# Dunford-Pettis properties in projective tensor products


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Workshop on Functional Analysis
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Valencia, June 2013

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The space $L^1(\mu)$ satisfies the DPP.

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$X^*$ has the DPP $\Rightarrow$ $X$ has the DPP.

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Despite of early contributions by N. Dunford, B.J. Pettis, A. Grothendieck, S. Kakutani, K. Yosida, M. Talagrand, J. Bourgain and several others, the Dunford-Pettis property is not completely understood. An irritating example is the projective tensor product of Banach spaces satisfying the DPP.

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Necessary Conditions: Since the DPP is inherited by complemented subspaces, it follows that $X$ and $Y$ satisfy the DPP whenever $X\hat{\otimes}_\pi Y$ has this property.

A negative answer:

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**[M. Talagrand, Israel J. Math.'1983]**

There exists a Banach space $X$ such that $X^*$ has the Schur property and $X^* \hat{\otimes}_\pi L^1[0,1]$ does not satisfy the DPP.

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Positive answers:

**[R. Ryan, Bull. Polish Acad. Sci. Math.'1987]**

The projective tensor product $X \hat{\otimes}_\pi Y$ satisfies the DPP and contains no copies of $\ell_1$ whenever $X$ and $Y$ have both properties.

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**[J. Becerra, A.M. Peralta, Math. Z.'2005]**

Let $X$ and $Y$ be two infinite-dimensional Banach spaces satisfying DPP and property $(V)$. Then $X\hat{\otimes}_\pi Y$ fails DPP whenever $X$ or $Y$ contains a copy of $\ell_1$.

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Let $X$ and $Y$ be Banach spaces. The following are equivalent:

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(b) $X$ and $Y$ have both properties and contain no copies of $\ell_1$.

When particularized to the classes of C$^*$-algebras and JB$^*$-triples (a wide class of complex Banach spaces defined by the "*good*" holomorphic properties of their open unit balls), and recalling that these spaces satisfy property $(V)$, we have:

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Let $A$ and $B$ be two C$^*$-algebras. The following statements are equivalent:

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A Banach space $X$ has the alternative Dunford-Pettis property (DP1 in the sequel) if whenever $x_n \to x$ weakly in $X$, with $\|x_n\| = \|x\| = 1$, and $\varphi_n \to 0$ weakly in $X^*$, we have $\varphi_n(x_n) \to 0$.

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A Banach space satisfies the KKP if weak sequential convergence in the unit sphere of $X$ implies norm convergence

## Map of relations:

Dunford-Pettis property

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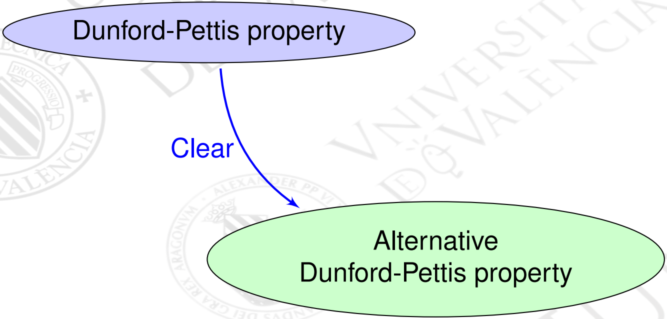

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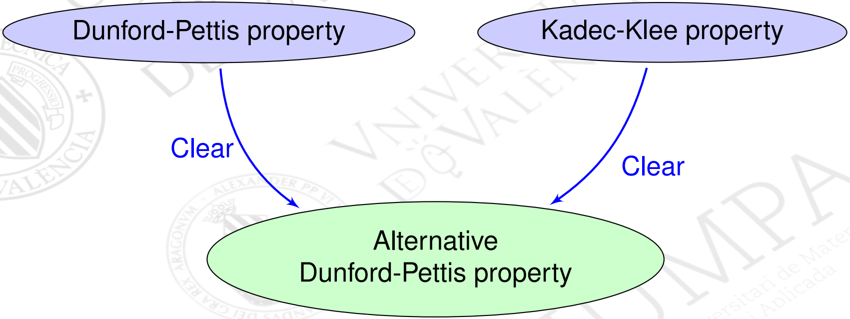

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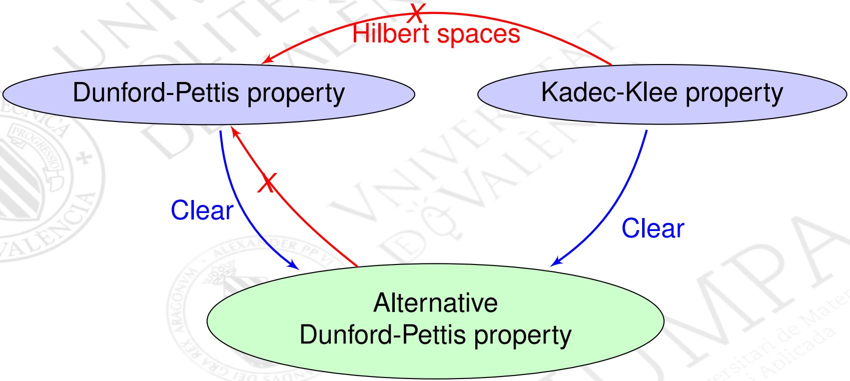

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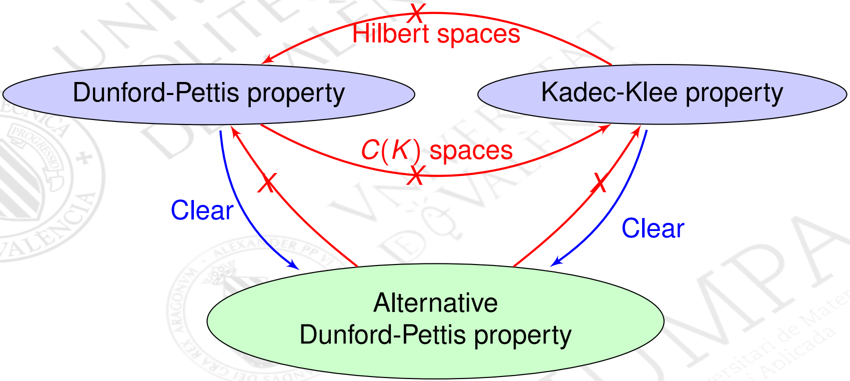

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When $X$ is reflexive, $X$ satisfies DP1 if and only if $X$ has KKP.

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A von Neumann algebra is a $C^*$-algebra which is also a dual Banach space.

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**Question:**

Is the above statement true for $C^*$-algebras?

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DPP and DP1 are equivalent for general $C^*$-algebras.

**[W. Freedman, Studia'1997, M.D. Acosta, A.M. Peralta, Quart. J. Math.'2001, L. Bunce, A.M. Peralta, Studia Math.'2004]**

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Once the first and basic results to understand the DP1 are given, it seemed more and more natural to explore the DP1 on projective tensor products of Banach spaces, and in particular of C$^*$-algebras and JB$^*$-triples.

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By confining the DP condition to the unit sphere of norm one elements the class of Banach spaces DP1 is strictly wider but we impose a metric condition which makes harder the study on projective tensor products.

## The inspiration: Complemented copies of $\ell_2$ in projective tensor products

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*"Unexpected subspaces of tensor products"*

**[F. Cabello, D. Pérez-García, I. Villanueva, J. London Math. Soc.'2006]**

When the projective tensor product of two infinite dimensional $C(K)$-spaces fails the DPP it also fails a weaker property, that is, in such a case it contains a complemented copy of $\ell_2$.

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*"Unexpected subspaces of tensor products"*

**[F. Cabello, D. Pérez-García, I. Villanueva, J. London Math. Soc.'2006]**

When the projective tensor product of two infinite dimensional $C(K)$-spaces fails the DPP it also fails a weaker property, that is, in such a case it contains a complemented copy of $\ell_2$.

**[A.M. Peralta, I. Villanueva, Math. Z.'2006]**

Let $E, F$ be Banach spaces such that $E$ contains $c_0$ and $F$ contains a $C(K)$ space $G$ containing $\ell_1$. Then $E\hat{\otimes}_\pi F$ contains a complemented copy of $\ell_2$.

For our purposes:

**[A.M. Peralta, I. Villanueva, Math. Z.'2006]**

Let $E, F$ be JB$^*$-triples such that $E$ is not reflexive and $F$ contains $\ell_1$. Then $E\hat{\otimes}_\pi F$ contains a complemented copy of $\ell_2$.

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**Key tool:**

$\ell_2 \otimes^\infty \ell_2$ does not satisfy the DP1.

Complemented subspaces of the projective tensor product take us to our goal:

**[A.M. Peralta, I. Villanueva, Math. Z.'2006]**

Let $E, F$ be two Banach spaces such that $E$ contains an isometric copy of $c_0$ and $F$ contains and isometric copy of $C[0,1]$. Then $E\hat{\otimes}_\pi F$ does not have DP1.

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A and $B$ be infinite dimensional C*-algebras. Then the following are
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Complemented subspaces of the projective tensor product take us to our goal:

**[A.M. Peralta, I. Villanueva, Math. Z.'2006]**

Let $A$ and $B$ be infinite dimensional C*-algebras. Then the following are equivalent:

(*a*) $A\hat{\otimes}_\pi B$ satisfies the DP1;

(*b*) $A$ and $B$ satisfy the DPP and do not contain $\ell_1$;

(*c*) $A\hat{\otimes}_\pi B$ satisfies the DPP;

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$(c)$ $A\hat{\otimes}_\pi B$ satisfies the DPP;

## Corollary

Let $K_1$ and $K_2$ be infinite compact Hausdorff spaces. Then the following are equivalent:

(a) $C(K_1)\hat{\otimes}_\pi C(K_2)$ satisfies the DP1;

(b) $C(K_1)$ and $C(K_2)$ satisfy the DPP and do not contain $\ell_1$;

(c) $C(K_1)\hat{\otimes}_\pi C(K_2)$ satisfies the DPP.

## Finally...

## Finally. . .

On behalf of those mathematicians (like me) who learnt from your contributions and will continue doing so . . . Many thanks Andreas!!