Constructing matrix-valued coordinate transformations from scratch

Bert Janssen

Universidad de Granada & CAFPE

In collaboration with: J. Adam, W. Troost and W. Van Herck (K.U.Leuven)

References: arXiv:0712.0918 [hep-th].

Motivation

A lot has been learned about the dynamics of multiple D-branes in the last past years:

- $U(1)^N \rightarrow U(N)$ symmetry enhancement
- effective actions describing the non-Abelian dynamics
- many applications of non-Abelian effects in modern string theory

Motivation

A lot has been learned about the dynamics of multiple D-branes in the last past years:

- $U(1)^N \to U(N)$ symmetry enhancement
- effective actions describing the non-Abelian dynamics
- many applications of non-Abelian effects in modern string theory

Question: Is the action invariant under background gauge transformations ?

[Adam, Gheerardyn, BJ, Y. Lozano][Adam, Illán, B.J.][Adam]

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]} \longrightarrow \delta V_a = -\Sigma_\mu D_a X^\mu, \qquad \delta X^\mu = \Sigma_\rho [X^\rho, X^\mu]$$

Motivation

A lot has been learned about the dynamics of multiple D-branes in the last past years:

- $U(1)^N \rightarrow U(N)$ symmetry enhancement
- effective actions describing the non-Abelian dynamics
- many applications of non-Abelian effects in modern string theory

Question: Is the action invariant under background gauge transformations ?

[Adam, Gheerardyn, BJ, Y. Lozano][Adam, Illán, B.J.][Adam]

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]} \longrightarrow \delta V_a = -\Sigma_\mu D_a X^\mu, \qquad \delta X^\mu = \Sigma_\rho [X^\rho, X^\mu]$$

Question: Is the action invariant under background general coord transf ?

- Worldvolume coordinates : Abelian theory $\longrightarrow$ trivial
- Target space coordinates : manifest as adjoint scalars X^μ in worldvolume
  How do background diffeomorphisms affect matrix coordinates X^μ?

Outlook

Outlook

1. Introduction
   - Physics of multiple branes
   - Problems with diffeomorphisms

Outlook

1. Introduction
   - Physics of multiple branes
   - Problems with diffeomorphisms
2. Matrix-valued differential geometry from scratch
   - Matrix scalars
   - Matrix contravariant vectors
   - Matrix covariant vectors
   - Matrix tensors and form fields
   - Scalar product & problems

Outlook

1. Introduction
   - Physics of multiple branes
   - Problems with diffeomorphisms
2. Matrix-valued differential geometry from scratch
   - Matrix scalars
   - Matrix contravariant vectors
   - Matrix covariant vectors
   - Matrix tensors and form fields
   - Scalar product & problems
3. Conclusions

1. Non-Abelian physics of multiple parallel branes

The physics of N separated parallel Dp-branes is very different from physics of N coinciding Dp-branes.

- separated: → Abelian theory
- coinciding: → non-Abelian theory

1. Non-Abelian physics of multiple parallel branes

The physics of N separated parallel Dp-branes is very different from physics of N coinciding Dp-branes.

- separated: $\rightarrow$ Abelian theory
- coinciding: $\rightarrow$ non-Abelian theory

This non-Abelian character has numerous manifestations in modern string theory:

- Dielectric effect [Myers]
- Gravity duals of (confining) gauge theories [Polchinsky, Strassler]
- Enhançons [Johnson]
- Matrix models in non-trivial backgrounds [Berenstein, Maldacena, Nastase]
- Microscopic description of giant gravitons [B.J., Lozano, Rodríguez]
- ...

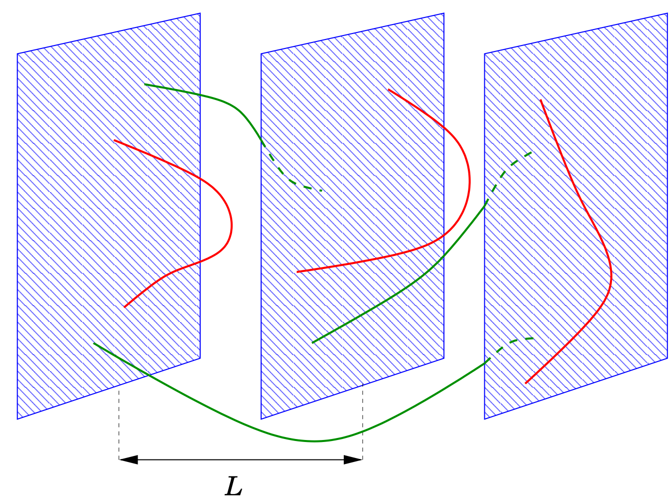
Difference in degrees of freedom

- separated: $N\; U(1)$ vector fields V_a^I, $(9-p)N$ scalars X^{iI}
  $\rightarrow$ Abelian worldvolume action
- coinciding: $1\; U(N)$ Yang-Mills vector V_a^I, N adjoint scalars X^{iI}
  $\rightarrow$ non-Abelian worldvolume action

Difference in degrees of freedom

- separated: $N\ U(1)$ vector fields V_a^I, $(9-p)N$ scalars X^{iI}
  $\rightarrow$ Abelian worldvolume action
- coinciding: $1\ U(N)$ Yang-Mills vector V_a^I, N adjoint scalars X^{iI}
  $\rightarrow$ non-Abelian worldvolume action

Extra degrees of freedom come from massless strings stretched between coinciding strings: [Witten]



strings between same brane: $m \sim 0$

strings between different branes: $m \sim L$

As $L \rightarrow 0$:

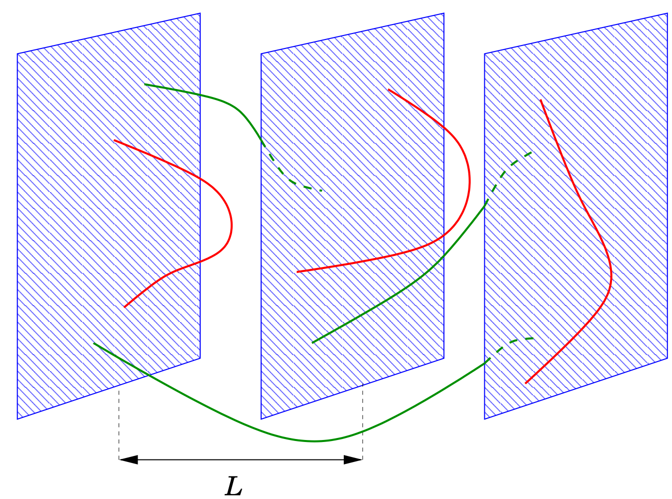
$\Longrightarrow N + N(N-1) = N^2$ degrees of freedom

$\Longrightarrow U(1)^N \rightarrow U(N)$ gauge enhancement

Difference in degrees of freedom

- separated: $N\ U(1)$ vector fields V_a^I, $(9-p)N$ scalars X^{iI}
  $\rightarrow$ Abelian worldvolume action
- coinciding: $1\ U(N)$ Yang-Mills vector V_a^I, N adjoint scalars X^{iI}
  $\rightarrow$ non-Abelian worldvolume action

Extra degrees of freedom come from massless strings stretched between coinciding strings: [Witten]



strings between same brane: $m \sim 0$
strings between different branes: $m \sim L$
As $L \rightarrow 0$:
$\Longrightarrow N + N(N-1) = N^2$ degrees of freedom
$\Longrightarrow U(1)^N \rightarrow U(N)$ gauge enhancement

Extra degrees of freedom and non-Abelian symmetries
$\Longrightarrow$ non-Abelian worldvolume action

Role of the scalars

- Abelian: $X^{\mu I}(\sigma)$ indicate position of Ith brane in direction x^μ
  $\longrightarrow$ rearrange in diagonal matrix $X^\mu(\sigma) = \mathrm{diag}\Big(X^{\mu 1}(\sigma), ..., X^{\mu N}(\sigma)\Big)$

Role of the scalars

- Abelian: $X^{\mu I}(\sigma)$ indicate position of Ith brane in direction x^μ
  $\longrightarrow$ rearrange in diagonal matrix $X^\mu(\sigma) = \mathrm{diag}\Big(X^{\mu 1}(\sigma), ..., X^{\mu N}(\sigma)\Big)$
- non-Abelian: X^μ non-Abelian scalars in adjoint of $U(N)$
  $\longrightarrow$ eigenvalues indicate position of Ith brane in direction x^μ

Role of the scalars

- Abelian: $X^{\mu I}(\sigma)$ indicate position of Ith brane in direction x^μ
  $\longrightarrow$ rearrange in diagonal matrix $X^\mu(\sigma) = \mathrm{diag}\Big(X^{\mu 1}(\sigma), ..., X^{\mu N}(\sigma)\Big)$
- non-Abelian: X^μ non-Abelian scalars in adjoint of $U(N)$
  $\longrightarrow$ eigenvalues indicate position of Ith brane in direction x^μ
  $\longrightarrow$ not simultaneously diagonisable: $[X^\mu, X^\nu] \neq 0$
  $\longrightarrow$ non-commutative geometry

Role of the scalars

- Abelian: $X^{\mu I}(\sigma)$ indicate position of Ith brane in direction x^μ
  $\longrightarrow$ rearrange in diagonal matrix $X^\mu(\sigma) = \mathrm{diag}\Big(X^{\mu 1}(\sigma), ..., X^{\mu N}(\sigma)\Big)$
- non-Abelian: X^μ non-Abelian scalars in adjoint of $U(N)$
  $\longrightarrow$ eigenvalues indicate position of Ith brane in direction x^μ
  $\longrightarrow$ not simultaneously diagonisable: $[X^\mu, X^\nu] \neq 0$
  $\longrightarrow$ non-commutative geometry
- Background fields: $\Phi = \Phi(x) \longrightarrow \Phi = \Phi(X)$
  $\longrightarrow$ $\Phi(X) = \sum_n \frac{1}{n!}\partial_{\mu_1}...\partial_{\mu_n}\Phi(x)|_{x^\lambda=0}X^{\mu_1}...X^{\mu_n}$ [Douglas][Garousi, Myers]
  $\longrightarrow$ Symmetrized trace prescription [Tseytlin]

Role of the scalars

- Abelian: $X^{\mu I}(\sigma)$ indicate position of Ith brane in direction x^μ
  $\longrightarrow$ rearrange in diagonal matrix $X^\mu(\sigma) = \mathrm{diag}\Big(X^{\mu 1}(\sigma), ..., X^{\mu N}(\sigma)\Big)$

- non-Abelian: X^μ non-Abelian scalars in adjoint of $U(N)$
  $\longrightarrow$ eigenvalues indicate position of Ith brane in direction x^μ
  $\longrightarrow$ not simultaneously diagonisable: $[X^\mu, X^\nu] \neq 0$
  $\longrightarrow$ non-commutative geometry

- Background fields: $\Phi = \Phi(x) \longrightarrow \Phi = \Phi(X)$
  $\longrightarrow$ $\Phi(X) = \sum_n \frac{1}{n!}\partial_{\mu_1}...\partial_{\mu_n}\Phi(x)|_{x^\lambda=0}X^{\mu_1}...X^{\mu_n}$ [Douglas][Garousi, Myers]
  $\longrightarrow$ Symmetrized trace prescription [Tseytlin]

- coordinate transf: $g'_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, g_{\mu\nu} \longrightarrow X' = X'(X)$ such that

$$S[g, X] \equiv S[g', X']$$

Question: What is $X' = X'(X)$?

$\longrightarrow$ Problems when combining ordering and transitivity

[de Boer, Schalm]

Abelian composition:

$$y^\mu \;=\; x^\mu \;+\; a^\mu_{\nu\rho}\, x^\nu x^\rho \;+\; b^\mu_{\nu\rho\lambda}\, x^\nu x^\rho x^\lambda \;+\; ...$$

$$\begin{aligned} z^\mu \;&=\; y^\mu \;+\; \tilde{a}^\mu_{\nu\rho}\, y^\nu y^\rho \;+\; \tilde{b}^\mu_{\nu\rho\lambda}\, y^\nu y^\rho y^\lambda \;+\; ... \\ &=\; x^\mu \;+\; (a^\mu_{\nu\rho} + \tilde{a}^\mu_{\nu\rho})\, x^\nu x^\rho \;+\; (b^\mu_{\nu\rho\lambda} + 2a^\mu_{\nu\sigma}\tilde{a}^\sigma_{\rho\lambda} + \tilde{b}^\mu_{\nu\rho\lambda})\, x^\nu x^\rho x^\lambda \;+\; ... \end{aligned}$$

Abelian composition:

$$y^\mu \;=\; x^\mu \;+\; a^\mu_{\nu\rho}\, x^\nu x^\rho \;+\; b^\mu_{\nu\rho\lambda}\, x^\nu x^\rho x^\lambda \;+\; ...$$

$$\begin{aligned} z^\mu \;&=\; y^\mu \;+\; \tilde{a}^\mu_{\nu\rho}\, y^\nu y^\rho \;+\; \tilde{b}^\mu_{\nu\rho\lambda}\, y^\nu y^\rho y^\lambda \;+\; ... \\ &=\; x^\mu \;+\; (a^\mu_{\nu\rho} + \tilde{a}^\mu_{\nu\rho})\, x^\nu x^\rho \;+\; (b^\mu_{\nu\rho\lambda} + 2a^\mu_{\nu\sigma}\tilde{a}^\sigma_{\rho\lambda} + \tilde{b}^\mu_{\nu\rho\lambda})\, x^\nu x^\rho x^\lambda \;+\; ... \end{aligned}$$

Non-abelian composition:

$$Y^\mu \;=\; X^\mu \;+\; F^\mu_{\nu\rho}(a)\, X^\nu X^\rho \;+\; G^\mu_{\nu\rho\lambda}(a,b)\, X^\nu X^\rho X^\lambda \;+\; ...$$

$$\begin{aligned} Z^\mu \;&=\; Y^\mu \;+\; F^\mu_{\nu\rho}(\tilde{a})\, Y^\nu Y^\rho \;+\; G^\mu_{\nu\rho\lambda}(\tilde{a},\tilde{b})\, Y^\nu Y^\rho Y^\lambda \;+\; ... \\ &=\; X^\mu \;+\; \Big[F^\mu_{\nu\rho}(a) + F^\mu_{\nu\rho}(\tilde{a})\Big]\, X^\nu X^\rho \\ &\quad\; + \Big[G^\mu_{\nu\rho\lambda}(a,b) + F^\mu_{\nu\sigma}(\tilde{a})F^\sigma_{\rho\lambda}(a) + F^\mu_{\sigma\lambda}(\tilde{a})F^\sigma_{\nu\rho}(a) + \tilde{G}^\mu_{\nu\rho\lambda}(\tilde{a},\tilde{b})\Big]\, X^\nu X^\rho X^\lambda \;+\; ... \end{aligned}$$

Abelian composition:

$$y^\mu \;=\; x^\mu \;+\; a^\mu_{\nu\rho}\, x^\nu x^\rho \;+\; b^\mu_{\nu\rho\lambda}\, x^\nu x^\rho x^\lambda \;+\; ...$$

$$\begin{aligned} z^\mu \;&=\; y^\mu \;+\; \tilde{a}^\mu_{\nu\rho}\, y^\nu y^\rho \;+\; \tilde{b}^\mu_{\nu\rho\lambda}\, y^\nu y^\rho y^\lambda \;+\; ... \\ &=\; x^\mu \;+\; (a^\mu_{\nu\rho} + \tilde{a}^\mu_{\nu\rho})\, x^\nu x^\rho \;+\; (b^\mu_{\nu\rho\lambda} + 2a^\mu_{\nu\sigma}\tilde{a}^\sigma_{\rho\lambda} + \tilde{b}^\mu_{\nu\rho\lambda})\, x^\nu x^\rho x^\lambda \;+\; ... \end{aligned}$$

Non-abelian composition:

$$Y^\mu \;=\; X^\mu \;+\; F^\mu_{\nu\rho}(a)\, X^\nu X^\rho \;+\; G^\mu_{\nu\rho\lambda}(a,b)\, X^\nu X^\rho X^\lambda \;+\; ...$$

$$\begin{aligned} Z^\mu \;&=\; Y^\mu \;+\; F^\mu_{\nu\rho}(\tilde{a})\, Y^\nu Y^\rho \;+\; G^\mu_{\nu\rho\lambda}(\tilde{a},\tilde{b})\, Y^\nu Y^\rho Y^\lambda \;+\; ... \\ &=\; X^\mu \;+\; \Big[F^\mu_{\nu\rho}(a) + F^\mu_{\nu\rho}(\tilde{a})\Big]\, X^\nu X^\rho \\ &\quad\; + \Big[G^\mu_{\nu\rho\lambda}(a,b) + F^\mu_{\nu\sigma}(\tilde{a})F^\sigma_{\rho\lambda}(a) + F^\mu_{\sigma\lambda}(\tilde{a})F^\sigma_{\nu\rho}(a) + \tilde{G}^\mu_{\nu\rho\lambda}(\tilde{a},\tilde{b})\Big]\, X^\nu X^\rho X^\lambda \;+\; ... \end{aligned}$$

$$\equiv U\Big(X^\mu + \; F^\mu_{\nu\rho}(a+\tilde{a})\, X^\nu X^\rho \;+\; G^\mu_{\nu\rho\lambda}(a+\tilde{a}, b + 2a\tilde{a} + \tilde{b})\, X^\nu X^\rho X^\lambda \;+\; ...\Big)U^\dagger$$

$\longrightarrow \nexists\; F^\mu_{\nu\rho}$ and $G^\mu_{\nu\rho\lambda}$ such that above is true

$$f(x) \longrightarrow F(X), \qquad g(x) \longrightarrow G(X)$$

$$h(x) = (g \circ f)(x) \longrightarrow H(X) \neq (G \circ F)(X)$$ for any ordering!

Abelian diffeomorphisms is not a subset of Non-Abelian DIFFEOMORPHISMS

[de Boer, Schalm]

No solution for $S[g, X] = S[g'(g), X'(X)]$ ...

$f(x) \longrightarrow F(X)$, $\qquad g(x) \longrightarrow G(X)$

$h(x) = (g \circ f)(x) \longrightarrow H(X) \neq (G \circ F)(X)$ for any ordering!

Abelian diffeomorphisms is not a subset of Non-Abelian DIFFEOMORPHISMS

[de Boer, Schalm]

No solution for $\quad S[g, X] = S[g'(g), X'(X)]$ ...

... unless there is an explicit metric dependence in $X'(X)$

$\longrightarrow$ Real invariance: $\quad S[g, X] = S[g'(g), X'(g, X)]$ [de Boer, Schalm]

$$f(x) \longrightarrow F(X), \qquad g(x) \longrightarrow G(X)$$

$$h(x) = (g \circ f)(x) \longrightarrow H(X) \neq (G \circ F)(X)$$ for any ordering!

Abelian diffeomorphisms is not a subset of Non-Abelian DIFFEOMORPHISMS

[de Boer, Schalm]

No solution for $S[g, X] = S[g'(g), X'(X)]$ ...

... unless there is an explicit metric dependence in $X'(X)$

$\longrightarrow$ Real invariance: $S[g, X] = S[g'(g), X'(g, X)]$ [de Boer, Schalm]

Two strategies:

- Look for inclusion function via Noether procedure

$$f(x) \longrightarrow F(X), \qquad g(x) \longrightarrow G(X)$$

$$h(x) = (g \circ f)(x) \longrightarrow H(X) \neq (G \circ F)(X)$$ for any ordering!

Abelian diffeomorphisms is not a subset of Non-Abelian DIFFEOMORPHISMS

[de Boer, Schalm]

No solution for $S[g, X] = S[g'(g), X'(X)]$ ...

... unless there is an explicit metric dependence in $X'(X)$

$\longrightarrow$ Real invariance: $S[g, X] = S[g'(g), X'(g, X)]$ [de Boer, Schalm]

Two strategies:

- Look for inclusion function via Noether procedure
- Construct matrix differential geometry from scratch

$\longrightarrow$ rest of the talk

2. Matrix differential geometry from scratch

Construct an algebra for matrix coord transf on its own,
independend of group of Abelian diffeomorphisms

2. Matrix differential geometry from scratch

Construct an algebra for matrix coord transf on its own, independend of group of Abelian diffeomorphisms

- no homomorphism between **diff** and **DIFF**

  $\longrightarrow$ no inclusion function

2. Matrix differential geometry from scratch

Construct an algebra for matrix coord transf on its own,
independend of group of Abelian diffeomorphisms

- no homomorphism between **diff** and **DIFF**
  $\longrightarrow$ no inclusion function
- require homomorphism between **DIFF** and **diff**
  $\longrightarrow$ projection onto abelian functions via diagonal matrices

2. Matrix differential geometry from scratch

Construct an algebra for matrix coord transf on its own, independend of group of Abelian diffeomorphisms

- no homomorphism between **diff** and **DIFF**
  $\longrightarrow$ no inclusion function
- require homomorphism between **DIFF** and **diff**
  $\longrightarrow$ projection onto abelian functions via diagonal matrices
- Construct matrix functions, scalars, vectors, tensors, ...
  by analogy of Abelian counterparts

2. Matrix differential geometry from scratch

Construct an algebra for matrix coord transf on its own,
independend of group of Abelian diffeomorphisms

- no homomorphism between **diff** and **DIFF**
  $\longrightarrow$ no inclusion function
- require homomorphism between **DIFF** and **diff**
  $\longrightarrow$ projection onto abelian functions via diagonal matrices
- Construct matrix functions, scalars, vectors, tensors, ...
  by analogy of Abelian counterparts

Matrix function, defined by expansion series:

$$F(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \, f_{\lambda_1 \ldots \lambda_k} \, X^{\lambda_1} \ldots X^{\lambda_k}$$

where $f_{\lambda_1 \ldots \lambda_k}$ complex coefficients without specific symmetries

$\longrightarrow$ sum, product and composition gives matrix function

2.1 Substitution operator and matrix scalar

Coordinate transformation:

$$\text{Abelian}: \quad x^\mu \rightarrow y^\mu = x^\mu - \xi^\mu(x) \qquad \text{with} \quad \xi(x) = \Big(\xi^0(x), ..., \xi^{D-1}(x)\Big)$$

$$\text{non}-\text{Abelian}: \quad X^\mu \rightarrow Y^\mu = X^\mu - \Xi^\mu(X) \qquad \text{with} \quad \Xi(x) = \Big(\Xi^0(x), ..., \Xi^{D-1}(x)\Big)$$

2.1 Substitution operator and matrix scalar

Coordinate transformation:

$$\text{Abelian}: \quad x^\mu \to y^\mu = x^\mu - \xi^\mu(x) \qquad \text{with} \quad \xi(x) = \Big(\xi^0(x), ..., \xi^{D-1}(x)\Big)$$

$$\text{non}-\text{Abelian}: \quad X^\mu \to Y^\mu = X^\mu - \Xi^\mu(X) \qquad \text{with} \quad \Xi(x) = \Big(\Xi^0(x), ..., \Xi^{D-1}(x)\Big)$$

Abelian scalars:

$$f'(y) = f(x) \qquad \Longleftrightarrow \qquad \delta_\xi f(x) = f'(x) - f(x)$$

To first order in ξ

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\partial_{\lambda_1}...\partial_{\lambda_k} f\Big)(0) \; x^{\lambda_1}...x^{\lambda_k}$$

$$\delta_\xi f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\partial_{\lambda_1}...\partial_{\lambda_k} f\Big)(0)\Big(\xi^{\lambda_1}(x)x^{\lambda_2}...x^{\lambda_k} + ... + x^{\lambda_1}...x^{\lambda_{k-1}}\xi^{\lambda_k}(x)\Big)$$

$$= \xi^\rho \partial_\rho f(x)$$

Non-Abelian scalars:

$$F'(Y) = F(X) \qquad \Longleftrightarrow \qquad \delta_\xi F(X) = F'(X) - F(X)$$

To first order in ξ

$$F(X) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, f_{\lambda_1 \dots \lambda_k}\, X^{\lambda_1} \dots X^{\lambda_k}$$

$$\delta_\xi F(X) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, f_{\lambda_1 \dots \lambda_k} \Big(\Xi^{\lambda_1}(X) X^{\lambda_2} \dots X^{\lambda_k} + \dots + X^{\lambda_1} \dots X^{\lambda_{k-1}} \Xi^{\lambda_k}(X) \Big)$$

$$\equiv \; \overline{\Xi^\rho \partial_\rho} F(X)$$

Non-Abelian scalars:

$$F'(Y) = F(X) \qquad \Longleftrightarrow \qquad \delta_\xi F(X) = F'(X) - F(X)$$

To first order in ξ

$$F(X) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$$

$$\delta_\xi F(X) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} \Big(\Xi^{\lambda_1}(X) X^{\lambda_2} \ldots X^{\lambda_k} + \ldots + X^{\lambda_1} \ldots X^{\lambda_{k-1}} \Xi^{\lambda_k}(X) \Big)$$

$$\equiv \overline{\Xi^\rho \partial_\rho} F(X)$$

where we defined $\overline{\Xi^\rho \partial_\rho}$ as:

"take each X^μ in $F(X)$ in turn and sustitute by $\Xi^\mu(X)$ and sum over everything"

Hence matrix scalars are defined by

$$\delta_\Xi F(X) = \overline{\Xi^\rho \partial_\rho}\, F(X)$$

$$[\delta_1, \delta_2] F(X) \;=\; \overline{\Xi^\lambda \partial_\lambda} \Big(\overline{\Sigma^\rho \partial_\rho} F(X) \Big) \;-\; \overline{\Sigma^\lambda \partial_\lambda} \Big(\overline{\Xi^\rho \partial_\rho} F(X) \Big)$$

$$
\begin{aligned}
[\delta_1, \delta_2] F(X) &= \overline{\Xi^\lambda \partial_\lambda} \Big(\overline{\Sigma^\rho \partial_\rho} F(X) \Big) - \overline{\Sigma^\lambda \partial_\lambda} \Big(\overline{\Xi^\rho \partial_\rho} F(X) \Big) \\
&= \Big(\overline{\Xi^\lambda \partial_\lambda} \Big| \overline{\Sigma^\rho \partial_\rho} \Big) F(X) + \overline{\Big(\overline{\Xi^\lambda \partial_\lambda} \Sigma^\rho \Big) \partial_\rho} F(X) \\
&\quad - \Big(\overline{\Sigma^\lambda \partial_\lambda} \Big| \overline{\Xi^\rho \partial_\rho} \Big) F(X) - \overline{\Big(\overline{\Sigma^\lambda \partial_\lambda} \Xi^\rho \Big) \partial_\rho} F(X)
\end{aligned}
$$

$$[\delta_1, \delta_2]F(X) = \overline{\Xi^\lambda \partial_\lambda}\Big(\overline{\Sigma^\rho \partial_\rho} F(X)\Big) - \overline{\Sigma^\lambda \partial_\lambda}\Big(\overline{\Xi^\rho \partial_\rho} F(X)\Big)$$

$$= \Big(\overline{\Xi^\lambda \partial_\lambda}\Big|\overline{\Sigma^\rho \partial_\rho}\Big) F(X) + \overline{\Big(\overline{\Xi^\lambda \partial_\lambda}\Sigma^\rho\Big)\partial_\rho} F(X)$$

$$- \Big(\overline{\Sigma^\lambda \partial_\lambda}\Big|\overline{\Xi^\rho \partial_\rho}\Big) F(X) - \overline{\Big(\overline{\Sigma^\lambda \partial_\lambda}\Xi^\rho\Big)\partial_\rho} F(X)$$

$$= \overline{\Big(\overline{\Xi^\lambda \partial_\lambda}\Sigma^\rho\Big)\partial_\rho} F(X) - \overline{\Big(\overline{\Sigma^\lambda \partial_\lambda}\Xi^\rho\Big)\partial_\rho} F(X)$$

$$= \overline{\Lambda^\rho \partial_\rho} F(X)$$

where

$$\Lambda^\rho = \overline{\Xi^\lambda \partial_\lambda}\Sigma^\rho - \overline{\Sigma^\lambda \partial_\lambda}\Xi^\rho$$

$$
\begin{aligned}
[\delta_1, \delta_2]F(X) &= \overline{\Xi^\lambda\partial_\lambda}\Big(\overline{\Sigma^\rho\partial_\rho}F(X)\Big) - \overline{\Sigma^\lambda\partial_\lambda}\Big(\overline{\Xi^\rho\partial_\rho}F(X)\Big) \\
&= \Big(\overline{\Xi^\lambda\partial_\lambda}\Big|\overline{\Sigma^\rho\partial_\rho}\Big)F(X) + \overline{\Big(\overline{\Xi^\lambda\partial_\lambda}\Sigma^\rho\Big)\partial_\rho}F(X) \\
&\quad - \Big(\overline{\Sigma^\lambda\partial_\lambda}\Big|\overline{\Xi^\rho\partial_\rho}\Big)F(X) - \overline{\Big(\overline{\Sigma^\lambda\partial_\lambda}\Xi^\rho\Big)\partial_\rho}F(X) \\
&= \overline{\Big(\overline{\Xi^\lambda\partial_\lambda}\Sigma^\rho\Big)\partial_\rho}F(X) - \overline{\Big(\overline{\Sigma^\lambda\partial_\lambda}\Xi^\rho\Big)\partial_\rho}F(X) \\
&= \overline{\Lambda^\rho\partial_\rho}F(X)
\end{aligned}
$$

where

$$
\Lambda^\rho = \overline{\Xi^\lambda\partial_\lambda}\Sigma^\rho - \overline{\Sigma^\lambda\partial_\lambda}\Xi^\rho
$$

- $\delta_\Xi F(X) = \overline{\Xi^\rho\partial_\rho}\, F(X)$ forms an algebra
- Leibnitz rule: $\overline{\Xi^\rho\partial_\rho}(F \cdot G) = \overline{\Xi^\rho\partial_\rho}F \cdot G + F \cdot \overline{\Xi^\rho\partial_\rho}G$

2.2 Contravariant matrix vectors

Abelian: $$\delta_\xi a^\mu = \xi^\rho \partial_\rho a^\mu - a^\rho \partial_\rho \xi^\mu$$

2.2 Contravariant matrix vectors

Abelian: $$\delta_\xi a^\mu = \xi^\rho \partial_\rho a^\mu - a^\rho \partial_\rho \xi^\mu$$

Non-Abelian: $$A^\mu = \sum_{k=0}^\infty \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$$

$$\delta_\Xi A^\mu = \overline{\Xi^\rho \partial_\rho} A^\mu - \overline{A^\rho \partial_\rho} \Xi^\mu$$

2.2 Contravariant matrix vectors

Abelian: $\delta_\xi a^\mu = \xi^\rho \partial_\rho a^\mu - a^\rho \partial_\rho \xi^\mu$

Non-Abelian: $A^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} \, a^\mu_{\lambda_1 \ldots \lambda_k} \, X^{\lambda_1} \ldots X^{\lambda_k}$

$$\delta_\Xi A^\mu = \overline{\Xi^\rho \partial_\rho} A^\mu - \overline{A^\rho \partial_\rho} \Xi^\mu$$

Commutator:

$$\begin{aligned}
[\delta_1, \delta_2] A^\mu &= \overline{\Xi^\rho \partial_\rho}\Big(\overline{\Sigma^\sigma \partial_\sigma} A^\mu - \overline{A^\rho \partial_\rho} \Sigma^\mu\Big) - \overline{\Sigma^\rho \partial_\rho}\Big(\overline{\Xi^\sigma \partial_\sigma} A^\mu - \overline{A^\rho \partial_\rho} \Xi^\mu\Big) \\
&= \overline{\Big(\overline{\Xi^\rho \partial_\rho} \Sigma^\sigma - \overline{\Sigma^\rho \partial_\rho} \Xi^\sigma\Big) \partial_\sigma} A^\mu - \overline{A^\rho \partial_\rho}\Big(\overline{\Xi^\sigma \partial_\sigma} \Sigma^\mu - \overline{\Sigma^\sigma \partial_\sigma} \Xi^\mu\Big)
\end{aligned}$$

2.2 Contravariant matrix vectors

Abelian: $$\delta_\xi a^\mu = \xi^\rho \partial_\rho a^\mu - a^\rho \partial_\rho \xi^\mu$$

Non-Abelian: $$A^\mu = \sum_{k=0}^\infty \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$$

$$\delta_\Xi A^\mu = \overline{\Xi^\rho \partial_\rho A^\mu} - \overline{A^\rho \partial_\rho \Xi^\mu}$$

Commutator:

$$\begin{aligned}
[\delta_1, \delta_2] A^\mu &= \overline{\Xi^\rho \partial_\rho}\Big(\overline{\Sigma^\sigma \partial_\sigma A^\mu} - \overline{A^\rho \partial_\rho \Sigma^\mu}\Big) - \overline{\Sigma^\rho \partial_\rho}\Big(\overline{\Xi^\sigma \partial_\sigma A^\mu} - \overline{A^\rho \partial_\rho \Xi^\mu}\Big) \\
&= \overline{\Big(\overline{\Xi^\rho \partial_\rho \Sigma^\sigma} - \overline{\Sigma^\rho \partial_\rho \Xi^\sigma}\Big)\partial_\sigma A^\mu} - \overline{A^\rho \partial_\rho}\Big(\overline{\Xi^\sigma \partial_\sigma \Sigma^\mu} - \overline{\Sigma^\sigma \partial_\sigma \Xi^\mu}\Big)
\end{aligned}$$

Scalar multiplication:

$$\begin{aligned}
\delta_\Xi (F \cdot A^\mu) &= \delta_\Xi F \cdot A^\mu + F \cdot \delta_\Xi A^\mu = \overline{\Xi^\rho \partial_\rho}(F \cdot A^\mu) - F \cdot \overline{A^\rho \partial_\rho} \Xi \\
&\neq \overline{\Xi^\rho \partial_\rho}(F \cdot A^\mu) - \overline{(F \cdot A^\rho) \partial_\rho} \Xi^\mu
\end{aligned}$$

$\longrightarrow$ $A^\mu(X)$ do not form vector space!

$\longrightarrow$ Give up Leibnitz rule? Enhance/restrict definition of matrix vector?

2.3 Covariant matrix vectors

Abelian: $df = dx^\rho \partial_\rho f$

2.3 Covariant matrix vectors

Abelian: $df = dx^\rho \partial_\rho f$

Non-Abelian:

$$\underline{d}F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} \Big(\underline{d}X^{\lambda_1} X^{\lambda_2} \ldots X^{\lambda_k} + \ldots + X^{\lambda_1} \ldots X^{\lambda_{k-1}} \underline{d}X^{\lambda_k} \Big)$$

$$\equiv \overline{\underline{d}X^\rho \, \partial_\rho} \, F$$

where we defined $\overline{\underline{d}X^\rho \, \partial_\rho}$ as

"take each X^μ in $F(X)$ in turn and sustitute by $\underline{d}X^\mu$ and sum over everything"

2.3 Covariant matrix vectors

Abelian: $$df = dx^\rho \partial_\rho f$$

Non-Abelian:

$$\underline{d}F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \dots \lambda_k} \left(\underline{d}X^{\lambda_1} X^{\lambda_2} \dots X^{\lambda_k} + \dots + X^{\lambda_1} \dots X^{\lambda_{k-1}} \underline{d}X^{\lambda_k} \right)$$

$$\equiv \overline{\underline{d}X^\rho \, \partial_\rho} \, F$$

where we defined $\overline{\underline{d}X^\rho \, \partial_\rho}$ as

"take each X^μ in $F(X)$ in turn and sustitute by $\underline{d}X^\mu$ and sum over everything"

Variation:

$$\delta_\Xi \left(\underline{d}F \right) = \underline{d} \left(\delta_\Xi F \right) = \underline{d} \left(\overline{\Xi^\mu \, \partial_\mu} \, F \right) = \overline{\underline{d}\Xi^\mu \, \partial_\mu} \, F + \overline{\Xi^\mu \, \partial_\mu} \, \underline{d}F$$

2.3 Covariant matrix vectors

Abelian: $$df = dx^\rho \partial_\rho f$$

Non-Abelian:

$$\underline{d}F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} \left(\underline{d}X^{\lambda_1} X^{\lambda_2} \ldots X^{\lambda_k} + \ldots + X^{\lambda_1} \ldots X^{\lambda_{k-1}} \underline{d}X^{\lambda_k} \right)$$

$$\equiv \overline{\underline{d}X^\rho \, \partial_\rho} \, F$$

where we defined $\overline{\underline{d}X^\rho \, \partial_\rho}$ as

"take each X^μ in $F(X)$ in turn and sustitute by $\underline{d}X^\mu$ and sum over everything"

Variation:

$$\delta_\Xi \left(\underline{d}F \right) = \underline{d} \left(\delta_\Xi F \right) = \underline{d} \left(\overline{\Xi^\mu \, \partial_\mu} \, F \right) = \overline{\underline{d}\Xi^\mu \, \partial_\mu} \, F + \overline{\Xi^\mu \, \partial_\mu} \, \underline{d}F$$

$\longrightarrow$ possible to write in terms of $\underline{d}F$ only?
(necessary for defining one-forms)

$\overline{\underline{d}\Xi^\mu}\,\partial_\mu\, F$ reads

"take each X^μ in $F(X)$ in turn and sustitute by $\underline{d}\Xi^\mu$ and sum over everything"

$$\Longrightarrow\ \overline{\underline{d}\Xi^\mu}\,\partial_\mu\, F \;=\; \Big(\overline{\underline{d}\Xi^\mu\underline{\partial}_\mu}\;\circ\;\underline{d}\Big)F \;=\; \overline{\underline{d}\Xi^\mu\underline{\partial}_\mu}\Big(\underline{d}F\Big)$$

where we defined $\overline{\underline{d}\Xi^\mu\underline{\partial}_\mu}$ as

"take each $\underline{d}X^\mu$ in $\underline{d}F(X)$ in turn (leaving X^ρ's untouched) and sustitute by $\underline{d}\Xi^\mu$ and sum over everything"

$\overline{\underline{d}\Xi^\mu}\,\partial_\mu\, F$ reads

"take each X^μ in $F(X)$ in turn and sustitute by $\underline{d}\Xi^\mu$ and sum over everything"

$$\Longrightarrow\ \overline{\underline{d}\Xi^\mu}\,\partial_\mu\, F \;=\; \Big(\overline{\underline{d}\Xi^\mu \underline{\partial}_\mu} \,\circ\, \underline{d}\Big) F \;=\; \overline{\underline{d}\Xi^\mu \underline{\partial}_\mu}\Big(\underline{d}F\Big)$$

where we defined $\overline{\underline{d}\Xi^\mu \underline{\partial}_\mu}$ as

"take each $\underline{d}X^\mu$ in $\underline{d}F(X)$ in turn (leaving X^ρ's untouched)
and sustitute by $\underline{d}\Xi^\mu$ and sum over everything"

Hence the variation of $\underline{d}F$ is

$$\underline{d}F \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, f_{\lambda_1 \dots \lambda_k} \Big(\underline{d}X^{\lambda_1} X^{\lambda_2} \dots X^{\lambda_k} + \dots + X^{\lambda_1} \dots X^{\lambda_{k-1}} \underline{d}X^{\lambda_k}\Big)$$

$$\delta_{\underline{\Xi}}(\underline{d}F) \;=\; \overline{\underline{d}\Xi^\mu \underline{\partial}_\mu}(\underline{d}F) \;+\; \overline{\Xi^\mu \underline{\partial}_\mu}(\underline{d}F)$$

$\longrightarrow$ Generalize to one-forms that are not $\underline{d}F$

Matrix one-forms

$$B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$$

$$\delta_{\Xi}(B) = \overline{\underline{d}\Xi^{\mu} \underline{\partial}_{\mu}}(B) + \overline{\Xi^{\mu} \partial_{\mu}}(B)$$

Matrix one-forms

$$B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$$

$$\delta_{\Xi}(B) = \overline{\underline{d}\Xi^\mu \underline{\partial}_\mu}(B) + \overline{\Xi^\mu \partial_\mu}(B)$$

Comments

- $[\delta_1, \delta_2]B = \overline{\underline{d}\Lambda^\mu \underline{\partial}_\mu}(B) + \overline{\Lambda^\mu \partial_\mu}(B)$

  with $\Lambda^\mu = \overline{\Xi^\rho \partial_\rho} \Sigma^\mu - \overline{\Sigma^\rho \partial_\rho} \Xi^\mu$

Matrix one-forms

$$B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$$

$$\delta_{\Xi}(B) = \overline{\underline{d}\Xi^\mu \underline{\partial}_\mu}(B) + \overline{\Xi^\mu \partial_\mu}(B)$$

Comments

- $[\delta_1, \delta_2]B = \overline{\underline{d}\Lambda^\mu \underline{\partial}_\mu}(B) + \overline{\Lambda^\mu \partial_\mu}(B)$

  with $\Lambda^\mu = \overline{\Xi^\rho \partial_\rho} \Sigma^\mu - \overline{\Sigma^\rho \partial_\rho} \Xi^\mu$

- Not decomposable in components $B \neq B_\mu \underline{d} X^\mu$

Matrix one-forms

$$B = \sum_{k=1}^{\infty}\sum_{j=1}^{k} b^{(j)}{}_{\lambda_1\ldots\lambda_k} X^{\lambda_1}\ldots X^{\lambda_{j-1}} \underline{d}X^{\lambda_j} X^{\lambda_{j+1}}\ldots X^{\lambda_k}$$

$$\delta_{\Xi}(B) = \overline{\underline{d}\Xi^{\mu}\underline{\partial}_{\mu}}(B) + \overline{\Xi^{\mu}\partial_{\mu}}(B)$$

Comments

- $[\delta_1, \delta_2]B = \overline{\underline{d}\Lambda^{\mu}\underline{\partial}_{\mu}}(B) + \overline{\Lambda^{\mu}\partial_{\mu}}(B)$

  with $\Lambda^{\mu} = \overline{\Xi^{\rho}\partial_{\rho}}\Sigma^{\mu} - \overline{\Sigma^{\rho}\partial_{\rho}}\Xi^{\mu}$

- Not decomposable in components $B \neq B_{\mu}\underline{d}X^{\mu}$

- B matrix one-form, F, F' matrix scalars:

  $\longrightarrow$ $F \cdot B \cdot F'$ is matrix one-form

  $\longrightarrow$ Matrix one-forms span vector space

2.4 Covariant matrix tensors

Abelian: $C = C_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$\longrightarrow$ Place of dx^μ and dx^ν is important

2.4 Covariant matrix tensors

Abelian: $C = C_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$\longrightarrow$ Place of dx^μ and dx^ν is important

Non-Abelian: $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{k} c^{(i,j)}_{\mu_1 \ldots \mu_k}\, X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k}$

where $\underline{d}_{(i)} X^\mu$ indicates ith factor of tensor space

2.4 Covariant matrix tensors

Abelian: $C = C_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$\longrightarrow$ Place of dx^μ and dx^ν is important

Non-Abelian: $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{k} c^{(i,j)}_{\mu_1\ldots\mu_k}\, X^{\mu_1}\ldots \underline{d}_{(1)}X^{\mu_i}\ldots \underline{d}_{(2)}X^{\mu_j}\ldots X^{\mu_k}$

where $\underline{d}_{(i)}X^\mu$ indicates ith factor of tensor space

Transformation: $\delta_\Xi C \;=\; \overline{\Xi^\rho\, \partial_\rho}\, C \;+\; \overline{\underline{d}\Xi^\rho\, \underline{\partial}_\rho}\, C$

Cfr: $\delta C_{\mu\nu} = \xi^\rho \partial_\rho C_{\mu\nu} \;+\; C_{\rho\nu}\partial_\mu \xi^\rho \;+\; C_{\mu\rho}\partial_\nu \xi^\rho$

2.4 Covariant matrix tensors

Abelian: $C = C_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$\longrightarrow$ Place of dx^μ and dx^ν is important

Non-Abelian: $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{k} c^{(i,j)}_{\mu_1\ldots\mu_k}\, X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k}$

where $\underline{d}_{(i)} X^\mu$ indicates ith factor of tensor space

Transformation: $\delta_\Xi C \;=\; \overline{\Xi^\rho\, \partial_\rho}\, C \;+\; \overline{\underline{d}\Xi^\rho\, \underline{\partial}_\rho}\, C$

Cfr: $\delta C_{\mu\nu} = \xi^\rho \partial_\rho C_{\mu\nu} \;+\; C_{\rho\nu} \partial_\mu \xi^\rho \;+\; C_{\mu\rho} \partial_\nu \xi^\rho$

Anti-symmetry: $c^{(i,j)}_{\mu_1\ldots\mu_k} \;=\; -c^{(j,i)}_{\mu_1\ldots\mu_k}$

$$C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{k} c^{(i,j)}_{\mu_1\ldots\mu_k} \Big(X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k} \;-\; X^{\mu_1} \ldots \underline{d}_{(2)} X^{\mu_i} \ldots \underline{d}_{(1)} X^{\mu_j} \ldots X^{\mu_k} \Big)$$

$$
\begin{aligned}
\underline{d}\, B \;\equiv\; \sum_{k=2}^{\infty} \Big[\, & b^{(1)}_{\mu_1 \ldots \mu_k} \Big(\underline{d}_{(1)} X^{\mu_1}\, \underline{d}_{(2)} X^{\mu_2}\, X^{\mu_3} \ldots X^{\mu_k} \;+\; \underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2}\, \underline{d}_{(2)} X^{\mu_3} \ldots X^{\mu_k} \\
& \qquad + \ldots + \underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2} \ldots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \Big) \\
& - b^{(1)}_{\mu_1 \ldots \mu_k} \Big(\underline{d}_{(2)} X^{\mu_1}\, \underline{d}_{(1)} X^{\mu_2}\, X^{\mu_3} \ldots X^{\mu_k} \;+\; \underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2}\, \underline{d}_{(1)} X^{\mu_3} \ldots X^{\mu_k} \\
& \qquad + \ldots + \underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2} \ldots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \Big) \\
& + \ldots + \\
& + b^{(k)}_{\mu_1 \ldots \mu_k} \Big(\underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2} \ldots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \;+\; X^{\mu_1}\, \underline{d}_{(2)} X^{\mu_2}\, X^{\mu_3} \ldots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \\
& \qquad + \ldots + X^{\mu_1} \ldots X^{\mu_{k-2}}\, \underline{d}_{(2)} X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \Big) \\
& - b^{(k)}_{\mu_1 \ldots \mu_k} \Big(\underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2} \ldots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \;+\; X^{\mu_1}\, \underline{d}_{(1)} X^{\mu_2}\, X^{\mu_3} \ldots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \\
& \qquad + \ldots + X^{\mu_1} \ldots X^{\mu_{k-2}}\, \underline{d}_{(1)} X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \Big) \Big]
\end{aligned}
$$

$$
\begin{aligned}
\underline{d}\, B \;\equiv\; \sum_{k=2}^{\infty} \Big[\, & b^{(1)}_{\mu_1 \dots \mu_k} \Big(\underline{d}_{(1)} X^{\mu_1}\, \underline{d}_{(2)} X^{\mu_2}\, X^{\mu_3} \dots X^{\mu_k} \;+\; \underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2}\, \underline{d}_{(2)} X^{\mu_3} \dots X^{\mu_k} \\
& \qquad + \dots + \underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2} \dots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \Big) \\
& - b^{(1)}_{\mu_1 \dots \mu_k} \Big(\underline{d}_{(2)} X^{\mu_1}\, \underline{d}_{(1)} X^{\mu_2}\, X^{\mu_3} \dots X^{\mu_k} \;+\; \underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2}\, \underline{d}_{(1)} X^{\mu_3} \dots X^{\mu_k} \\
& \qquad + \dots + \underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2} \dots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \Big) \\
& + \dots + \\
& + b^{(k)}_{\mu_1 \dots \mu_k} \Big(\underline{d}_{(2)} X^{\mu_1}\, X^{\mu_2} \dots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \;+\; X^{\mu_1}\, \underline{d}_{(2)} X^{\mu_2}\, X^{\mu_3} \dots X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \\
& \qquad + \dots + X^{\mu_1} \dots X^{\mu_{k-2}}\, \underline{d}_{(2)} X^{\mu_{k-1}}\, \underline{d}_{(1)} X^{\mu_k} \Big) \\
& - b^{(k)}_{\mu_1 \dots \mu_k} \Big(\underline{d}_{(1)} X^{\mu_1}\, X^{\mu_2} \dots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \;+\; X^{\mu_1}\, \underline{d}_{(1)} X^{\mu_2}\, X^{\mu_3} \dots X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \\
& \qquad + \dots + X^{\mu_1} \dots X^{\mu_{k-2}}\, \underline{d}_{(1)} X^{\mu_{k-1}}\, \underline{d}_{(2)} X^{\mu_k} \Big) \Big]
\end{aligned}
$$

NB: $B = \underline{d}F \iff \underline{d}B = 0$

$$\longrightarrow \underline{d}\,(\underline{d}\;.) = 0$$

2.5 Scalar product

$$F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k} \qquad \text{(scalar)}$$

$$A^{\mu} = \sum_{k=0}^{\infty} \frac{1}{k!} a^{\mu}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k} \qquad \text{(contrav. vector)}$$

$$B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k} \qquad \text{(covar. vector)}$$

$\longrightarrow$ can we define a scalar product between covar and contrav vectors?

2.5 Scalar product

$$F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k} \qquad \text{(scalar)}$$

$$A^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k} \qquad \text{(contrav. vector)}$$

$$B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k} \qquad \text{(covar. vector)}$$

$\longrightarrow$ can we define a scalar product between covar and contrav vectors?

$\longrightarrow$ sustitute $\underline{d}X^\lambda$ in B by A^λ:

$$A \cdot B = \overline{A^\mu \underline{\partial}_\mu} B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} A^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k} = F(X)$$

where $\overline{A^\mu \underline{\partial}_\mu}$ reads

"take each $\underline{d}X^\mu$ in B in turn and sustitute by A^μ and sum over everything"

Transformation: use Leibniz rule

$$\delta_{\underline{\Xi}}(A \cdot B) = \delta_{\underline{\Xi}} A \cdot B + A \cdot \delta_{\underline{\Xi}} B$$

$$= \overline{(\overline{\underline{\Xi}^\rho\, \partial_\rho}\, A^\mu - \overline{A^\rho\, \partial_\rho}\, \underline{\Xi}^\mu)\, \underline{\partial}_\mu}\, B \;+\; \overline{A^\mu\, \underline{\partial}_\mu}\, (\overline{\underline{\Xi}^\rho\, \partial_\rho}\, B + \overline{\underline{d}\underline{\Xi}^\rho\, \underline{\partial}_\rho}\, B)$$

Transformation: use Leibniz rule

$$\delta_{\underline{\Xi}}(A \cdot B) = \delta_{\underline{\Xi}} A \cdot B + A \cdot \delta_{\underline{\Xi}} B$$

$$= \overline{(\overline{\underline{\Xi}^\rho \partial_\rho} A^\mu - \overline{A^\rho \partial_\rho} \underline{\Xi}^\mu)\, \underline{\partial}_\mu} B + \overline{A^\mu \underline{\partial}_\mu} (\overline{\underline{\Xi}^\rho \partial_\rho} B + \overline{\underline{d}\underline{\Xi}^\rho \underline{\partial}_\rho} B)$$

$$\text{NB}: \overline{(\overline{A^\rho \partial_\rho} \underline{\Xi}^\mu)\, \underline{\partial}_\mu} B = \overline{A^\mu \underline{\partial}_\mu} (\overline{\underline{d}\underline{\Xi}^\rho \underline{\partial}_\rho} B)$$

Transformation: use Leibniz rule

$$\delta_{\underline{\Xi}}(A \cdot B) = \delta_{\underline{\Xi}} A \cdot B + A \cdot \delta_{\underline{\Xi}} B$$

$$= \overline{\left(\overline{\Xi^\rho \, \partial_\rho} \, A^\mu - \overline{A^\rho \, \partial_\rho} \, \Xi^\mu\right) \underline{\partial}_\mu} \, B \; + \; \overline{A^\mu \, \underline{\partial}_\mu} \left(\overline{\Xi^\rho \, \partial_\rho} \, B + \overline{\underline{d}\Xi^\rho \, \underline{\partial}_\rho} \, B\right)$$

$$\text{NB}: \; \overline{\left(\overline{A^\rho \, \partial_\rho} \, \Xi^\mu\right) \underline{\partial}_\mu} \, B \; = \; \overline{A^\mu \, \underline{\partial}_\mu} \left(\overline{\underline{d}\Xi^\rho \, \underline{\partial}_\rho} \, B\right)$$

$$= \overline{\left(\overline{\Xi^\rho \, \partial_\rho} \, A^\mu\right) \underline{\partial}_\mu} \, B \; + \; \overline{A^\mu \, \underline{\partial}_\mu} \left(\overline{\Xi^\rho \, \partial_\rho} \, B\right)$$

$$= \overline{\Xi^\rho \, \partial_\rho} \, (A \cdot B) \qquad \longrightarrow \; A^\mu \text{ and } B \text{ are dual objects}$$

Transformation: use Leibniz rule

$$\delta_{\Xi}(A \cdot B) = \delta_{\Xi}A \cdot B + A \cdot \delta_{\Xi}B$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu - \overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B + \overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$\text{NB}: \;\overline{(\overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B \;=\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B)$$

$$= \overline{\Xi^\rho\,\partial_\rho}\,(A \cdot B) \qquad \longrightarrow \quad A^\mu \text{ and } B \text{ are dual objects}$$

Comments and problems

- Scalar transformation: $\overline{\Xi^\rho\,\partial_\rho}\,F \;=\; \overline{\Xi^\rho\,\underline{\partial}_\rho}\,\underline{d}F \;=\; \Xi \cdot \underline{d}F$

Transformation: use Leibniz rule

$$\delta_\Xi(A\cdot B) = \delta_\Xi A\cdot B + A\cdot\delta_\Xi B$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu - \overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B + \overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$\text{NB}:\ \overline{(\overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B = \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B)$$

$$= \overline{\Xi^\rho\,\partial_\rho}\,(A\cdot B) \qquad \longrightarrow \quad A^\mu \text{ and } B \text{ are dual objects}$$

Comments and problems

- Scalar transformation: $\overline{\Xi^\rho\,\partial_\rho}\,F = \overline{\Xi^\rho\,\underline{\partial}_\rho}\,\underline{d}F = \Xi\cdot\underline{d}F$
- Every form dual to $\overline{A^\rho\,\underline{\partial}_\rho}$ is of the form

$$B = \sum_{k=1}^{\infty}\sum_{j=1}^{k} b^{(j)}{}_{\lambda_1\ldots\lambda_k} X^{\lambda_1}\ldots X^{\lambda_{j-1}}\underline{d}X^{\lambda_j}X^{\lambda_{j+1}}\ldots X^{\lambda_k}$$

Transformation: use Leibniz rule

$$\delta_\Xi(A\cdot B) = \delta_\Xi A\cdot B + A\cdot\delta_\Xi B$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu - \overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B + \overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$\text{NB}:\; \overline{(\overline{A^\rho\,\partial_\rho}\,\Xi^\mu)\,\underline{\partial}_\mu}\,B \;=\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\underline{d}\Xi^\rho\,\underline{\partial}_\rho}\,B)$$

$$= \overline{(\overline{\Xi^\rho\,\partial_\rho}\,A^\mu)\,\underline{\partial}_\mu}\,B \;+\; \overline{A^\mu\,\underline{\partial}_\mu}\,(\overline{\Xi^\rho\,\partial_\rho}\,B)$$

$$= \overline{\Xi^\rho\,\partial_\rho}\,(A\cdot B) \qquad \longrightarrow \; A^\mu \text{ and } B \text{ are dual objects}$$

Comments and problems

- Scalar transformation: $\overline{\Xi^\rho\,\partial_\rho}\,F = \overline{\Xi^\rho\,\underline{\partial}_\rho}\,\underline{d}F = \Xi\cdot\underline{d}F$
- Every form dual to $\overline{A^\rho\,\underline{\partial}_\rho}$ is of the form
  $B = \sum_{k=1}^{\infty}\sum_{j=1}^{k} b^{(j)}{}_{\lambda_1\ldots\lambda_k} X^{\lambda_1}\ldots X^{\lambda_{j-1}}\underline{d}X^{\lambda_j}X^{\lambda_{j+1}}\ldots X^{\lambda_k}$
- More operators than $\overline{A^\rho\,\underline{\partial}_\rho}$ dual to B ($\longrightarrow$ old problem for vectors)

Problem:

Old problem: $F \cdot A^\mu \cdot F'$ does not transforms as vector

Problem:

Old problem: $F \cdot A^\mu \cdot F'$ does not transforms as vector

New problem: Define $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$, acting as:

$$(F \cdot \overline{A^\rho \underline{\partial}_\rho} \cdot F')\, B \;\equiv\; F \cdot (\overline{A^\mu \;\, \underline{\partial}_\mu}\, B) \cdot F' \;=\; F''$$

$\longrightarrow$ Space of objects dual to B contains not only $\overline{A^\mu \underline{\partial}_\mu}$,
but also $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$ $\longrightarrow$ much bigger space!

Problem:

Old problem: $F \cdot A^\mu \cdot F'$ does not transforms as vector

New problem: Define $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$, acting as:

$$(F \cdot \overline{A^\rho \underline{\partial}_\rho} \cdot F')\, B \;\equiv\; F \cdot (\overline{A^\mu \;\underline{\partial}_\mu B}) \cdot F' \;=\; F''$$

$\longrightarrow$ Space of objects dual to B contains not only $\overline{A^\mu \underline{\partial}_\mu}$,
but also $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$ $\longrightarrow$ much bigger space!

New problem 2: What objects are dual to $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$?

$$(G \cdot B \cdot G')(F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F') \;=\; G \cdot F \cdot \overline{A^\mu \underline{\partial}_\mu B} \cdot F' \cdot G'$$

Problem:

Old problem: $F \cdot A^\mu \cdot F'$ does not transforms as vector

New problem: Define $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$, acting as:

$$(F \cdot \overline{A^\rho \underline{\partial}_\rho} \cdot F')\, B \;\equiv\; F \cdot (\overline{A^\mu \;\; \underline{\partial}_\mu} B) \cdot F' \;=\; F''$$

$\longrightarrow$ Space of objects dual to B contains not only $\overline{A^\mu \underline{\partial}_\mu}$,
but also $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$ $\longrightarrow$ much bigger space!

New problem 2: What objects are dual to $F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F'$?

$$(G \cdot B \cdot G')(F \cdot \overline{A^\mu \underline{\partial}_\mu} \cdot F') \;=\; G \cdot F \cdot \overline{A^\mu \underline{\partial}_\mu} B \cdot F' \cdot G'$$

New problem 3: What objects are dual to $(G \cdot B \cdot G')$?

New problem 4: ...

No clear solution at this point...

3. Conclusions

- $F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $A^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$

  $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} c^{(i,j)}_{\mu_1 \ldots \mu_k} X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k}$

3. Conclusions

- $F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $A^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$

  $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} c^{(i,j)}_{\mu_1 \ldots \mu_k} X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k}$

- $\delta F = \overline{\Xi^\rho \partial_\rho} F(X)$

  $\delta A^\mu = \overline{\Xi^\rho \partial_\rho} A^\mu - \overline{A^\rho \partial_\rho} \Xi^\mu$

  $\delta B = \overline{\Xi^\mu \partial_\mu} B + \overline{\underline{d} \Xi^\mu \underline{\partial}_\mu} B$

  $\delta C = \overline{\Xi^\rho \partial_\rho} C + \overline{\underline{d} \Xi^\rho \underline{\partial}_\rho} C$

3. Conclusions

- $F = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $A^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} a^\mu_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_k}$

  $B = \sum_{k=1}^{\infty} \sum_{j=1}^{k} b^{(j)}{}_{\lambda_1 \ldots \lambda_k} X^{\lambda_1} \ldots X^{\lambda_{j-1}} \underline{d} X^{\lambda_j} X^{\lambda_{j+1}} \ldots X^{\lambda_k}$

  $C = \sum_{k=2}^{\infty} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} c^{(i,j)}_{\mu_1 \ldots \mu_k} X^{\mu_1} \ldots \underline{d}_{(1)} X^{\mu_i} \ldots \underline{d}_{(2)} X^{\mu_j} \ldots X^{\mu_k}$

- $\delta F \ = \ \overline{\Xi^\rho \partial_\rho} \, F(X)$

  $\delta A^\mu = \overline{\Xi^\rho \partial_\rho} A^\mu - \overline{A^\rho \partial_\rho} \Xi^\mu$

  $\delta B = \overline{\Xi^\mu \partial_\mu} B + \overline{\underline{d} \Xi^\mu \underline{\partial}_\mu} B$

  $\delta C \ = \ \overline{\Xi^\rho \, \partial_\rho} \, C \ + \ \overline{\underline{d} \Xi^\rho \, \underline{\partial}_\rho} \, C$

- de Rahm operator $\underline{d}$

  scalar product $(A \cdot B) \ \longrightarrow$ problems

- Tower of ever bigger dual spaces

  $\#\{A^\mu\} \; < \; \#\{B\} \; < \; \#\{F \cdot A^\mu \cdot F'\} \; < \; \#\{G \cdot B \cdot G'\} \; < \; \ldots$

- Tower of ever bigger dual spaces
  $\#\{A^\mu\} \ < \ \#\{B\} \ < \ \#\{F \cdot A^\mu \cdot F'\} \ < \ \#\{G \cdot B \cdot G'\} \ < \ \ldots$

$\longrightarrow$ no isomorphism between covariant and contravariant vectors

- Tower of ever bigger dual spaces
  $\#\{A^\mu\} \; < \; \#\{B\} \; < \; \#\{F \cdot A^\mu \cdot F'\} \; < \; \#\{G \cdot B \cdot G'\} \; < \; ...$

  $\longrightarrow$ no isomorphism between covariant and contravariant vectors

  $\longrightarrow$ Not clear how to generalise metric?

$$a_\mu = g_{\mu\nu} a^\nu, \qquad a^\mu = g^{\mu\nu} a_\nu$$

- Tower of ever bigger dual spaces

  $\#\{A^\mu\} \;<\; \#\{B\} \;<\; \#\{F\cdot A^\mu \cdot F'\} \;<\; \#\{G\cdot B\cdot G'\} \;<\; \ldots$

  $\longrightarrow$ no isomorphism between covariant and contravariant vectors

  $\longrightarrow$ Not clear how to generalise metric?

$$a_\mu = g_{\mu\nu} a^\nu, \qquad a^\mu = g^{\mu\nu} a_\nu$$

- Possible ways out:
  - Give up Leibnitz rule?
  - Enhance definition for vector & give up transformation rule?
  - ...?