The Palatini formalism in higher-curvature gravity

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References: JCAP 0811 (2008) 008, `arXiv:0804.4440 [hep-th]`.

Outlook

1. The Levi-Civita connection: what and why?

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2. Palatini formalism for Einstein-Hilbert
3. Higher-curvature corrections and Lovelock gravity
4. More general Lagrangians
   - Metric Formalism
   - Palatini formalism
   - Comparison
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1 The Levi-Civita connection: what and why?

Main lessons of General Relativity:

Gravitational interaction is related to curvature of spacetime,
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Metric: measures distances in manifold and angles in tangent space

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Metric: measures distances in manifold and angles in tangent space

Connection: map between tangent spaces that defines parallel transport

Mathematically metric and connection are independent quantities

However if $\Gamma^{\rho}_{\mu\nu}$ satisfies the properties

- symmetric: $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$ $\quad (\Leftrightarrow \; T^{\rho}_{\mu\nu} = 0)$
- metric compatible: $\nabla_\mu g_{\nu\rho} = 0$

then $\Gamma^{\rho}_{\mu\nu}$ is uniquely determined by the metric

$$\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2}\, g^{\rho\lambda}\Big(\partial_\mu g_{\lambda\nu} + \partial_\mu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big)$$

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- uniqueness: see above
- simplicity: tensor identities simplify
- physical grounds: Equivalence principle & geodesics

Simplicity: many curvature tensor identities simplify for Levi-Civita

- Bianchi identity:

$$0 = R_{\mu\nu\rho}{}^{\lambda} + R_{\nu\rho\mu}{}^{\lambda} + R_{\rho\mu\nu}{}^{\lambda}$$
$$- \nabla_\mu T^\lambda_{\nu\rho} - \nabla_\nu T^\lambda_{\rho\mu} - \nabla_\rho T^\lambda_{\mu\nu} + T^\sigma_{\mu\nu} T^\lambda_{\rho\sigma} + T^\sigma_{\nu\rho} T^\lambda_{\mu\sigma} + T^\sigma_{\rho\mu} T^\lambda_{\nu\sigma}$$

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- Symmetry of Ricci tensor:

$$R_{[\mu\nu]} = \tfrac{1}{2} R_{\mu\nu\lambda}{}^{\lambda} - \tfrac{1}{2} \nabla_{\lambda} T_{\mu\nu}^{\lambda} - \nabla_{[\mu} T_{\nu]\lambda}^{\lambda} + \tfrac{1}{2} T_{\mu\nu}^{\rho} T_{\lambda\rho}^{\lambda}$$

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- Divergence of Einstein tensor:

$$\nabla_\mu G_\nu{}^{\mu} = T_{\mu\nu}^{\rho} R_\rho{}^{\mu} - \tfrac{1}{2} T_{\mu\rho}^{\lambda} R_{\nu\lambda}{}^{\mu\rho} + \tfrac{1}{2} \nabla_\rho g^{\mu\lambda} \Big[\delta_\mu^\rho R_{\nu\lambda} + \delta_\nu^\rho R_{\mu\lambda} + R_{\mu\nu\lambda}{}^{\rho} \Big]$$

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- Partial integration:

$$\begin{aligned} S & = \int d^D x \sqrt{|g|}\, \nabla_{\mu} A_{\nu}\, B^{\mu\nu} \\ & = - \int d^D x \sqrt{|g|} \Big[\nabla_{\mu} B^{\mu\nu} A_{\nu} + \Big(\tfrac{1}{2} g^{\sigma\tau} \nabla_{\mu} g_{\sigma\tau} + T_{\mu\sigma}^{\sigma} \Big) A_{\nu} B^{\mu\nu} \Big] \end{aligned}$$

- ...

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Affine geodesics and metric geodesics do not coincide,
unless for Levi-Civita

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- mathematically more rigorous reason? Palatini formalism!

2 Palatini formalism for Einstein-Hilbert

2.1 Metric formalism

$$S(g) = \int d^D x \, \sqrt{|g|} \, g^{\mu\nu} R_{\mu\nu}(g)$$

where

$$\begin{aligned} R_{\mu\nu} &= \partial_\mu \Gamma^\lambda_{\lambda\nu} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} \\ \Gamma^\rho_{\mu\nu} &= \tfrac{1}{2}\, g^{\rho\lambda}\Big(\partial_\mu g_{\lambda\nu} + \partial_\mu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big) \end{aligned}$$

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- $S(g)$ non-linear and second order in $g_{\mu\nu}$
  $\longrightarrow$ expect fourth-order diff eqns
- explicit calculation: only $\sqrt{|g|} \, g^{\mu\nu}$ contributes to $\delta S(g)$

$$R_{\mu\nu}(g) - \tfrac{1}{2} g_{\mu\nu} R(g) = 0$$

2.2 Palatini formalism

Consider metric and connection as independent:

$$S(g,\Gamma) = \int d^Dx \sqrt{|g|}\, g^{\mu\nu} R_{\mu\nu}(\Gamma)$$

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$$\nabla_\mu g_{\nu\rho} \neq 0 \qquad \text{and/or} \qquad \Gamma^\rho_{\mu\nu} \neq \Gamma^\rho_{\nu\mu}$$

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$$\begin{aligned} 0 \equiv \delta S(g,\Gamma) = \int d^D x \sqrt{|g|} (\delta\Gamma^\lambda_{\mu\nu}) \Big[& \nabla_\lambda g^{\mu\nu} + \Big(\tfrac{1}{2} g^{\sigma\tau}\nabla_\lambda g_{\sigma\tau} + T^\sigma_{\lambda\sigma}\Big) g^{\mu\nu} \\ & - \nabla_\rho g^{\rho\nu}\delta^\mu_\lambda - \Big(\tfrac{1}{2} g^{\sigma\tau}\nabla_\rho g_{\sigma\tau} + T^\sigma_{\rho\sigma}\Big) g^{\rho\nu}\delta^\mu_\lambda + g^{\rho\nu} T^\mu_{\rho\lambda} \Big] \end{aligned}$$

$$\Longleftrightarrow \quad \Gamma^\lambda_{\mu\nu} = \text{Levi} - \text{Civita}$$

Advantages of Palatini formalism:

- Practical: Einstein eqn $\frac{\delta S}{\delta g^{\mu\nu}}$ easy to calculate if $R_{\mu\nu\rho}{}^{\lambda} = R_{\mu\nu\rho}{}^{\lambda}(\Gamma)$

$$S = \int d^D x \, \sqrt{|g|} \, \Big[g^{\mu\nu} R_{\mu\nu}(\Gamma) + \mathcal{L}(\phi, g_{\mu\nu}) \Big]$$

$$\longrightarrow \quad R_{\mu\nu}(\Gamma) - \tfrac{1}{2} g_{\mu\nu} R(\Gamma) = T_{\mu\nu}$$

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- Question: how much remains valid in presence of curvature corrections?

3 Higher-curvature corrections & Lovelock gravity

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- Lanczos (1938):

$$S = \int d^5x \sqrt{|g|} \Big[R^2 \ - \ 4R_{\mu\nu}R^{\mu\nu} \ + \ R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} \Big]$$

$$H_{\mu\nu} = 2RR_{\mu\nu} + 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} + 2R_{\mu\rho\lambda\sigma}R_{\nu}{}^{\rho\lambda\sigma} - 4R_{\mu\rho}R_{\nu}{}^{\rho}$$

$$-\tfrac{1}{2}g_{\mu\nu}\Big[R^2 - 4R_{\rho\lambda}R^{\rho\lambda} + R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}\Big]$$

$\longrightarrow$ second order, divergence-free modified Einstein eqns

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$\longrightarrow$ second order, divergence-free modified Einstein eqns

NB: $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ is Gauss-Bonnet term

$\longrightarrow$ topological invariant in $D = 4$
identically zero in $D < 4$
dynamical in $D > 4$

- Lovelock (1970): generalise Lanczos to arbitrary D
  second order, divergence-free Einstein eqns for

$$S = \int d^D x \sqrt{|g|} \Big[\Lambda \, + \, R \, + \, \mathcal{L}_{\rm GB} \, + \, ... \, + \, \mathcal{L}_{[D/2]} \Big]$$

  where

$$\mathcal{L}_n = \det \begin{vmatrix} g^{\mu_1\nu_1} & \dots & g^{\mu_{2n}\nu_1} \\ \vdots & \ddots & \vdots \\ g^{\mu_1\nu_{2n}} & \dots & g^{\mu_{2n}\nu_{2n}} \end{vmatrix} R_{\mu_1\mu_2\nu_1\nu_2} \dots R_{\mu_{2n-1}\mu_{2n}\nu_{2n-1}\nu_{2n}}$$

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  $\mathcal{L}_n$ is Euler character in $D = 2n$

  $\longrightarrow$ identically zero in $D < 2n$

  topological invariant in $D = 2n$

  dynamical in $D > 2n$

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- Not all stringy corrections are Lovelock, but dilaton might cure the problem.
- Higher-curvature terms are popular in cosmology

  $\longrightarrow$ modify FRW dynamics that mimic dark matter/energy or cause late time acceleration?

  $\longrightarrow$ increasingly more difficult to derive eqns of motion

  $\longrightarrow$ Palatini come to help?

4 More general Lagrangians

4.1 Metric formalism

$$S(g) = \int d^Dx \sqrt{|g|}\, \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}(g))$$

with $\mathcal{L}$ functional of (Riem), but not of ∇(Riem)

Gravitational tensor via chain rule

$$H_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta g^{\mu\nu}}\Big|_{\text{expl}} + \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta R_{\alpha\beta\gamma}{}^{\delta}} \frac{\delta R_{\alpha\beta\gamma}{}^{\delta}}{\delta \Gamma^{\sigma}_{\rho\lambda}} \frac{\delta \Gamma^{\sigma}_{\rho\lambda}}{\delta g^{\mu\nu}}$$

where

$$\delta R_{\mu\nu\rho}{}^{\lambda} = \nabla_\mu(\delta\Gamma^{\lambda}_{\nu\rho}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\mu\rho}),$$

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$$
\begin{aligned}
H_{\mu\nu} \;\; = \;\; & \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \; - \; \tfrac{1}{2} g_{\mu\nu} \mathcal{L} \; + \; \tfrac{1}{2} [\nabla_\alpha, \nabla_\beta] \Big(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}} \Big) g_{\rho(\mu} \delta^{\lambda}_{\nu)} \\
& - \tfrac{1}{2} \nabla_\rho \nabla_\alpha \Big(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}} \Big) g_{\beta(\mu} \delta^{\lambda}_{\nu)} + \tfrac{1}{2} \nabla_\rho \nabla_\beta \Big(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}} \Big) g_{\alpha(\mu} \delta^{\lambda}_{\nu)} \\
& + \tfrac{1}{2} \nabla^\lambda \nabla_\alpha \Big(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}} \Big) g_{\beta(\mu} g_{\nu)\rho} \; - \; \tfrac{1}{2} \nabla^\lambda \nabla_\beta \Big(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}} \Big) g_{\alpha(\mu} g_{\nu)\rho}
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$$

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Examples:

- $\mathcal{L} = R$: $\quad H_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$

- $\mathcal{L} = R^2$: $\quad H_{\mu\nu} \;=\; 2\nabla_\mu \partial_\nu R \;-\; 2 g_{\mu\nu} \nabla^2 R \;+\; 2 R_{\mu\nu} R \;-\; \tfrac{1}{2} g_{\mu\nu} R^2$

- $\mathcal{L} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$:

$$
H_{\mu\nu} = 2RR_{\mu\nu} + 4R_{\rho\mu\nu\lambda}R^{\rho\lambda} + 2R_{\mu\rho\lambda\sigma}R_{\nu}{}^{\rho\lambda\sigma} - 4R_{\mu\rho}R_{\nu}{}^{\rho} - \tfrac{1}{2} g_{\mu\nu}\mathcal{L}
$$

4.2 Palatini formalism

$$S(g,\Gamma) = \int d^Dx \sqrt{|g|}\, \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}(\Gamma))$$

with $\mathcal{L}$ functional of (Riem), but not of ∇(Riem)

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$$R_{\mu\rho} \equiv R_{\mu\lambda\rho}{}^{\lambda} \qquad \tilde{R}_{\mu}{}^{\lambda} \equiv g^{\nu\rho} R_{\mu\nu\rho}{}^{\lambda}, \qquad \bar{R}_{\mu\nu} \equiv R_{\mu\nu\lambda}{}^{\lambda}$$

$$R_{\mu\nu}R^{\mu\nu},\ \ R_{\mu\nu}R^{\nu\mu},\ \ R_{\mu\nu}\tilde{R}^{\mu\nu}, \ldots \quad \longrightarrow \quad R_{\mu\nu}R^{\mu\nu}$$

$$\bar{R}_{\mu\nu}\bar{R}^{\mu\nu},\ \ \bar{R}_{\mu\nu}R^{\mu\nu},\ \ \bar{R}_{\mu\nu}\tilde{R}^{\mu\nu}, \ldots \quad \longrightarrow \quad 0$$

$\longrightarrow$ what is the equivalent of Lovelock gravity?

Gravitational tensor & connection tensor:

$$\tilde{H}_{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g,\Gamma)}{\delta g^{\mu\nu}}, \qquad K_\lambda^{\mu\nu} \equiv \frac{1}{\sqrt{|g|}} \frac{\delta S(g,\Gamma)}{\delta \Gamma_{\mu\nu}^\lambda}$$

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Examples:

- $\mathcal{L} = R$: $\quad \tilde{H}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \qquad K_{\mu\nu}^\lambda = \mathcal{O}(\nabla^\lambda g_{\mu\nu}, T_{\mu\nu}^\lambda)$

- $\mathcal{L} = R^2$:

$$\tilde{H}_{\mu\nu} = 2 R_{\mu\nu} R,$$

$$K_{\mu\nu}^\lambda = 2\Big[g_{\mu\nu} \nabla^\lambda R - \delta_\nu^\lambda \nabla_\mu R \Big] + \mathcal{O}(\nabla^\lambda g_{\mu\nu}, T_{\mu\nu}^\lambda)$$

- $\mathcal{L} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}$:

$$\begin{aligned}
\tilde{H}_{\mu\nu} \;&=\; 2 R_{(\mu\nu)} R \;-\; 8 R_{(\mu|\lambda|} R_{\nu)}{}^\lambda \;+\; R_{\rho\lambda\sigma(\mu} R^{\rho\lambda\sigma}{}_{\nu)} \\
&\quad + 2 R_{(\mu|\lambda\sigma\rho|} R_{\nu)}{}^{\lambda\sigma\rho} \;-\; R_{\rho\lambda(\mu|\sigma|} R^{\rho\lambda}{}_{\nu)}{}^\sigma \;-\; \tfrac{1}{2} g_{\mu\nu} \mathcal{L}, \\
K_{\mu\nu}^\lambda \;&=\; 2\Big[g_{\mu\nu} \nabla^\lambda R - \delta_\nu^\lambda \nabla_\mu R \Big] - 8 \Big[\nabla^\lambda R_{\mu\nu} - \delta_\nu^\lambda \nabla^\sigma R_{\mu\sigma} \Big] \\
&\quad + 4 \nabla^\sigma R_{\nu\sigma}{}^\lambda{}_\mu \;+\; \mathcal{O}(\nabla^\lambda g_{\mu\nu}, T_{\mu\nu}^\lambda)
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- Comparison: take Levi-Civita as Ansatz and substitute in $\tilde{H}_{\mu\nu}$ and $K_\lambda^{\mu\nu}$

$$\Longrightarrow \quad \tilde{R}_{\mu\nu} \longrightarrow -R_{\mu\nu} \qquad R_{\nu\mu} \longrightarrow R_{\mu\nu} \qquad \bar{R}_{\mu\nu} \longrightarrow 0$$

$$\Longrightarrow \quad \mathcal{H}_{\mu\nu} = \tilde{H}_{\mu\nu}\Big|_{\text{Levi-Civita}} \qquad \mathcal{K}_{\mu\nu}^{\lambda} \equiv K_{\mu\nu}^{\lambda}\Big|_{\text{Levi-Civita}}$$

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- $\mathcal{L} = R^2$: $\quad \mathcal{H}_{\mu\nu} = 2 R_{\mu\nu} R, \qquad \mathcal{K}_{\mu\nu}^{\lambda} = 2\Big[g_{\mu\nu}\nabla^\lambda R - \delta_\nu^\lambda \nabla_\mu R\Big]$
- $\mathcal{L} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$:

$$\mathcal{H}_{\mu\nu} = 2R_{\mu\nu}R - 8R_{\mu\lambda}R_\nu{}^\lambda + 2R_{\mu\lambda\sigma\rho}R_\nu{}^{\lambda\sigma\rho} - \frac{1}{2} g_{\mu\nu}\mathcal{L},$$

$$\mathcal{K}_{\mu\nu}^{\lambda} = 2\Big[g_{\mu\nu}\nabla^\lambda R - \delta_\nu^\lambda \nabla_\mu R\Big] - 8\Big[\nabla^\lambda R_{\mu\nu} - \delta_\nu^\lambda \nabla^\sigma R_{\mu\sigma}\Big] + 4\nabla^\sigma R_{\nu\sigma}{}^\lambda{}_\mu$$

4.3 Comparison

$$H_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \; - \; \tfrac{1}{2} g_{\mu\nu}\mathcal{L} \; + \; \tfrac{1}{2}[\nabla_\alpha, \nabla_\beta]\Big(\frac{\delta\mathcal{L}}{\delta R_{\alpha\beta\rho}{}^{\lambda}}\Big) g_{\rho(\mu}\delta^{\lambda}_{\nu)}$$
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$$\boxed{H_{\mu\nu} \; = \; \mathcal{H}_{\mu\nu} \; - \; \tfrac{1}{2}\nabla_\lambda \mathcal{K}^{\lambda}_{(\mu\nu)} \; + \; \tfrac{1}{4} g_{\lambda\mu}\nabla^\rho \mathcal{K}^{\lambda}_{\rho\nu} \; + \; \tfrac{1}{4} g_{\lambda\nu}\nabla^\rho \mathcal{K}^{\lambda}_{\rho\nu}}$$

- Palatini formalism:

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$$H_{\mu\nu} \ = \ \mathcal{H}_{\mu\nu} - \tfrac{1}{2}\nabla_\rho \mathcal{K}^{\rho}_{(\mu\nu)} + \tfrac{1}{2} g_{\lambda\mu}\nabla^\rho \mathcal{K}^{\lambda}_{(\nu\rho)} + \tfrac{1}{2} g_{\lambda\nu}\nabla^\rho \mathcal{K}^{\lambda}_{(\mu\rho)} \ = \ T_{\mu\nu}$$

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  $\longrightarrow$ These are Lovelock gravities! [Exirifard, Sheikh-Jabbari]

5 Conclusions

For an action

$$S = \int d^D x \sqrt{|g|} \Big[\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}) \; + \; \mathcal{L}(g_{\mu\nu}, \phi) \Big]$$

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- Advantages are obvious when equivalence: Lovelock is very special!

- Recall:

$$H_{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta g^{\mu\nu}}\Big|_{\rm expl} + \frac{1}{\sqrt{|g|}} \frac{\delta S(g)}{\delta R_{\alpha\beta\gamma}{}^{\delta}} \frac{\delta R_{\alpha\beta\gamma}{}^{\delta}}{\delta \Gamma_{\rho\lambda}^{\sigma}} \frac{\delta \Gamma_{\rho\lambda}^{\sigma}}{\delta g^{\mu\nu}}$$

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- $H_{\mu\nu} = \mathcal{H}_{\mu\nu} - \frac{1}{2}\nabla_\rho \mathcal{K}^{\rho}_{(\mu\nu)} + \frac{1}{2} g_{\lambda\mu} \nabla^\rho \mathcal{K}^{\lambda}_{(\nu\rho)} + \frac{1}{2} g_{\lambda\nu} \nabla^\rho \mathcal{K}^{\lambda}_{(\mu\rho)}$ reflects structure of the chain rule

- Lovelock gravities $\Longleftrightarrow$ no ∇(Riem) terms

  $\Longleftrightarrow$ $\mathcal{K}^{\lambda}_{\mu\nu} \equiv 0$

  $\Longleftrightarrow$ equivalence between metric & Palatini