_2 \rangle \\ \langle \varphi_2 | \varphi_0 \rangle & \langle \varphi_2 | \varphi_1 \rangle & \langle \varphi_2 | \varphi_2 \rangle \end{pmatrix} = \begin{pmatrix} 4 & 0 & \frac{20}{9} \\ 0 & \frac{20}{9} & 0 \\ \frac{20}{9} & 0 & \frac{164}{81} \end{pmatrix}$$

$$G^{-1} = \begin{pmatrix} \frac{41}{64} & 0 & -\frac{45}{64} \\ 0 & \frac{9}{20} & 0 \\ -\frac{45}{64} & 0 & \frac{81}{64} \end{pmatrix} \Rightarrow a = G^{-1}\cdot b \simeq \begin{pmatrix} 0'9952\ldots \\ 1'1595\ldots \\ 0'5479\ldots \end{pmatrix}$$

$$u(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x) \simeq 0'9952 + 1'1595x + 0'5479x^2$$

<u>Ejercicio</u> Repetir el ejemplo anterior usando una base ortonormal obtenida a partir de $B(P_2)=\{1,x,x^2\}$ por Gram-Schmidt.

$$\tilde{\varphi}_0 = \varphi_0\,,\quad \tilde{\varphi}_1 = \varphi_1 - \frac{\cancelto{0}{\overbrace{\langle \varphi_1 | \tilde{\varphi}_0 \rangle}^{-1\cdot1-\frac{1}{3}\cdot1+\frac{1}{3}\cdot1+1\cdot1=0}}}{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle}\tilde{\varphi}_0 = \varphi_1$$

$$\tilde{\varphi}_2 = \varphi_2 - \frac{\overbrace{\langle \varphi_2 | \tilde{\varphi}_0 \rangle}^{20/9}}{\underbrace{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle}_{4}}\tilde{\varphi}_0 - \frac{\cancelto{0}{\langle \varphi_2 | \tilde{\varphi}_1 \rangle}}{\langle \tilde{\varphi}_1 | \tilde{\varphi}_1 \rangle}\tilde{\varphi}_1 = \varphi_2 - \frac{5}{9}\varphi_0,\quad \tilde{\varphi}_2(x) = x^2 - \frac{5}{9}$$

$$u(x) = \tilde{a}_0 \tilde{\varphi}_0(x) + \tilde{a}_1 \tilde{\varphi}_1(x) + \tilde{a}_2 \tilde{\varphi}_2$$

$$\tilde{a}_0 = \frac{\langle f | \tilde{\varphi}_0 \rangle}{\langle \tilde{\varphi}_0 | \tilde{\varphi}_0 \rangle} = \frac{f(-1)\tilde{\varphi}_0(-1) + f(-\frac{1}{3})\tilde{\varphi}_0(-\frac{1}{3}) + f(\frac{1}{3})\tilde{\varphi}_0(\frac{1}{3}) + f(1)\tilde{\varphi}(1)}{4} = \frac{e^{-1} + e^{-1/3} + e^{1/3} + e}{4} \simeq 1'2996\ldots$$

$$\tilde{a}_1 = \frac{\langle f | \tilde{\varphi}_1 \rangle}{\langle \tilde{\varphi}_1 | \tilde{\varphi}_1 \rangle} = \frac{e^{-1} \cdot (-1) + e^{-1/3} \cdot (-\frac{1}{3}) + e^{1/3} \cdot (\frac{1}{3}) + e^{1} \cdot (1)}{20/9} \simeq 1'1595\ldots$$

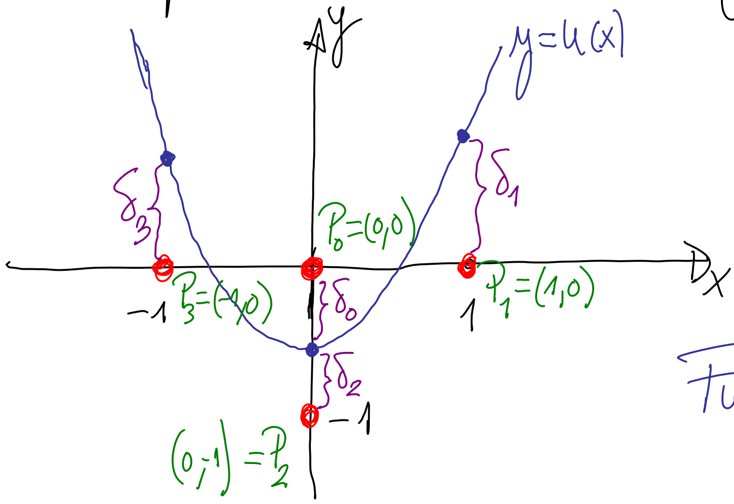
$$\tilde{a}_2 = \frac{\langle f | \tilde{\varphi}_2 \rangle}{\langle \tilde{\varphi}_2 | \tilde{\varphi}_2 \rangle} = \frac{\langle f | \varphi_2 - \frac{5}{9}\varphi_0 \rangle}{\langle \varphi_2 - \frac{5}{9}\varphi_0 | \varphi_2 - \frac{5}{9}\varphi_0 \rangle} = \frac{\langle f | \varphi_2 \rangle - \frac{5}{9}\langle f | \varphi_0 \rangle}{\langle \varphi_2 | \varphi_2 \rangle - \frac{10}{9}\langle \varphi_2 | \varphi_0 \rangle + \frac{25}{81}\langle \varphi_0 | \varphi_0 \rangle}$$

$$= \frac{e^{-1} \cdot (-1)^2 + e^{-1/3} \cdot (-\frac{1}{3})^2 + e^{1/3} \cdot (\frac{1}{3})^2 + e^{1} \cdot (1)^2 - \frac{5}{9}\left(e^{-1} + e^{-1/3} + e^{1/3} + e\right)}{\frac{164}{81} - \frac{10}{9}\frac{20}{9} + \frac{25}{81}4} \simeq 0'5479\ldots$$

$$u(x) \simeq 1'2996\,\tilde{\varphi}_0(x) + 1'1595\,\tilde{\varphi}_1(x) + 0'5479\,\tilde{\varphi}_2(x)$$

MINIMOS CUADRADOS DISCRETO COMO UN PROBLEMA DE EXTREMOS RELATIVOS

Ejemplo Encontrar la parábola $u(x) = a_0 + a_1 x + a_2 x^2$ que pasa más cerca de los siguientes 4 puntos $P_i = (x_i, y_i^*)$, $i = 1,2,3,4$



$$\delta_i = |u(x_i) - y_i^*|$$

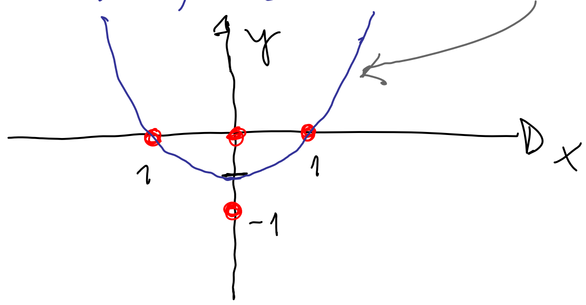
Función "distancia total":

$$f(a_0, a_1, a_2) = \delta_0^2 + \delta_1^2 + \delta_2^2 + \delta_3^2 = (u(0)-0)^2 + (u(1)-0)^2 + (u(0)-(-1))^2 + (u(-1)-0)^2 =$$

$$= (a_0)^2 + (a_0 + a_1 + a_2)^2 + (a_0 + 1)^2 + (a_0 - a_1 + a_2)^2$$

$$\left.\begin{aligned}
\frac{\partial f}{\partial a_0} &= 2a_0 + 2(a_0+a_1+a_2) + 2(a_0+1) + 2(a_0-a_1+a_2) = 4(a_2+2a_0)+2 = 0 \\
\frac{\partial f}{\partial a_1} &= 2(a_0+a_1+a_2) - 2(a_0-a_1+a_2) = 4a_1 = 0 \Rightarrow a_1 = 0 \\
\frac{\partial f}{\partial a_2} &= 2(a_0+a_1+a_2) + 2(a_0-a_1+a_2) = 4(a_0+a_2) = 0 \Rightarrow a_0 = -a_2
\end{aligned}\right\}$$

$$a_0 = -1/2, \quad a_1 = 0, \quad a_2 = 1/2 \Rightarrow u(x) = \frac{1}{2}x^2 - \frac{1}{2}$$



$\varphi_0(x)$ $\varphi_1(x)$

<u>Ejercicio</u> Aproximar $f(x)=e^x$ en $P_1 = \text{Lin}\{1, x\}$ de 2 formas:

1) Con producto escalar $\langle f|g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$

2) Como un problema de extremos relativos, calculando la recta que pasa más cerca de los 3 puntos:

$P_1 = (-1, f(-1))$, $P_2 = (0, f(0))$, $P_3 = (1, f(1))$

¿Obtenemos la misma solución, $u(x) = a_0 + a_1 x$, en los 2 casos?