New advances on the Grothendieck’s inequality problem for bilinear forms on JB*-triples

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Abstract

1 Introduction

The results known as Grothendieck’s inequalities began with the famous paper [8] in which A. Grothendieck proved the so-called “Grothendieck’s inequalities” for commutative C*-álgebras. These inequalities were generalized by G. Pisier [18] and U. Haagerup [10, 9] to the setting of C*-algebras. Every C*-algebra belongs to a more general class of Banach spaces known as JB*-triples (see definition and examples below). JB*-triples were introduced by Kaup [14] in the study of bounded symmetric domains in complex Banach spaces. The class of JB*-triples has been intensively developed in the last twenty years. In the setting of JB*-triples, Grothendieck’s inequalities were studied by T. Barton and Y. Friedman [1], C.-H. Chu, B. Iochum and G. Loupias [3], A. M. Peralta [15] and A. M. Peralta and A. Rodríguez Palacios [16, 17].

The natural prehilbertian seminorms associated derived from states in a C*-algebra do not make sense in a JB*-triple because the latter needs not have, in general, a natural order structure. In the setting of JB*-triples, the prehilbertian seminorms associated to norm-one functionals are constructed as follows: Let \( \varphi \) be a norm-one element in the dual space of a JB*-triple

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E. Let \( z \) be a norm-one element in \( E \) (or in \( E^{**} \)) such that \( \varphi(z) = 1 \). By [1, Proposition 1.2] the mapping \( (x, y) \mapsto \varphi\{x, y, z\} \) defines a positive sesquilinear form on \( E \) which does not depend on the element \( z \). Thus the law \( x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}} \) \((x \in E)\) defines a prehilbert seminorm on \( E \).

The main contribution of [15, 16] is the discovery that some technical result from the Banach space geometry on weak*-continuous bilinear forms is not true (see [16, Example 1 and comments before]). Therefore, previously published results on Grothendieck’s inequalities for JB*-triples in [1, 3] cannot be considered fully proved. In the amendment provided in [16, Corollaries 1 and 7] it is shown that the assertions in [1, Theorems 1.3 and 1.4] remains true when the seminorms of the form \( \|x\|_\varphi \) are replaced by seminorms of the form \( \|x\|_{\varphi_1, \varphi_2} = \sqrt{\|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2} \). More precisely, there exists a universal constant \( M > 0 \) such that for every pair of JB*-triples \((E, F)\) and every bounded bilinear form \( V \) on \( E \times F \) there exist norm-one functionals \( \varphi_1, \varphi_2 \in E^* \) and \( \psi_1, \psi_2 \in F^* \) satisfying

\[
|V(x, y)| \leq M \|V\| \|x\|_{\varphi_1, \varphi_2} \|y\|_{\psi_1, \psi_2}
\]

for all \((x, y) \in E \times F\). However, until this moment we do not know a counterexample to the version of Grothendieck’s inequality for JB*-triples established by Barton and Friedman. Therefore, it is natural to ask whether the seminorms of the form \( \|x\|_{\varphi_1, \varphi_2} \) appearing in (1) can be replaced by seminorms of the form \( \|x\|_\varphi \), as it is established in [1]. More concretely, let \( \mathcal{G} \) denote the set of all bounded bilinear forms \( V \) on \( E \times F \) such that there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in F^* \) satisfying

\[
|V(x, y)| \leq M \|V\| \|x\|_\varphi \|y\|_\psi,
\]

for all \((x, y) \in E \times F\). Although it is known that \( \mathcal{G} \) is norm-dense in \( L^2(E \times F) \), the space of all bounded bilinear forms on \( E \times F \) (see [16, Theorem 1]), we do not know if \( \mathcal{G} \) coincides or not with whole space \( L^2(E \times F) \).

When \( E \) and \( F \) are JBW*-triples (JB*-triples which are dual Banach spaces) and the bilinear form is assumed to be separately weak*-continuous it seems natural to request that the functionals appearing in (1) belong to the preduals of \( E \) and \( F \), respectively.

In the present paper we present a big class of JB*-triples where the above problem have a positive answer. We shall show that this class includes all Cartan factors and all atomic JBW*-triples.

Let \( X \) and \( Y \) be Banach spaces. Throughout the paper, \( L(X, Y) \) will stand for the Banach space of all bounded linear operators from \( X \) to \( Y \). We usually write \( L(X) \) instead of \( L(X, X) \).
A JB*-triple is a complex Banach space $E$ equipped with a continuous triple product
\[ \{.,.,.\} : E \otimes E \otimes E \to E \]
\[ (x, y, z) \mapsto \{x, y, z\} \]
which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (Jordan Identity)
\[ L(x, y)L(a, b) - L(a, b)L(x, y) = L(L(x, y)a, b) - L(a, L(y, x)b), \]
for all $x, y, a, b \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with non-negative spectrum for all $x \in E$;

(c) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the triple product
\[ \{x, y, z\} = 2^{-1}(xy^*z + zy^*x). \]

Every JB*-algebra is a JB*-triple with triple product given by
\[ \{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*. \]

The (classical) Cartan factors constitute an interesting variety of examples of JB*-triples. Cartan factors are defined as follows (see [13] for more details): Let $H$ and $K$ be complex Hilbert spaces. A type 1 Cartan factor is a JB*-triple of the form $L(H, K)$ with operator norm and triple product defined by
\[ \{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x). \]

Let $j : H \to H$ be a conjugation (conjugate linear isometry of period 2) on $H$. For each $x \in L(H)$ we define $x^t = jx^*j$. The the law $x \mapsto x^t$ defines linear isometry of period 2 on $L(H)$. $S_n := \{x \in L(H) : x^t = -x\}$ with product (2) and operator norm is a Cartan factor of type 2 or of symplectic type and $H_n := \{x \in L(H) : x^t = x\}$ with product (2) and operator norm is a Cartan factor of type 3 or of symplectic type.
A type-4 Cartan factor, (also called spin factor) is a complex Hilbert space provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|\overline{z})\overline{y},$$

and norm given by $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\overline{y})|^2}$.

The type 6 Cartan factor is the space $H_3(\mathbb{O})$ of all $3 \times 3$ hermitian matrices over the complex Cayley algebra $\mathbb{O}$ with product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \quad (3)$$

where $a \circ b = \frac{1}{2}(ab + ba)$. The type 5 Cartan factor consists of all $1 \times 2$ matrices over $\mathbb{O}$ and can be regarded as a JB*-subtriple of the Cartan factor of type 6.

A JBW*-triple is a JB*-triple which is also a dual Banach space. The bidual of a JB*-triple is a JBW*-triple with respect to a triple product extending the one of $E$ [4]. Every JBW*-triple has a unique predual and its triple product is separately weak* continuous [2].

Let $E$ be a JB*-triple. An element $e \in E$ is said to be a tripotent if $\{e, e, e\} = e$. The set of all tripotents of $E$ is denoted by $\text{Tri}(E)$. Given a tripotent $e \in E$ there exist a decomposition of $E$ in terms of the eigenspaces of $L(e, e)$ given by

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e), \quad (4)$$

where $E_k(e) := \{x \in E : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of $E$ ($k : 0, 1, 2$). The natural projection of $E$ onto $E_k(e)$ will be denoted by $P_k(e)$. The following rules are also satisfied

$$\{E_k(e), E_l(e), E_m(e)\} \subseteq E_{k-l+m}(e),$$

$$\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = 0,$$

where $E_{k-l+m}(e) = 0$ whenever $k - l + m$ is not in $\{0, 1, 2\}$. It is also known that $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ y = \{x, e, y\}$ and $x^* = \{e, x, e\}$, respectively. When $E$ is a JBW*-triple then $E_2(e)$ is a JBW*-algebra.

For background about JB- and JBW-algebras the reader is referred to [11]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [21] (respectively, [5]).
2 Grothendieck’s Inequalities

The natural strategy to approach Grothendieck’s inequalities in the setting of JB*-triples is based on the study of the so called “Little Grothendieck’s Theorem” for JB*-triples. The results in [16] provide a new approach to Grothendieck’s inequalities for JB*-triples, which allows us to avoid some difficulties in the proofs of [1, Theorems 1.3 and 1.4] and [3, Proposition 4, Theorem 6]. In [16, Corollary 1] it is proved the following Little Grothendieck’s Theorem:

**Theorem 2.1.** Let \( \mathcal{W} \) be a complex JBW*-triple and \( T \) a weak*-continuous linear operator from \( \mathcal{W} \) to a complex Hilbert space. Then there exist norm-one functionals \( \varphi_1, \varphi_2 \in \mathcal{W}_* \) such that, for every \( x \in \mathcal{W} \), we have

\[
\|T(x)\| \leq 2\|T\|\|x\|_{\varphi_1,\varphi_2}.
\]

The question if in the above Theorem we can replace the seminorm \( \|\cdot\|_{\varphi_1,\varphi_2} \) by a seminorm of the form \( \|\cdot\|_{\varphi} \) remains open. The aim of this section is to give an affirmative answer to the above question in the case of an atomic JBW*-triple.

**Remark 2.2.** Let \( E \) be a finite dimensional JB*-triple and let \( T \) be a bounded linear operator from \( E \) to a complex Hilbert space \( \mathcal{H} \). Since \( T \) attains its norm we conclude from [16, Lemma 3] that there exists a norm-one functional \( \varphi \in E_* \) satisfying

\[
\|T(x)\| \leq \sqrt{2} \|T\| \|x\|_{\varphi},
\]

for all \( x \in E \).

Let \( H \) and \( K \) be Hilbert spaces. Let \( h \) in \( H \) and \( k \) in \( K \) we denote by \( k \otimes h \) the element in \( L(H, K) \) given by \( k \otimes h(x) := (x|h)k \) (\( x \in H \)).

**Proposition 2.3.** Let \( H \) be a complex Hilbert space and let \( p \) be a projection in \( L(H) \). Suppose that \( p(H) = K \) is infinite dimensional. Let \( E = L(H, K) \) be the JBW*-subtriple of \( L(H) \) of all bounded linear operators from \( H \) to \( K \). Then for every normal state \( \phi \in L(H)_* \) there exists a norm-one element \( \varphi \in E_* \) satisfying

\[
\|x\|_{\phi} \leq \sqrt{6} \|x\|_{\varphi}
\]

for all \( x \in E \).
Proof. Let \( \phi \) be a normal state of \( L(H) \). Write \( q = 1 - p \). Let \( x \in L(H) \).

By the Cauchy-Schwarz inequality we deduce that

\[
|\phi(px^*qx)|^2 = |\phi(qx^*xp)|^2 \leq \phi(px^*xp) \phi(qx^*xq),
\]

which implies that

\[
\phi(x^*x) = \phi(px^*xp) + \phi(px^*xq) + \phi(qx^*xp) + \phi(qx^*xq)
\]

\[
\leq \phi(px^*xp) + \phi(qx^*xq) + 2 \sqrt{\phi(px^*xp) \phi(qx^*xq)} \leq 2(\phi(px^*xp) + \phi(qx^*xq)).
\]

Write \( \varphi_1(x) := \phi(px)p \) and \( \varphi_2(x) := \phi(qx)q \). Then \( \varphi_1 \) and \( \varphi_2 \) are positive normal functionals of \( L(H) \), \( \|\varphi_1\| + \|\varphi_2\| = \varphi_1(1) + \varphi_2(1) = \phi(1) \), \( \varphi_1(p) = \varphi_1(1) \), \( \varphi_2(q) = \varphi_2(1) \), and for every positive element \( y \in L(H) \) we have

\[
\phi(y) \leq 2(\varphi_1(y) + \varphi_2(y)). \tag{5}
\]

Since \( \varphi_1 \) is a positive normal functional of \( pL(H)p \), it follows that

\[
\varphi_1(x) := \sum_{n \in \mathbb{N}} \lambda_n(x(\eta_n)|\eta_n) \quad (x \in L(H)), \tag{6}
\]

where \( (\eta_n) \) is an orthonormal sequence in \( p(H) = K \) and \( (\lambda_n) \) is a sequence of non-negative real numbers with \( \sum_{n \in \mathbb{N}} \lambda_n = \|\varphi_1\| = \varphi_1(1) \) (compare [20, Corollary 1.15.4]). Analogously we deduce that

\[
\varphi_2(x) := \sum_{n \in \mathbb{N}} \mu_n(x(\xi_n)|\xi_n) \quad (x \in L(H)), \tag{7}
\]

where \( (\xi_n) \) is an orthonormal sequence in \( p(H) = K \) and \( (\mu_n) \) is a sequence of non-negative real numbers with \( \sum_{n \in \mathbb{N}} \mu_n = \|\varphi_2\| = \varphi_2(1) \).

If \( \|\varphi_2\| = 0 \) then \( \phi = \varphi_1 \) is a norm-one element in \( (pL(H)p)_* = (Ep)_* \subseteq E_* \), which gives the desired conclusion for \( \varphi = \phi \).

If \( \|\varphi_1\| = 0 \) then \( \phi = \varphi_2 \) is a normal state of \( qL(H)q \). Since \( p(H) = K \) is infinite dimensional we can choose an orthonormal sequence \( (\nu_n) \) in \( K \). Let \( \phi \) be the norm-one functional in \( E_* \) defined by

\[
\varphi(x) = \sum_{n \in \mathbb{N}} \mu_n(x(\xi_n)|\nu_n) \quad (x \in E). \]
Let us denote by $e$ the tripotent in $E$ given by $e = \sum_{n \in \mathbb{N}} \nu_n \otimes \xi_n$. Let $x$ be an arbitrary element in $E$. Since $\varphi(e) = 1$ and $\phi(q) = 1$, we deduce that

$$
\|x\|_{\tilde{\varphi}}^2 = \varphi\{x, x, e\} = \frac{1}{2} \varphi(x x^* e + e x^* x) = \sum_{n \in \mathbb{N}} \frac{\mu_n}{2} ((x x^* e + e x^* x)(\xi_n)\nu_n)
$$

and

$$
\|x\|_{\phi}^2 = \phi\{x, x, q\} = \frac{1}{2} \phi(x x^* q + q x^* x) = \frac{1}{2} \phi(q x^* x)
$$

$$
= \sum_{n \in \mathbb{N}} \frac{\mu_n}{2} ((q x^* x)(\xi_n))(\xi_n) = \sum_{n \in \mathbb{N}} \frac{\mu_n}{2} (\|x(\xi_n)\|^2).
$$

From the above expressions we see that

$$
\|x\|_{\phi}^2 \leq \|x\|_{\tilde{\varphi}}^2,
$$

for all $x \in E$.

Finally we assume $\|\varphi_1\|, \|\varphi_2\| \neq 0$. Set $\phi_i := \|\varphi_i\|^{-1} \varphi_i$ ($1 \leq i \leq 2$) and $\tilde{\varphi} = 2^{-1}(\varphi_1 + \varphi_2)$. It is clear that $\tilde{\varphi}$ is a normal state of $L(H)$. Since for each $x \in E$, $\{x, x, 1\}$ is a positive element in $L(H)$, we conclude from (5) that the inequality

$$
\|x\|_{\tilde{\varphi}}^2 = \phi\{x, x, 1\} \leq 2(\phi_1 + \phi_2)\{x, x, 1\} = 4\tilde{\phi}\{x, x, 1\} = 4\|x\|_{\tilde{\varphi}}^2 \quad (8)
$$

holds for each $x \in E$. From (6) and (7) we see that the expression

$$
\|x\|_{\tilde{\varphi}}^2 = \tilde{\phi}\{x, x, 1\} = \frac{1}{2} \tilde{\phi}(x x^* + x^* x)
$$

$$
= \sum_{n \in \mathbb{N}} \left( \frac{\lambda_n}{4\|\varphi_1\|} \left(\|x^* (\eta_n)\|^2 + \|x (\eta_n)\|^2\right) + \frac{\mu_n}{4\|\varphi_2\|} \|x(\xi_n)\|^2 \right), \quad (9)
$$

holds for all $x \in E$.

Let $\varphi$ be the norm-one functional in $E_*$ given by

$$
\varphi(x) = \sum_{n \in \mathbb{N}} \delta_n(x(\xi'_n)|\eta_n) \quad (x \in E),
$$

where $(\xi'_n)$ is the orthonormal sequence in $H$ defined by $\xi'_{2k} = \eta_k$ and $\xi'_{2k-1} = \xi_k$ ($\forall k \in \mathbb{N}$), and $(\delta_n)$ is the sequence in $\mathbb{R}^+_0$ given by $\delta_{2k} = \frac{\lambda_k + \lambda_k}{2\|\varphi_1\|}$. 7
and \( \delta_{2k-1} = \frac{\mu_k}{\|\varphi_2\|^{-1} + \lambda_{2k-1}} \|\varphi_1\|^{-1} \) (\( \forall k \in \mathbb{N} \)). It is not hard to check from the above definition that \( 3\delta_{2n} \geq \lambda_n \|\varphi_1\|^{-1}, 3\delta_{2n-1} \geq \mu_n \|\varphi_2\|^{-1} \), \( 3\delta_n \geq \frac{\lambda_n}{\|\varphi_1\|} \), and \( \sum_{n \in \mathbb{N}} \delta_n = 1 \). Set \( e = \sum_{n \in \mathbb{N}} \eta_n \otimes \xi_n \in E \). It is easy to see that \( \varphi(e) = 1 \).

Thus, for each \( x \in E \) we get

\[
\|x\|_\varphi^2 = \varphi\{x, x, e\} = \sum_{n \in \mathbb{N}} \delta_{2n} \|x^*(\eta_n)\|^2 + \sum_{n \in \mathbb{N}} \delta_{2n-1} \|x(\xi_n)\|^2 + \sum_{n \in \mathbb{N}} \delta_n \|x^*(\eta_n)\|^2.
\]

From (8), (9) and the above expression we deduce that

\[
\|x\|_\varphi^2 \leq 4 \|x\|^2 \varphi \leq 6 \|x\|^2 \varphi,
\]

for all \( x \in E \).

**Corollary 2.4.** Let \( E = L(H, K) \) be a type 1 Cartan factor with \( H \) and \( K \) infinite dimensional and let \( \mathcal{H} \) be a complex Hilbert space. Then for every weak*-continuous linear operator \( T \) from \( E \) to \( \mathcal{H} \) there exists a norm-one functional \( \varphi \in E^* \) satisfying

\[
\|T(x)\| \leq 2\sqrt{3} \|T\| \|x\|_\varphi,
\]

for all \( x \in E \).

**Proof.** Since \( L(H, K) \) and \( L(K, H) \) are triple isomorphic we may assume that \( K \) is a Hilbert subspace of \( H \). Let \( p \) be a projection in \( L(H) \) such that \( p(H) = K \). Let \( T : E \to \mathcal{H} \) be a weak*-continuous linear operator. The law \( z \mapsto T(pz) \) defines a weak*-continuous linear operator \( \tilde{T} \) from \( L(H) \) to \( \mathcal{H} \) which satisfies \( \tilde{T}(x) = T(x) \) for all \( x \in E \). By [10, Proposition 2.3] (see also [16, Remark 1]) there exists a normal state \( \phi \in L(H)_\times \) satisfying

\[
\|\tilde{T}(z)\| \leq \sqrt{2} \|T\| \|z\|_\phi,
\]

for all \( z \in L(H) \). From Proposition 2.3 it follows that there exists a norm-one functional \( \varphi \in E_* \) satisfying

\[
\|x\|_\phi \leq \sqrt{6} \|x\|_\varphi,
\]

for all \( x \in E \). Therefore

\[
\|T(x)\| \leq 2\sqrt{3} \|T\| \|x\|_\varphi,
\]

for all \( x \in E \). \( \square \)
The case of a type 1 Cartan factor $E = L(H, K)$ with $\dim(K)$ finite will need a special development.

**Remark 2.5.** Let $E$ be a JBW*-triple. From [16, Remark 3, (i) $\iff$ (iii)] it follows that the following assertions are equivalent:

(a) There is a universal constant $G$ such that, for every couple $(\varphi_1, \varphi_2)$ of norm-one functionals in $E_* \times E_*$, we can find a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\varphi_i} \leq G\|x\|_{\varphi}$$

for every $x \in E$ and $i = 1, 2$.

(b) There is a universal constant $\tilde{G}$ such that for every weak*-continuous linear operator $T$ from $E$ to a complex Hilbert space, there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|T(x)\| \leq \tilde{G} \|T\| \|x\|_{\varphi}$$

for all $x \in W$.

Moreover, in the implication (a) $\Rightarrow$ (b) we can take $\tilde{G} = 2\sqrt{2}G$, and in (b) $\Rightarrow$ (a) we can choose $G = \sqrt{2}\tilde{G}$ (compare [16, Corollary 1]).

Let $V$ and $W$ be JBW*-triples satisfying one of the above equivalent statements. From [16, Remark 3 (ii) $\iff$ (i)] (see also the proof of [16, Theorem 6]), we deduce that there is a universal constant $\hat{G} = \tilde{G}^2(1 + 2\sqrt{3})$ such that for every separately weak*-continuous bilinear form $U$ on $V \times W$, there are norm-one functionals $\varphi \in V_*$, and $\psi \in W_*$ satisfying

$$|U(x, y)| \leq \hat{G} \|U\| \|x\|_{\varphi} \|y\|_{\psi}$$

for all $(x, y) \in V \times W$.

The following result describes the hilbertian semi-norms of the form $\|\cdot\|_{\varphi}$ in a type 1 Cartan factor.

Let $A$ be a C*-algebra with involution $\ast$. Let $\circ$ denote the natural Jordan product on $A$ defined by $x \circ y = \frac{1}{2}(xy + yx)$. It is well known that $A$ has a JB*-algebra structure with respect to the product $\circ$, the involution $\ast$, and the natural norm. The JB*-algebra $(A, \circ, \ast)$ will be denoted by $A^+$. Moreover $A^+$ is a JBW*-algebra whenever $A$ is a von Neumann algebra. It is also known that $A^+$ and $A$ has the same normal states.
Lemma 2.6. Let $E = L(H,K)$ be a type 1 Cartan factor. Suppose that there exists a projection $p \in L(H)$ with $p(H) = K$. Let $\varphi$ be a norm-one functional in $E^*$. Then there exists a partial isometry $e \in L(H)$ such that $pe = e \in E$, an orthonormal sequence $(\xi_n)$ in $e^*(H)$, and a sequence of non-negative real numbers $(\lambda_n)$ such that $\sum_{n \in \mathbb{N}} \lambda_n = 1$ and

$$\varphi(x) = \sum_{n \in \mathbb{N}} \lambda_n \langle x(\xi_n)|e(\xi_n)\rangle,$$

for all $x \in E$. As a consequence, for each $x \in E$ we have

$$\|x\|^2_\varphi = \sum_{n \in \mathbb{N}} \frac{\lambda_n}{2} \left( \|x^*(e(\xi_n))\|^2 + \|x(\xi_n)\|^2 \right).$$

Proof. It is immediate that every tripotent $e \in E$ is also a tripotent in $L(H)$, since $E = pL(H)$ is a JBW*-subtriple of $L(H)$. Thus every tripotent $e \in E$ is a partial isometry in $L(H)$ satisfying $pe = e$. Let $e$ be a tripotent in $E$. Then $ee^* = p_1$ and $e^*e = q_1$ are projections in $L(H)$ with $p_1 \leq p$ and $e|_{e^*(H)} : e^*(H) = p_1(H) \to e(H) = q_1(H)$ is a surjective isometry. It is easy to check that $E_2(e) = p_1L(H)q_1 = p_1L(H)q_1$.

Let us denote by $\cdot_e$ and $\sharp_e$ the product and involution on $E_2(e)$ given by

$$x \cdot_e y := xe^*y \quad (x,y \in E_2(e))$$

and

$$x^{\sharp_e} = ex^*e \quad (x \in E_2(e)),$$

respectively. It is clear that $(E_2(e), \cdot_e, \sharp_e)$ is a von Neumann algebra and the mapping

$$E_2(e) \to L(e^*(H))$$

$$x \mapsto e^*x$$

is a $^*$-isomorphism from $(E_2(e), \cdot_e, \sharp_e)$ to $L(e^*(H))$.

Let $\varphi$ be a norm-one functional in $E^*$. By [6, Proposition 2], there exists a tripotent $e \in E$ such that $\varphi = \varphi P_2(e)$ and $\varphi|_{E_2(e)}$ is a positive normal functional on the JBW*-algebra $(E_2(e), \circ, \ast) = (E_2(e), \cdot_e, \sharp_e)^+$. Therefore, by [20, Corollary 1.15.4], there exists an orthonormal sequence $(\xi_n)$ in $e^*(H)$, and a sequence of non-negative real numbers $(\lambda_n)$ such that $\sum_{n \in \mathbb{N}} \lambda_n = 1$ and

$$\varphi(x) = \sum_{n \in \mathbb{N}} \lambda_n \langle x(\xi_n)|e(\xi_n)\rangle,$$

for all $x \in E_2(e)$. Finally, the above expression remains valid for all $x \in E$, since $\varphi = \varphi P_2(e)$ and $P_2(e)(x) = ee^*xe^e (\forall x \in E)$. \qed
Remark 2.7. Let $E = L(H, K)$ be a type 1 Cartan factor with $\dim(H) \geq \dim(K)$. Let $\varphi$ be a norm-one element in the predual of $E$ and let $e$ be the tripotent given in $E$ given by Lemma 2.6 above. We claim that we can always assume that $ee^*$ coincide with the orthogonal projection of $H$ onto $K$ (i.e., $ee^*(H) = e(H) = K$). Indeed, from the above proposition we deduce that there is an orthonormal sequence $(\xi_n)$ in $e^*(H)$, and a sequence of non-negative real numbers $(\lambda_n)$ such that $\sum_{n \in \mathbb{N}} \lambda_n = 1$ and

$$\varphi(x) = \sum_{n \in \mathbb{N}} \lambda_n \langle x(\xi_n) | e(\xi_n) \rangle,$$

for all $x \in E$. If $e(H) \neq K$ we write $K_1 = (e(H))^\perp \cap K$. Since $\dim(H) \geq \dim(K)$, there exists a Hilbert subspace $H_1 \subseteq (e^*(H))^\perp \cap H$ and a surjective isometry $e_1$ mapping $H_1$ to $K_1$. Then, when $e_1$ is regarded as a tripotent in $E$ it follows that $u = e + e_1$ is a tripotent in $E$ satisfying $\varphi(u) = 1$ and $u(H) = K$.

Proposition 2.8. Let $K$ be a finite dimensional subspace of a Hilbert space $H$. Let $E = L(H, K)$ be a type 1 Cartan factor. Then for every couple of norm-one functionals $\varphi_1, \varphi_2 \in E_*$ there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\varphi_i} \leq 2\sqrt{2} \|x\|_{\varphi},$$

for all $x \in E$, $i \in \{1, 2\}$.

Proof. Let $p$ denote the orthogonal projection of $H$ onto $K$. Let $\varphi_1, \varphi_2$ norm-one functionals in $E_*$. By Lemma 2.6 there are partial isometries $e_1, e_2 \in L(H)$ such that $pe_i = e_i \in E$ ($i \in \{1, 2\}$), orthonormal sequences $(\xi_n) \subset e_1^*(H)$ and $(\eta_n) \subset e_2^*(H)$, and sequences of non-negative real numbers $(\lambda_n)$ and $(\mu_n)$ such that $\sum_{n \in \mathbb{N}} \lambda_n = \sum_{n \in \mathbb{N}} \mu_n = 1$,

$$\varphi_1(x) = \sum_{n \in \mathbb{N}} \lambda_n \langle x(\xi_n) | e_1(\xi_n) \rangle \quad (\forall x \in E),$$

and

$$\varphi_2(x) = \sum_{n \in \mathbb{N}} \mu_n \langle x(\eta_n) | e_2(\eta_n) \rangle \quad (\forall x \in E),$$

As a consequence, for each $x \in E$ we have

$$\|x\|_{\varphi_1}^2 = \sum_{n \in \mathbb{N}} \frac{\lambda_n}{2} \left( \|x^*(e_1(\xi_n))\|^2 + \|x(\xi_n)\|^2 \right)$$

(12)
and

\[
\|x\|_{\varphi^2}^2 = \sum_{n \in \mathbb{N}} \frac{\mu_n}{2} \left( \|x^*(\eta_n)\|^2 + \|x(\eta_n)\|^2 \right). \tag{13}
\]

Let \(H_1\) be the subspace of \(H\) generated by \(e_1^*(H)\) and \(e_2^*(H)\) and let \(p_1\) be the orthogonal projection of \(H\) onto \(H_1\). Since \(K\) is finite dimensional and for each \(i \in \{1, 2\}\), \(\varphi_i|_{e_i^*(H)} : e_i^*(H) \to e_i(H) \subseteq K\) is a surjective isometry, we conclude that \(H_1\) is finite dimensional. Set \(F = Ep_1 = pL(H)p_1\). Then \(F\) is a finite dimensional JBW*-subtriple of \(E\) and \(e_1, e_2 \in F\). Since \(\|\cdot\|_{\varphi_1,\varphi_2}|F\) comes from a suitable separately weak*-continuous positive sesquilinear form \((\cdot,\cdot)\) on \(F\) given by the equality \(\|x\|_{\varphi_1,\varphi_2}^2 := (x,x)\), it follows from the proof of [19, Corollary] that there exists a weak*-continuous linear operator \(T\) from \(F\) to a Hilbert space satisfying \(\|T(x)\| = \|x\|_{\varphi_1,\varphi_2}\) for each \(x \in F\). Since \(F\) is finite dimensional, it follows from Remark 2.2 that there exists a norm-one functional \(\varphi \in F_* \subseteq E_*\) satisfying

\[
\|y\|_{\varphi_1,\varphi_2}^2 \leq 2\|y\|_{\varphi}^2, \tag{14}
\]

for all \(y \in F\). Let \(e\) be a tripotent in \(F\) such that \(\varphi(e) = 1\). We note that \(F_2(e) = E_2(e)\) and \(Ep_1 = e\). We may also assume \(ee^* = p\) (see Remark 2.7).

Write \(q_1 = 1 - p_1\). Then \(E = F \oplus Eq_1\). Let \(z_3 \in Eq_1\). Since \(\{z_3, z_3, e\}\) is a positive element in the von Neumann algebra \((E_2(e), \bullet, \sharp_e)\) (the latter is defined in the proof of 2.6), then there exists \(y \in E_2(e)\) satisfying \(yee^* = y\) and \(\{z_3, z_3, e\} = y \bullet_e y\). From the equality \(yee^* = yy^*e = y\) we deduce that \(ye^* = ey^*, e^*y = y^*e\) and hence \(\{y, y, e\} = yy^*e = \{z_3, z_3, e\} = yy^*e = \frac{1}{2}z_3z_3^*e\). As a consequence we get \(yy^* = yy^*ee^* = \frac{1}{2}z_3z_3^*ee^* = \frac{1}{2}z_3z_3^*\), and

\[
\|y\|_{\varphi} = \|z_3\|_{\varphi}.
\]

It follows from (14) that

\[
2\|z_3\|_{\varphi}^2 = 2\|y\|_{\varphi}^2 \geq \|y\|_{\varphi_i}^2, \quad (1 \leq i \leq 2).
\]

We compute now the right hand side of the above inequality. From (10) and (11) it is easily seen that \(\varphi_i(ei, yy^*y) \geq 0\) for all \(i \in \{1, 2\}\). Thus

\[
2\|z_3\|_{\varphi}^2 = 2\|y\|_{\varphi}^2 \geq \|y\|_{\varphi_i}^2 = \varphi_i \{y, y, e_i\} = \frac{1}{2} \varphi_i(yyy^*e_i + e_iyy^*) \\
\geq \frac{1}{2} \varphi_i(yyy^*e_i) = \frac{1}{4} \varphi_i(z_3z_3^*e_i) = \frac{1}{2} \varphi_i \{z_3, z_3, e_i\} = \frac{1}{2} \|z_3\|_{\varphi_i}^2.
\]
Therefore, for each $z_3 \in \text{Eq}_1$ and $i \in \{1, 2\}$ we get

$$\|z_3\|_\varphi^2 \leq 4\|z_3\|_\varphi^2$$  \hspace{1cm} (15)

Finally, let $x \in E$. Then $x = y + z_3$ for suitable $y \in F$ and $z_3 \in \text{Eq}_1$. From (14) and (15) we obtain:

$$\|x\|_\varphi \leq \|y\|_\varphi + \|z_3\|_\varphi \leq 2(\|y\|_\varphi + \|z_3\|_\varphi) \leq 2\sqrt{2} \sqrt{\|y\|_\varphi^2 + \|z_3\|_\varphi^2},$$

for all $i \in \{1, 2\}$. Since $e_z z_3^* = ep_1 q_1 z_3^* = 0 = yz_3^*$ we deduce that

$$\varphi \{z_3, y, e\} = \varphi \{y, z_3, e\} = \varphi(0) = 0$$

and hence

$$\|x\|_\varphi^2 = \|y\|_\varphi^2 + \|z_3\|_\varphi^2,$$

which implies

$$\|x\|_\varphi \leq 2\sqrt{2} \|x\|_\varphi,$$

for all $i \in \{1, 2\}$.  \hspace{1cm} $\square$

The following corollary shows that every rectangular type 1 Cartan factor satisfies the Little Grothendieck’s inequality.

**Corollary 2.9.** Let $E = L(H, K)$ be a type 1 Cartan factor with $\dim(H) \geq \dim(K)$. Then for every complex Hilbert space $\mathcal{H}$ and every weak*-continuous linear operator $T : E \to \mathcal{H}$ there exists a norm-one functional $\varphi \in E^*$ satisfying

$$\|T(x)\| \leq 8 \|T\| \|x\|_\varphi,$$

for all $x \in E$.

*Proof.* When $H$ and $K$ are finite dimensional then $E$ is finite dimensional and hence Remark 2.2 gives the desired conclusion. If $H$ and $K$ are infinite dimensional then the statement follows from Proposition 2.3. Finally, if $H$ is infinite dimensional and $K$ is finite dimensional the conclusion follows from Remark 2.5 and Proposition 2.8.  \hspace{1cm} $\square$

We have already proved the Little Grothendieck’s inequality in the particular case of a finite dimensional Cartan factor (see Remark 2.2) and in the case of a rectangular Cartan factor (Corollary 2.9). We shall discuss now the remaining Cartan factors.

Let $J$ be a JB*-triple. We recall that a tripotent $u \in J$ is said to be *unitary* if $L(u, u)$ coincides with the identity operator on $J$. In this case
$J = J_2(u)$ and hence $J$ is a JB*-algebra with product and involution given by $x \circ y = \{x, u, y\}$ and $x^* = \{u, x, u\}$, respectively. When $E$ is a JBW*-triple with a unitary element $u$ then $E$ is a JBW*-algebra with respect to the product and involution given above. We can now rephrase [16, Theorem 4] as follows.

**Proposition 2.10.** Let $M > 2$ and let $E$ be a JBW*-triple with a unitary element $u$. Then for every complex Hilbert space and every weak*-continuous linear operator $T : E \to H$ there exists a norm-one functional $\varphi \in E_*$ such that

$$\|T(x)\| \leq M \|T\| \|x\|_{\varphi},$$

for all $x \in E$.

**Proof.** Let $T$ be a weak*-continuous linear operator from $E$ to a complex Hilbert space. Since $E$ contains a unitary element $u$, then $E$ is a JBW*-algebra with product and involution given by $x \circ y = \{x, u, y\}$ and $x^* = \{u, x, u\}$, respectively. By [16, Theorem 4], there exists a norm-one positive linear functional $\varphi \in E_*$ such that

$$\|T(x)\| \leq M \|T\| (\varphi(x \circ x^*))^{\frac{1}{2}},$$

for all $x \in E$. Since $\varphi$ is norm-one and positive then $\varphi(u) = 1 = \|\varphi\|$, and hence for each $x \in E$ we have $\|x\|^2_{\varphi} = \varphi\{x, x, u\} = \varphi(x \circ x^*)$, which completes the proof. \qed

Let $S$ be a spin factor and let $u$ be a norm-one element in $S$ satisfying $u = \overline{u}$. It is easily seen that $L(u, u)$ coincides with the identity operator on $S$ and hence $u$ is a unitary element in $S$. It is also known that every Cartan factor of type 1 with $\dim(H) = \dim(K)$, every Cartan factor of type 2 with $\dim(H)$ even, or infinite, every Cartan factors of type 3 and every type 6 Cartan factor contains a unitary element (see for instance [12, Proposition 2]). As a consequence, we can assure that when $C$ is one of the above Cartan factors and $\Omega$ is a hyperstonean compact Hausdorff space then $C(\Omega, C)$ is a JBW*-triple containing a unitary element.

**Corollary 2.11.** Let $E = C(\Omega, C)$, where $\Omega$ is a hyperstonean Hausdorff space and $C$ is a Cartan factor of type 1 with with $\dim(H) = \dim(K)$, or a Cartan factor of type 2 with $\dim(H)$ even, or infinite, or a Cartan factors of type 3, or a spin factor, or a type 6 Cartan factor. Let $M > 2$. Then for every complex Hilbert space and every weak*-continuous linear operator $T : E \to H$ there exists a norm-one functional $\varphi \in E_*$ such that

$$\|T(x)\| \leq M \|T\| \|x\|_{\varphi},$$

14
for all $x \in E$.

The next theorem shows that the family of all JBW*-triples satisfying the Little Grothendieck’s inequality is stable by $\ell_\infty$-sums.

**Theorem 2.12.** Let $M > 0$. Let $\{E_\alpha\}_{\alpha \in \Lambda}$ be a family of JBW*-triples such that for every $\alpha \in \Lambda$ and every weak*-continuous linear operator $T$ from $E_\alpha$ to a complex Hilbert space $H$ there exists a norm-one functional $\varphi_\alpha \in (E_\alpha)_*$ satisfying that

$$\|T(x)\| \leq M \|T\| \|x\|_{\varphi_\alpha},$$

for all $x \in E_\alpha$. Let $E = \bigoplus_{\alpha \in \Lambda} E_\alpha$. Then for every complex Hilbert space $H$ and every weak*-continuous linear operator $T : E \to H$ there exists a norm-one functional $\varphi \in E_*$ such that

$$\|T(x)\| \leq 4\sqrt{2} M \|T\| \|x\|_{\varphi},$$

for all $x \in E$.

**Proof.** By [16, Remark 3] (see also Remark 2.5 above) it suffices to prove that for every pair $(\varphi_1, \varphi_2)$ of norm-one functionals in $E_* \times E_*$ there exists a norm-one functional $\varphi \in E_*$ satisfying

$$\|x\|_{\varphi_1, \varphi_2} \leq 2M \|x\|_{\varphi},$$

for all $x \in E$. Let $\varphi_1, \varphi_2$ norm-one functionals in $E_* = \bigoplus_{\alpha \in \Lambda} (E_\alpha)_*$. Then there are countably subsets $\Lambda_1, \Lambda_2 \subseteq \Lambda$ such that

$$\varphi_1 = \sum_{\alpha \in \Lambda_1} \mu_\alpha \phi_1^\alpha,$$

$$\varphi_2 = \sum_{\alpha \in \Lambda_2} \nu_\alpha \phi_2^\alpha,$$

where $(\mu_\alpha) \in \ell_1(\Lambda_1)$, $(\nu_\alpha) \in \ell_1(\Lambda_2)$, $\mu_\alpha, \nu_\alpha \geq 0$, $\phi_j^\alpha$ are norm-one elements in $(E_\alpha)_*$ $\forall j : 1, 2, \forall \alpha$ and $\|\varphi_1\| = \sum_{\alpha \in \Lambda_1} \mu_\alpha$ and $\|\varphi_2\| = \sum_{\alpha \in \Lambda_2} \nu_\alpha$.

Let $I = \Lambda_1 \cap \Lambda_2$, $I_1 = (\Lambda \setminus \Lambda_2) \cap \Lambda_1$ and $I_2 = (\Lambda \setminus \Lambda_1) \cap \Lambda_2$. By hypothesis and [16, Remark 3] it follows that for each $\alpha \in I$ there exists a norm-one functional $\psi_\alpha \in (E_\alpha)_*$ such that

$$\|x\|_{\varphi_1, \varphi_2} \leq \sqrt{2}M \|x\|_{\psi_\alpha},$$

for all $x \in E_\alpha$. Let $\varphi$ be the norm-one functional in $E_*$ defined by

$$\varphi := \sum_{\alpha \in I} \frac{\mu_\alpha + \nu_\alpha}{2} \psi_\alpha + \sum_{\alpha \in I_1} \frac{\mu_\alpha}{2} \phi_1^\alpha + \sum_{\alpha \in I_2} \frac{\nu_\alpha}{2} \phi_2^\alpha.$$
It is not hard to see that in this case
\[ \|x\|_{\varphi_1,\varphi_2} \leq 2M \|x\|_{\varphi}, \]
for all \( x \in E \), which proves the theorem. \( \square \)

Let \( e \) be a tripotent in a JB*-triple \( J \). When \( J^2(e) = Ce \) we say that \( e \) is a minimal tripotent. A JBW*-triple \( E \) is called \textit{atomic} if \( E \) coincides with the weak*-closed ideal generated by all its minimal tripotents. From [7, Proposition 2] it follows that every atomic JBW*-triple coincides with an \( \ell_{\infty} \)-sum of Cartan factors.

\textbf{Theorem 2.13.} Let \( E \) be an atomic JBW*-triple. Then for every weak*-continuous linear operator \( T \) from \( E \) to a complex Hilbert space there exists a norm-one functional \( \varphi \in E^* \) satisfying
\[ \|T(x)\| \leq 32\sqrt{2} \|T\| \|x\|_{\varphi}, \]
for all \( x \in E \).

\textit{Proof.} Let \( E \) be an atomic JBW*-triple. We have already commented that \( E \) admits a decomposition in the form \( \bigoplus \ell_{\infty} C_\alpha \), where each \( C_\alpha \) is a Cartan factor. If we prove that each factor \( C_\alpha \) satisfies the hypothesis of Theorem 2.12 for \( M = 8 \), then the assertion will follow from the just quoted theorem.

Let \( T : C_\alpha \to H \) be a weak*-continuous linear operator from \( C_\alpha \) to a complex Hilbert space. If \( C_\alpha \) is a type 1 Cartan factor with \( \dim(H) > \dim(K) \), then Corollary 2.9 assures the existence of a norm-one functional \( \varphi_\alpha \in (C_\alpha)^* \) satisfying inequality (16) for \( M = 8 \). If \( C_\alpha \) is a Cartan factor of type 1 with \( \dim(H) = \dim(K) \), or a Cartan factor of type 2 with \( \dim(H) \) even, or infinite, or a Cartan factors of type 3, or a type 6 Cartan factor, then 2.11 gives the existence of a norm-one functional \( \varphi_\alpha \in (C_\alpha)^* \) satisfying (16) for \( M > 2 \). Finally, if \( C_\alpha \) is finite dimensional, then it follows from Remark 2.2 that there exists a norm-one functional \( \varphi_\alpha \in (C_\alpha)^* \) satisfying (16) for \( M = \sqrt{2} \). \( \square \)

Let \( E \) be a JB*-triple. We have already mentioned that \( E^{**} \) is a JBW*-triple. From [6, Theorems 1 and 2] it follows that \( E^{**} \) and \( E^* \) admit the following decompositions:
\[ E^{**} = A \oplus \ell_{\infty} N, \]
and
\[ E^* = A_\ast \oplus \ell_1 N_\ast, \]
where $A$ is the weak*-closed ideal of $E^{**}$ generated by all minimal tripotents of $E^{**}$, $N$ contains no minimal tripotents, $A_*$ is the predual of $A$ and coincides with the norm closure of the linear span of the extreme points of the closed unit ball of $E^*$, and the closed unit ball of $N_*$ contains no extreme points. $A$ is called the atomic part of $E^{**}$. Moreover, by [7, Proposition 2] we conclude that $A$ is an $\ell_\infty$-sum of Cartan factors.

**Corollary 2.14.** Let $E$ be a JB*-triple and let $A$ denote the atomic part of $E^{**}$. Then for every pair $(\varphi_1, \varphi_2)$ of norm-one functionals in $A_* \subseteq E^*$, there exists a norm-one functional $\varphi \in A_* \subseteq E^*$ satisfying
\[
\|x\|_{\varphi_1, \varphi_2} \leq 32 \sqrt{2} \|x\|_{\varphi},
\]
for all $x \in E$.

**Proof.** Let $(\varphi_1, \varphi_2)$ be a couple of norm-one functionals in $A_* \subseteq E^*$. By Theorem 2.13 there exists a norm-one functional $\varphi \in A_*$ satisfying
\[
\|z\|_{\varphi_1, \varphi_2} \leq 32 \sqrt{2} \|z\|_{\varphi},
\]
for all $z \in A$.

Let $i$ denote the canonical embedding of $E$ in its bidual and let $\pi$ be the natural projection of $E^{**}$ onto $A$. From the proof of [7, Proposition 1] we deduce that $\pi \circ i$ is a triple embedding of $E$ into $A$. Let $\phi$ be a norm-one functional in $A_*$. $\phi$ can be also regarded as a norm-one element in $E^*$. Since $\phi$ attains its norm at a tripotent in $A$, it is not hard to see that for each $x \in E$ we have
\[
\|\pi \circ i(x)\|_{\phi} = \|x\|_{\phi}.
\]
The conclusion of the corollary follows now from the above expression and inequality (17).

From the last part of Remark 2.5 and Theorem 2.13 we derive the following Grothendieck’s Theorem for atomic JBW* -triples.

**Theorem 2.15.** Let $V$ and $W$ be atomic JBW*-triples. Then for every separately weak*-continuous bilinear form $U$ on $V \times W$, there are norm-one functionals $\varphi \in V_*$, and $\psi \in W_*$ satisfying
\[
|U(x, y)| \leq 2^{11} (1 + 2\sqrt{3}) \|U\| \|x\|_{\varphi} \|y\|_{\psi}
\]
for all $(x, y) \in V \times W$. \(\square\)
References


