Palatini versus metric formalism in higher-curvature gravity

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Outlook

1. The Levi-Civita connection: what and why?

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7. Conclusions

1 The Levi-Civita connection: what and why?

Main lessons of General Relativity:

Gravitational interaction is related to curvature of spacetime,
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Metric: measures distances in manifold and angles in tangent space

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Mathematically metric and connection are independent quantities

However if $\Gamma_{\mu\nu}^{\rho}$ satisfies the properties

- symmetric: $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\nu\mu}^{\rho} \qquad (\Leftrightarrow \; T_{\mu\nu}^{\rho} = 0)$
- metric compatible: $\nabla_{\mu} g_{\nu\rho} = 0$

then $\Gamma_{\mu\nu}^{\rho}$ is uniquely determined by the metric

$$\Gamma_{\mu\nu}^{\rho} = \tfrac{1}{2}\, g^{\rho\lambda}\Big(\partial_{\mu} g_{\lambda\nu} + \partial_{\mu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu}\Big)$$

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- uniqueness: see above
- simplicity: tensor identities simplify
- physical grounds: Equivalence principle & geodesics

Simplicity: many curvature tensor identities simplify for Levi-Civita

- Bianchi identity:

$$0 = R_{\mu\nu\rho}{}^{\lambda} + R_{\nu\rho\mu}{}^{\lambda} + R_{\rho\mu\nu}{}^{\lambda}$$
$$- \nabla_\mu T^\lambda_{\nu\rho} - \nabla_\nu T^\lambda_{\rho\mu} - \nabla_\rho T^\lambda_{\mu\nu} + T^\sigma_{\mu\nu} T^\lambda_{\rho\sigma} + T^\sigma_{\nu\rho} T^\lambda_{\mu\sigma} + T^\sigma_{\rho\mu} T^\lambda_{\nu\sigma}$$

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- Symmetry of Ricci tensor:

$$R_{[\mu\nu]} = \tfrac{1}{2} R_{\mu\nu\lambda}{}^{\lambda} - \tfrac{1}{2} \nabla_\lambda T_{\mu\nu}^{\lambda} - \nabla_{[\mu} T_{\nu]\lambda}^{\lambda} + \tfrac{1}{2} T_{\mu\nu}^{\rho} T_{\lambda\rho}^{\lambda}$$

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- Divergence of Einstein tensor:

$$\nabla_\mu G_\nu{}^{\mu} = T_{\mu\nu}^{\rho} R_\rho{}^{\mu} - \tfrac{1}{2} T_{\mu\rho}^{\lambda} R_{\nu\lambda}{}^{\mu\rho} + \tfrac{1}{2}\nabla_\rho g^{\mu\lambda}\Big[\delta_\mu^\rho R_{\nu\lambda} + \delta_\nu^\rho R_{\mu\lambda} + R_{\mu\nu\lambda}{}^{\rho}\Big]$$

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- Partial integration:

$$\begin{aligned} S &= \int d^Dx \sqrt{|g|} \, \nabla_\mu A_\nu \, B^{\mu\nu} \\ &= -\int d^Dx \sqrt{|g|} \, \Big[\nabla_\mu B^{\mu\nu} A_\nu + \Big(\tfrac{1}{2} g^{\sigma\tau} \nabla_\mu g_{\sigma\tau} + T^{\sigma}_{\mu\sigma} \Big) A_\nu B^{\mu\nu} \Big] \end{aligned}$$

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Affine geodesics and metric geodesics do not coincide,
unless for Levi-Civita

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- mathematically more rigorous reason? Palatini formalism!

2 Palatini formalism for Einstein-Hilbert

2.1 Metric formalism

$$S(g) = \int d^Dx \, \sqrt{|g|} \, g^{\mu\nu} R_{\mu\nu}(g)$$

where

$$\begin{aligned} R_{\mu\nu} &= \partial_\mu \Gamma^\lambda_{\lambda\nu} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} \\ \Gamma^\rho_{\mu\nu} &= \tfrac{1}{2} \, g^{\rho\lambda}\Big(\partial_\mu g_{\lambda\nu} + \partial_\mu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big) \end{aligned}$$

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- $S(g)$ non-linear and second order in $g_{\mu\nu}$
  $\longrightarrow$ expect fourth-order diff eqns
- explicit calculation: only $\sqrt{|g|}\, g^{\mu\nu}$ contributes to $\delta S(g)$

$$R_{\mu\nu}(g) - \tfrac{1}{2} g_{\mu\nu} R(g) = 0$$

2.2 Palatini formalism

Consider metric and connection as independent:

$$S(g,\Gamma) = \int d^Dx\, \sqrt{|g|}\, g^{\mu\nu} R_{\mu\nu}(\Gamma)$$

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$$0 \equiv \delta S(g,\Gamma) = \int d^Dx \sqrt{|g|}(\delta\Gamma^\lambda_{\mu\nu}) \Big[\nabla_\lambda g^{\mu\nu} + \Big(\tfrac{1}{2} g^{\sigma\tau}\nabla_\lambda g_{\sigma\tau} + T^\sigma_{\lambda\sigma}\Big) g^{\mu\nu} - \nabla_\rho g^{\rho\nu}\delta^\mu_\lambda - \Big(\tfrac{1}{2} g^{\sigma\tau}\nabla_\rho g_{\sigma\tau} + T^\sigma_{\rho\sigma}\Big) g^{\rho\nu}\delta^\mu_\lambda + g^{\rho\nu} T^\mu_{\rho\lambda} \Big]$$

$$\Longleftrightarrow \quad \Gamma^\lambda_{\mu\nu} = \text{Levi} - \text{Civita}$$

Advantages of Palatini formalism:

- Practical: Einstein eqn $\frac{\delta S}{\delta g^{\mu\nu}}$ easy to calculate if $R_{\mu\nu\rho}{}^{\lambda} = R_{\mu\nu\rho}{}^{\lambda}(\Gamma)$

$$S = \int d^D x \sqrt{|g|} \left[g^{\mu\nu} R_{\mu\nu}(\Gamma) + \mathcal{L}(\phi, g_{\mu\nu}) \right]$$

$$\longrightarrow \quad R_{\mu\nu}(\Gamma) - \tfrac{1}{2} g_{\mu\nu} R(\Gamma) = T_{\mu\nu}$$

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- Question: how much remains valid in presence of curvature corrections?

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- Lanczos (1938):

$$S = \int d^5x \sqrt{|g|} \left[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} \right]$$

$$H_{\mu\nu} = 2RR_{\mu\nu} + 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} + 2R_{\mu\rho\lambda\sigma}R_{\nu}{}^{\rho\lambda\sigma} - 4R_{\mu\rho}R_{\nu}{}^{\rho}$$
$$-\tfrac{1}{2}g_{\mu\nu}\Big[R^2 - 4R_{\rho\lambda}R^{\rho\lambda} + R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big]$$

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NB: $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ is Gauss-Bonnet term

$\longrightarrow$ topological invariant in $D = 4$
identically zero in $D < 4$
dynamical in $D > 4$

- Lovelock (1970): generalise Lanczos to arbitrary D
  second order, divergence-free Einstein eqns for

$$S = \int d^Dx \sqrt{|g|} \Big[\Lambda \, + \, R \, + \, \mathcal{L}_{\rm GB} \, + \, ... \, + \, \mathcal{L}_{[D/2]} \Big]$$

  where

$$\mathcal{L}_n = \det \begin{vmatrix} g^{\mu_1\nu_1} & \dots & g^{\mu_{2n}\nu_1} \\ \vdots & \ddots & \vdots \\ g^{\mu_1\nu_{2n}} & \dots & g^{\mu_{2n}\nu_{2n}} \end{vmatrix} R_{\mu_1\mu_2\nu_1\nu_2} \dots R_{\mu_{2n-1}\mu_{2n}\nu_{2n-1}\nu_{2n}}$$

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  $\mathcal{L}_n$ is Euler character in $D = 2n$

  $\longrightarrow$ identically zero in $D < 2n$
  topological invariant in $D = 2n$
  dynamical in $D > 2n$

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  [Exirifard, Sheikh-Jabbari] state that Palatini and metric are equivalent for Lovelock gravity
- We will demonstrate that metric includes Palatini, but not vice versa

4 Palatini vs metric formalism for (Riem)^2 gravity

4.1 Metric formalism

$$S = \int d^Dx\sqrt{|g|}\Big[\alpha R^2 + 4\beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\Big]$$

$$= \int d^Dx\sqrt{|g|}\Big[\alpha\, g^{\mu\rho}g^{\sigma\alpha}\delta^\nu_\lambda\delta^\tau_\beta + 4\beta\, g^{\mu\sigma}g^{\rho\alpha}\delta^\nu_\lambda\delta^\tau_\beta + \gamma\, g^{\mu\sigma}g^{\nu\tau}g^{\rho\alpha}g_{\lambda\beta}\Big]R_{\mu\nu\rho}{}^\lambda R_{\sigma\tau\alpha}{}^\beta$$

Gravitational tensor via chain rule

$$H_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}}\frac{\delta S(g)}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{|g|}}\frac{\delta S(g)}{\delta g^{\mu\nu}}\Big|_{\rm expl} + \frac{1}{\sqrt{|g|}}\frac{\delta S(g)}{\delta R_{\alpha\beta\gamma}{}^\delta}\,\frac{\delta R_{\alpha\beta\gamma}{}^\delta}{\delta\Gamma^\sigma_{\rho\lambda}}\,\frac{\delta\Gamma^\sigma_{\rho\lambda}}{\delta g^{\mu\nu}}$$

where

$$\delta R_{\mu\nu\rho}{}^\lambda = \nabla_\mu(\delta\Gamma^\lambda_{\nu\rho}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\rho}),$$

$$\delta\Gamma^\rho_{\mu\nu} = \tfrac{1}{2}g^{\rho\lambda}\Big[\nabla_\mu(\delta g_{\lambda\nu}) + \nabla_\nu(\delta g_{\mu\lambda}) - \nabla_\lambda(\delta g_{\mu\nu})\Big]$$

$$
\begin{aligned}
H_{\mu\nu} = \ & \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big] \\
& + 4\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big] \\
& + \gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} \\
& \qquad +2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big].
\end{aligned}
$$

$$
\begin{aligned}
H_{\mu\nu} = \ & \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big] \\
& + 4\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big] \\
& + \gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} \\
& \qquad +2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big].
\end{aligned}
$$

- ∇^2(Riem) terms cancel for $(\alpha,\beta,\gamma) = (1,-1,1)$
  $\longrightarrow$ Gauss-Bonnet is 2nd Lovelock term $\mathcal{L}_2$

$$
\begin{aligned}
H_{\mu\nu} = \ & \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big] \\
& + 4\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big] \\
& + \gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} \\
& \qquad + 2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big].
\end{aligned}
$$

- ∇^2(Riem) terms cancel for $(\alpha, \beta, \gamma) = (1, -1, 1)$
  $\longrightarrow$ Gauss-Bonnet is 2nd Lovelock term $\mathcal{L}_2$

- $\nabla_\mu H^{\mu\nu} = 0$ for arbitrary (α, β, γ)
  $\longrightarrow$ energy conservation generally valid

$$
\begin{aligned}
H_{\mu\nu} = \ & \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big] \\
& + 4\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big] \\
& + \gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} \\
& \qquad + 2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big].
\end{aligned}
$$

- ∇^2(Riem) terms cancel for $(\alpha, \beta, \gamma) = (1, -1, 1)$
  $\longrightarrow$ Gauss-Bonnet is 2nd Lovelock term $\mathcal{L}_2$

- $\nabla_\mu H^{\mu\nu} = 0$ for arbitrary (α, β, γ)
  $\longrightarrow$ energy conservation generally valid

- Compare $H_{\mu\nu} = T_{\mu\nu}$ to corresponding eqns of Palatini formalism

4.2 Palatini formalism

$$S(g,\Gamma) = \int d^Dx\sqrt{|g|}\Big[\alpha R^2 + \beta_1 R_{\mu\nu}R^{\nu\mu} + 2\beta_2 R_{\mu\nu}\tilde{R}^{\nu\mu} + \beta_3 \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}$$
$$+ \beta_4 R_{\mu\nu}R^{\mu\nu} + 2\beta_5 R_{\mu\nu}\tilde{R}^{\mu\nu} + \beta_6 \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$$
$$+ \gamma_1 R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu} + \gamma_2 R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} + \gamma_3 R_{\mu\nu\rho\lambda}R^{\mu\nu\lambda\rho}\Big]$$

with $\quad R_{\mu\rho} \equiv R_{\mu\lambda\rho}{}^{\lambda} \qquad \tilde{R}_{\mu}{}^{\lambda} \equiv g^{\nu\rho}R_{\mu\nu\rho}{}^{\lambda}$

4.2 Palatini formalism

$$S(g,\Gamma) \;=\; \int d^Dx\sqrt{|g|}\,\Big[\alpha R^2 \;+\; \beta_1 R_{\mu\nu}R^{\nu\mu} + 2\beta_2 R_{\mu\nu}\tilde{R}^{\nu\mu} + \beta_3 \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}$$
$$+\;\beta_4 R_{\mu\nu}R^{\mu\nu} + 2\beta_5 R_{\mu\nu}\tilde{R}^{\mu\nu} + \beta_6 \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$$
$$+\;\gamma_1 R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu} + \gamma_2 R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} + \gamma_3 R_{\mu\nu\rho\lambda}R^{\mu\nu\lambda\rho}\Big]$$

with

$$R_{\mu\rho} \equiv R_{\mu\lambda\rho}{}^{\lambda} \qquad\qquad \tilde{R}_{\mu}{}^{\lambda} \equiv g^{\nu\rho}R_{\mu\nu\rho}{}^{\lambda}$$

Gravitational tensor: $\tilde{H}_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}}\frac{\delta S(g,\Gamma)}{\delta g^{\mu\nu}}$

$$\tilde{H}_{\mu\nu} \;=\; 2\alpha R_{\mu\nu}R \;+\; \beta_1\big(R_{\mu\lambda}R^{\lambda}{}_{\nu} + R_{\lambda\mu}R_{\nu}{}^{\lambda}\big) \;-\; \beta_2\big(2R_{(\nu|\lambda\sigma|\mu)}R^{\lambda\sigma} + \tilde{R}^{\lambda}{}_{\mu}R_{\nu\lambda} + \tilde{R}^{\lambda}{}_{\nu}R_{\mu\lambda}\big)$$
$$+\,2\beta_3 R_{\lambda(\nu|\rho|\mu)}\tilde{R}^{\lambda\rho} \;+\; \beta_4\big(R_{\mu\lambda}R_{\nu}{}^{\lambda} + R_{\lambda\mu}R^{\lambda}{}_{\nu}\big) \;-\; \beta_5\big(2R_{(\nu|\lambda\sigma|\mu)}R^{\sigma\lambda} + \tilde{R}^{\lambda}{}_{\mu}R_{\lambda\nu} + \tilde{R}^{\lambda}{}_{\nu}R_{\lambda\mu}$$
$$+\,2\beta_6 R_{\lambda(\nu|\rho|\mu)}\tilde{R}^{\rho\lambda} \;+\; \gamma_1\big(R_{\rho\lambda\sigma\mu}R_{\nu}{}^{\sigma\lambda\rho} + R_{\rho\mu\sigma\lambda}R^{\sigma\lambda\rho}{}_{\nu}\big) \;+\; 2\gamma_3 R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho\lambda\sigma}$$
$$+\,\gamma_2\big(R_{\rho\lambda\sigma\mu}R^{\rho\lambda\sigma}{}_{\nu} \;+ 2R_{\mu\lambda\sigma\rho}R_{\nu}{}^{\lambda\sigma\rho} \;-\, R_{\rho\lambda\mu\sigma}R^{\rho\lambda}{}_{\nu}{}^{\sigma}\big) \;-\, \tfrac{1}{2}g_{\mu\nu}\mathcal{L}$$

Connection tensor:

$$K_\lambda^{\mu\nu} \;\equiv\; \frac{1}{\sqrt{|g|}}\frac{\delta S(g,\Gamma)}{\delta\Gamma_{\mu\nu}^\lambda} = 2\alpha\Big[g_{\mu\nu}\nabla^\lambda R - \delta_\nu^\lambda\nabla_\mu R\Big] \;+\; 2\beta_1\Big[\nabla^\lambda R_{\nu\mu} - \delta_\nu^\lambda\nabla^\sigma R_{\sigma\mu}\Big]$$

$$-2\beta_2\Big[-\nabla^\lambda\tilde{R}_{\nu\mu} + \delta_\nu^\lambda\nabla^\sigma\tilde{R}_{\sigma\mu} + g_{\mu\nu}\nabla_\sigma R^{\sigma\lambda} - \nabla_\mu R_\nu{}^\lambda\Big] \;+\; 2\beta_3\Big[\nabla_\mu\tilde{R}_\nu{}^\lambda - g_{\mu\nu}\nabla_\sigma\tilde{R}^{\sigma\lambda}\Big]$$

$$+2\beta_4\Big[\nabla^\lambda R_{\mu\nu} - \delta_\nu^\lambda\nabla^\sigma R_{\mu\sigma}\Big] \;-\; 2\beta_5\Big[-\nabla^\lambda\tilde{R}_{\mu\nu} + \delta_\nu^\lambda\nabla^\sigma\tilde{R}_{\mu\sigma} + g_{\mu\nu}\nabla_\sigma R^{\lambda\sigma} - \nabla_\mu R^\lambda{}_\nu\Big]$$

$$+2\beta_6\Big[-g_{\mu\nu}\nabla_\sigma\tilde{R}^{\lambda\sigma} + \nabla_\mu\tilde{R}^\lambda{}_\nu\Big] \;+\; 2\gamma_1\nabla^\sigma\Big[R^\lambda{}_{\mu\nu\sigma} - R^\lambda{}_{\mu\sigma\nu}\Big] \;+\; 4\gamma_2\nabla^\sigma R_{\nu\sigma}{}^\lambda{}_\mu$$

$$+\;4\gamma_3\nabla^\sigma R_{\nu\sigma\mu}{}^\lambda \;+\; \mathcal{O}(\nabla^\lambda g_{\mu\nu}) \;+\; \mathcal{O}(T_{\mu\nu}^\lambda)$$

Connection tensor:

$$K_\lambda^{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g,\Gamma)}{\delta \Gamma_{\mu\nu}^\lambda} = 2\alpha\Big[g_{\mu\nu}\nabla^\lambda R - \delta_\nu^\lambda \nabla_\mu R\Big] \ + \ 2\beta_1\Big[\nabla^\lambda R_{\nu\mu} - \delta_\nu^\lambda \nabla^\sigma R_{\sigma\mu}\Big]$$

$$-2\beta_2\Big[-\nabla^\lambda \tilde{R}_{\nu\mu} + \delta_\nu^\lambda \nabla^\sigma \tilde{R}_{\sigma\mu} + g_{\mu\nu}\nabla_\sigma R^{\sigma\lambda} - \nabla_\mu R_\nu{}^\lambda\Big] \ + \ 2\beta_3\Big[\nabla_\mu \tilde{R}_\nu{}^\lambda - g_{\mu\nu}\nabla_\sigma \tilde{R}^{\sigma\lambda}\Big]$$

$$+2\beta_4\Big[\nabla^\lambda R_{\mu\nu} - \delta_\nu^\lambda \nabla^\sigma R_{\mu\sigma}\Big] \ - \ 2\beta_5\Big[-\nabla^\lambda \tilde{R}_{\mu\nu} + \delta_\nu^\lambda \nabla^\sigma \tilde{R}_{\mu\sigma} + g_{\mu\nu}\nabla_\sigma R^{\lambda\sigma} - \nabla_\mu R^\lambda{}_\nu\Big]$$

$$+2\beta_6\Big[-g_{\mu\nu}\nabla_\sigma \tilde{R}^{\lambda\sigma} + \nabla_\mu \tilde{R}^\lambda{}_\nu\Big] \ + \ 2\gamma_1 \nabla^\sigma\Big[R^\lambda{}_{\mu\nu\sigma} - R^\lambda{}_{\mu\sigma\nu}\Big] \ + \ 4\gamma_2 \nabla^\sigma R_{\nu\sigma}{}^\lambda{}_\mu$$

$$+ \ 4\gamma_3 \nabla^\sigma R_{\nu\sigma\mu}{}^\lambda \ + \ \mathcal{O}(\nabla^\lambda g_{\mu\nu}) \ + \ \mathcal{O}(T_{\mu\nu}^\lambda)$$

- $\tilde{H}_{\mu\nu}$ has no ∇^2(Riem) terms, even for arbitrary (α, β, γ)

Connection tensor:

$$K_\lambda^{\mu\nu} \;\equiv\; \frac{1}{\sqrt{|g|}}\frac{\delta S(g,\Gamma)}{\delta\Gamma_{\mu\nu}^\lambda} = 2\alpha\Big[g_{\mu\nu}\nabla^\lambda R - \delta_\nu^\lambda\nabla_\mu R\Big] \;+\; 2\beta_1\Big[\nabla^\lambda R_{\nu\mu} - \delta_\nu^\lambda\nabla^\sigma R_{\sigma\mu}\Big]$$

$$-2\beta_2\Big[-\nabla^\lambda\tilde{R}_{\nu\mu} + \delta_\nu^\lambda\nabla^\sigma\tilde{R}_{\sigma\mu} + g_{\mu\nu}\nabla_\sigma R^{\sigma\lambda} - \nabla_\mu R_\nu{}^\lambda\Big] \;+\; 2\beta_3\Big[\nabla_\mu\tilde{R}_\nu{}^\lambda - g_{\mu\nu}\nabla_\sigma\tilde{R}^{\sigma\lambda}\Big]$$

$$+2\beta_4\Big[\nabla^\lambda R_{\mu\nu} - \delta_\nu^\lambda\nabla^\sigma R_{\mu\sigma}\Big] \;-\; 2\beta_5\Big[-\nabla^\lambda\tilde{R}_{\mu\nu} + \delta_\nu^\lambda\nabla^\sigma\tilde{R}_{\mu\sigma} + g_{\mu\nu}\nabla_\sigma R^{\lambda\sigma} - \nabla_\mu R^\lambda{}_\nu\Big]$$

$$+2\beta_6\Big[-g_{\mu\nu}\nabla_\sigma\tilde{R}^{\lambda\sigma} + \nabla_\mu\tilde{R}^\lambda{}_\nu\Big] \;+\; 2\gamma_1\nabla^\sigma\Big[R^\lambda{}_{\mu\nu\sigma} - R^\lambda{}_{\mu\sigma\nu}\Big] \;+\; 4\gamma_2\nabla^\sigma R_{\nu\sigma}{}^\lambda{}_\mu$$

$$+\;4\gamma_3\nabla^\sigma R_{\nu\sigma\mu}{}^\lambda \;+\; \mathcal{O}(\nabla^\lambda g_{\mu\nu}) \;+\; \mathcal{O}(T_{\mu\nu}^\lambda)$$

- $\tilde{H}_{\mu\nu}$ has no ∇^2(Riem) terms, even for arbitrary (α,β,γ)
- $\nabla_\mu\tilde{H}^{\mu\nu} \neq 0 \quad\longrightarrow\quad$ inconsistency with $\tilde{H}_{\mu\nu} = T_{\mu\nu}$?

- Comparison: take Levi-Civita as Ansatz and substitute in $\tilde{H}_{\mu\nu}$ and $K_\lambda^{\mu\nu}$

$$\Longrightarrow \quad \tilde{R}_{\mu\nu} \longrightarrow -R_{\mu\nu} \qquad\qquad R_{\nu\mu} \longrightarrow R_{\mu\nu}$$

$$\Longrightarrow \quad \mathcal{H}_{\mu\nu} = \tilde{H}_{\mu\nu}\Big|_{\rm Levi-Civita} \qquad\qquad \mathcal{K}^\lambda_{\mu\nu} \equiv K^\lambda_{\mu\nu}\Big|_{\rm Levi-Civita}$$

- Comparison: take Levi-Civita as Ansatz and substitute in $\tilde{H}_{\mu\nu}$ and $K_\lambda^{\mu\nu}$

$$\Longrightarrow \quad \tilde{R}_{\mu\nu} \longrightarrow -R_{\mu\nu} \qquad\qquad R_{\nu\mu} \longrightarrow R_{\mu\nu}$$

$$\Longrightarrow \quad \mathcal{H}_{\mu\nu} = \tilde{H}_{\mu\nu}\Big|_{\text{Levi-Civita}} \qquad\qquad \mathcal{K}^\lambda_{\mu\nu} \equiv K^\lambda_{\mu\nu}\Big|_{\text{Levi-Civita}}$$

$$\begin{aligned}\mathcal{H}_{\mu\nu} \;&=\; 2\alpha R_{\mu\nu}R \;+\; 4\beta\Big(R_{\mu\rho}R_\nu{}^\rho + R_{\mu\rho\lambda\nu}R^{\rho\lambda}\Big) \;+\; 2\gamma R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} \\ &\qquad -\tfrac{1}{2}g_{\mu\nu}\Big[\alpha R^2 - \beta R_{\rho\lambda}R^{\rho\lambda} + \gamma R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big]\end{aligned}$$

$$\begin{aligned}\mathcal{K}^\lambda_{\mu\nu} \;&=\; 2(\alpha+\beta)g_{\mu\nu}\nabla^\lambda R \;+\; 4(\beta+\gamma)\nabla^\lambda R_{\mu\nu} \\ &\qquad -\,2(\alpha+\beta)\delta^\lambda_\nu\nabla_\mu R \;-\; 4(\beta+\gamma)\nabla_\mu R_\nu{}^\lambda\end{aligned}$$

Compare:

$$H_{\mu\nu} = \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big]$$

$$+\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big]$$

$$+\gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} + 2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big]$$

$$\begin{cases} \mathcal{H}_{\mu\nu} = 2\alpha R_{\mu\nu}R \;+\; 4\beta\Big(R_{\mu\rho}R_\nu{}^\rho + R_{\mu\rho\lambda\nu}R^{\rho\lambda}\Big) \;+\; 2\gamma R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} \\ \qquad\qquad -\tfrac{1}{2}g_{\mu\nu}\Big[\alpha R^2 - \beta R_{\rho\lambda}R^{\rho\lambda} + \gamma R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big] \\ \mathcal{K}^\lambda_{\mu\nu} \;=\; 2(\alpha+\beta)g_{\mu\nu}\nabla^\lambda R \;+\; 4(\beta+\gamma)\nabla^\lambda R_{\mu\nu} \\ \qquad\qquad -\;2(\alpha+\beta)\delta^\lambda_\nu\nabla_\mu R \;-\; 4(\beta+\gamma)\nabla_\mu R_\nu{}^\lambda \end{cases}$$

Compare:

$$H_{\mu\nu} = \alpha\Big[2\nabla_\mu\partial_\nu R - 2g_{\mu\nu}\nabla^2 R + 2R_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R^2\Big]$$
$$+\beta\Big[\nabla_\mu\partial_\nu R - \nabla^2 R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla^2 R - 2R_{\mu\nu\rho\lambda}R^{\rho\lambda} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda}\Big]$$
$$+\gamma\Big[2\nabla_\mu\partial_\nu R - 4\nabla^2 R_{\mu\nu} - 4R_{\mu\rho}R_\nu{}^\rho - 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} + 2R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} - \tfrac{1}{2}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big]$$

$$\begin{cases} \mathcal{H}_{\mu\nu} = 2\alpha R_{\mu\nu}R \;+\; 4\beta\Big(R_{\mu\rho}R_\nu{}^\rho + R_{\mu\rho\lambda\nu}R^{\rho\lambda}\Big) \;+\; 2\gamma R_{\mu\rho\lambda\sigma}R_\nu{}^{\rho\lambda\sigma} \\ \qquad -\tfrac{1}{2}g_{\mu\nu}\Big[\alpha R^2 - \beta R_{\rho\lambda}R^{\rho\lambda} + \gamma R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big] \\ \mathcal{K}^\lambda_{\mu\nu} \;=\; 2(\alpha+\beta)g_{\mu\nu}\nabla^\lambda R \;+\; 4(\beta+\gamma)\nabla^\lambda R_{\mu\nu} \\ \qquad -\,2(\alpha+\beta)\delta^\lambda_\nu\nabla_\mu R \;-\; 4(\beta+\gamma)\nabla_\mu R_\nu{}^\lambda \end{cases}$$

$$\boxed{H_{\mu\nu} \;=\; \mathcal{H}_{\mu\nu} \;-\; \tfrac{1}{2}\nabla_\lambda\mathcal{K}^\lambda_{(\mu\nu)} \;+\; \tfrac{1}{4}g_{\lambda\mu}\nabla^\rho\mathcal{K}^\lambda_{\rho\nu} \;+\; \tfrac{1}{4}g_{\lambda\nu}\nabla^\rho\mathcal{K}^\lambda_{\rho\nu}}$$

- $\mathcal{K}_{\mu\nu}^{\lambda} \;=\; 2(\alpha+\beta) g_{\mu\nu} \nabla^\lambda R \;+\; 4(\beta+\gamma)\nabla^\lambda R_{\mu\nu}$

  $- \, 2(\alpha+\beta)\delta_\nu^\lambda \nabla_\mu R \;-\; 4(\beta+\gamma)\nabla_\mu R_\nu{}^\lambda \;\equiv 0$ for $(\alpha, \beta, \gamma) = (1, -1, 1)$

  $\longrightarrow$ equivalence for Gauss-Bonnet (= Lovelock)

- $$\mathcal{K}_{\mu\nu}^{\lambda} = 2(\alpha+\beta)g_{\mu\nu}\nabla^{\lambda}R + 4(\beta+\gamma)\nabla^{\lambda}R_{\mu\nu} - 2(\alpha+\beta)\delta_{\nu}^{\lambda}\nabla_{\mu}R - 4(\beta+\gamma)\nabla_{\mu}R_{\nu}{}^{\lambda} \equiv 0 \quad \text{for} \quad (\alpha,\beta,\gamma)=(1,-1,1)$$

  $\longrightarrow$ equivalence for Gauss-Bonnet (= Lovelock)

- All Lagrangians with $(\alpha,\beta,\gamma)=(1,-1,1)$ reduce to Gauss-Bonnet, but not all have $\mathcal{K}_{\mu\nu}^{\lambda} \equiv 0$:

$$\mathcal{K}_{\mu\nu}^{\lambda} = 2(\alpha-\beta_2+\beta_3-\beta_5+\beta_6)g_{\mu\nu}\nabla^{\lambda}R + 4(\beta_1-\beta_2+\beta_4-\beta_5+\gamma)\nabla^{\lambda}R_{\mu\nu} - 2(\alpha+\beta_1-\beta_2+\beta_4-\beta_5)\delta_{\nu}^{\lambda}\nabla_{\mu}R - 4(-\beta_2+\beta_3-\beta_5+\beta_6+\gamma)\nabla_{\mu}R_{\nu}{}^{\lambda}$$

- $$\mathcal{K}_{\mu\nu}^{\lambda} = 2(\alpha+\beta) g_{\mu\nu} \nabla^{\lambda} R + 4(\beta+\gamma) \nabla^{\lambda} R_{\mu\nu} - 2(\alpha+\beta) \delta_{\nu}^{\lambda} \nabla_{\mu} R - 4(\beta+\gamma) \nabla_{\mu} R_{\nu}{}^{\lambda} \equiv 0 \quad \text{for} \quad (\alpha, \beta, \gamma) = (1, -1, 1)$$

  $\longrightarrow$ equivalence for Gauss-Bonnet (= Lovelock)

- All Lagrangians with $(\alpha, \beta, \gamma) = (1, -1, 1)$ reduce to Gauss-Bonnet, but not all have $\mathcal{K}_{\mu\nu}^{\lambda} \equiv 0$:

$$\mathcal{K}_{\mu\nu}^{\lambda} = 2(\alpha - \beta_2 + \beta_3 - \beta_5 + \beta_6) g_{\mu\nu} \nabla^{\lambda} R + 4(\beta_1 - \beta_2 + \beta_4 - \beta_5 + \gamma) \nabla^{\lambda} R_{\mu\nu} - 2(\alpha + \beta_1 - \beta_2 + \beta_4 - \beta_5) \delta_{\nu}^{\lambda} \nabla_{\mu} R - 4(-\beta_2 + \beta_3 - \beta_5 + \beta_6 + \gamma) \nabla_{\mu} R_{\nu}{}^{\lambda}$$

- Conditions for $\mathcal{K}_{\mu\nu}^{\lambda} \equiv 0$:

$$\alpha = \gamma \neq 0, \qquad \beta_1 + \beta_4 = \beta_3 + \beta_6, \qquad \alpha + \beta_1 + \beta_4 - \beta_2 - \beta_5 = 0$$

- Lagrangiangs with $\mathcal{K} \equiv 0$ are

$$S_1(g,\Gamma) = \int d^Dx\sqrt{|g|}\,\Big[R^2 - R_{\mu\nu}R^{\nu\mu} - 2R_{\mu\nu}\tilde{R}^{\nu\mu} - \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\Big]$$

$$S_2(g,\Gamma) = \int d^Dx\sqrt{|g|}\,\Big[R^2 - R_{\mu\nu}R^{\nu\mu} - \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\Big]$$

$$S_3(g,\Gamma) = \int d^Dx\sqrt{|g|}\,\Big[R^2 - 2R_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\Big]$$

- Lagrangiangs with $\mathcal{K} \equiv 0$ are

$$S_1(g,\Gamma) = \int d^Dx\sqrt{|g|}\left[R^2 - R_{\mu\nu}R^{\nu\mu} - 2R_{\mu\nu}\tilde{R}^{\nu\mu} - \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right]$$

$$S_2(g,\Gamma) = \int d^Dx\sqrt{|g|}\left[R^2 - R_{\mu\nu}R^{\nu\mu} - \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right]$$

$$S_3(g,\Gamma) = \int d^Dx\sqrt{|g|}\left[R^2 - 2R_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right]$$

- Lagrangians must mimic the symmetries of ${R_{\mu\nu\rho}}^{\lambda}$ in metric formalism

  Determinant definition of Lovelock is too restrictive

  More (Palatini) inequivalent Lagrangians give Lovelock conditions

$$H_{\mu\nu} = H_{\mu\nu}(g, g', g'') \qquad\qquad \nabla_\mu H^{\mu\nu} = 0$$

5 More general Lagrangians

$$S = \int d^Dx \sqrt{|g|}\, \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda})$$

with $\mathcal{L}$ functional of (Riem), but not of ∇(Riem)

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- Einstein-Hilbert, $\mathcal{L} = R$:

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6 Explicit example: FRW in Gauss-Bonnet gravity

$$S = \int d^D x \sqrt{|g|} \Big[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} \Big]$$

NB: $\beta_4 = 1,$ $\beta_1 = \beta_3 = \beta_6 = 0 \Longrightarrow$ Not Palatini-Lovelock!

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Solutions:

- $\omega = -1$: de Sitter space with $A(t) = \lambda_1 t$
- $\omega \neq -1$: power law solution with $e^{A(t)} = \lambda_2 \left(\frac{t}{t_0}\right)^{\frac{4}{(D-1)(1+\omega)}}$

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- Only de Sitter satisfies $G_{\mu\nu} = \Lambda g_{\mu\nu}$
  Only $\omega = -1$ has necessary symmetries

7 Conclusions

For an action

$$S = \int d^D x \sqrt{|g|} \Big[\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}) \; + \; \mathcal{L}(g_{\mu\nu}, \phi) \Big]$$

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  $\longrightarrow$ Lovelock in metric AND sufficient symmetries in Palatini

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- Advantages are obvious when equivalence: Lovelock is very special!

- Recall:

$$H_{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta g^{\mu\nu}}\Big|_{\rm expl} + \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta R_{\alpha\beta\gamma}{}^{\delta}} \frac{\delta R_{\alpha\beta\gamma}{}^{\delta}}{\delta \Gamma_{\rho\lambda}^{\sigma}} \frac{\delta \Gamma_{\rho\lambda}^{\sigma}}{\delta g^{\mu\nu}}$$

$$\tilde{H}_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g,\Gamma)}{\delta g^{\mu\nu}}, \qquad K_{\lambda}^{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g,\Gamma)}{\delta \Gamma_{\mu\nu}^{\lambda}}$$

- $H_{\mu\nu} = \mathcal{H}_{\mu\nu} - \frac{1}{2}\nabla_\rho \mathcal{K}^{\rho}_{(\mu\nu)} + \frac{1}{2} g_{\lambda\mu}\nabla^\rho \mathcal{K}^{\lambda}_{(\nu\rho)} + \frac{1}{2} g_{\lambda\nu}\nabla^\rho \mathcal{K}^{\lambda}_{(\mu\rho)}$ reflects structure of the chain rule

- Lovelock gravities $\Longleftrightarrow$ no ∇(Riem) terms

  $\Longleftrightarrow$ $\mathcal{K}^{\lambda}_{\mu\nu} \equiv 0$

  $\Longleftrightarrow$ equivalence between metric & Palatini