HYPERSURFACES WITH CONSTANT CURVATURE IN $\mathbb{R}^{N+1}$

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Abstract. We give some optimal estimates of the height, curvature and volume of compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant curvature bounding a planar closed $(n-1)$-submanifold.

1. Introduction. The compact hypersurfaces of constant positive curvature K in $\mathbb{R}^{n+1}$, (K-hypersurfaces), have been the principal objects of interaction between differential geometry theory of convex bodies and elliptic partial differential equations, specially those of Monge-Ampère type. Although many problems about their existence an uniqueness seem far from being understood, here we pose the problem of clarifying some properties about the geometry and topology of a K-hypersurface.

Since closed K-hypersurfaces are round spheres, the K-hypersurfaces of interest to us bound a connected submanifold of codimension 2 which lies in a hyperplane.

Let S be a compact n-manifold with a nonempty connected boundary ∂S and $x : S \longrightarrow \mathbb{R}^{n+1}$ be a K-hypersurface such that $\Gamma = x(\partial S)$ lies in a hyperplane P of $\mathbb{R}^{n+1}$. First, we recall some elementary facts about K-hypersurfaces. Let N and η be unit normal vector fields along S and ∂S in $\mathbb{R}^{n+1}$ and P, respectively. Then, (up to sign) we have that along ∂S

$$\langle dN, dx\rangle = \langle N, \eta\rangle\langle d\eta, dx\rangle,$$

along ∂S. This means that asymptotic directions on Γ are also asymptotic on $x(S)$. We conclude that Γ must be locally strictly convex and P meets $x(S)$ transversally. Moreover, if $n \geq 3$, then the normal vector field $\eta : \partial S \longrightarrow \mathbb{S}^{n-1}$ along ∂S in P is a global diffeomorphism and Γ must be a hyperovaloid in P. Now, if Γ is embedded, then by using for instance the results of Ghomi, see [6], we can find a connected hypersurface M in $\mathbb{R}^{n+1}$ such that $x(S) + M$ is a hyperovaloid in $\mathbb{R}^{n+1}$. Consequently, there exists a convex body U in $\mathbb{R}^{n+1}$ with $\partial U = x(S) + M$. Particularly, $x(S)$ must be embedded and it lies in one of the halfspaces determined by P.

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Our goal in this paper is to prove some optimal estimates of the height, curvature and enclosed volume of hypersurfaces with positive constant curvature.

In §2 we prove two elliptic PDE's associated with the second fundamental form of the immersion which help us get height estimates for K-hypersurfaces (Theorems 1 and 2).

In §3 we derive a balancing formula which lets us to obtain optimal curvature estimates of K-hypersurfaces bounding a connected $(n-1)$-hyperovaloid in P, (Theorem 3).

Finally in §4 we prove an estimation of the volume enclosed by a graph with constant curvature and boundary lying in a hyperplane.

2. Height estimates. In order to get an estimation of the maximum height at which a hypersurface with constant curvature can rise above a hyperplane, we calculate the laplacian respect to the second fundamental form of the immersion and its Gauss map.

Lemma 1. *Let S be an orientable n-manifold and $x : S \longrightarrow \mathbb{R}^{n+1}$ an immersion with Gauss map $N : S \longrightarrow \mathbb{S}^n$ and a non-degenerate second fundamental form, $\sigma = -\langle dN, dx\rangle$. Then, the curvature of the immersion is constant if and only if*

$$\Delta^\sigma x = n\,N, \qquad \Delta^\sigma N = -n\,H\,N, \tag{1}$$

where H is the mean curvature of the immersion and Δ^σ denotes the laplacian of the second fundamental form.

Proof. Let ∇ and ∇^σ be the Levi–Civita connections of the usual metric of $\mathbb{R}^{n+1}$ and σ, respectively, and consider $\{E_1, \dots, E_n\}$ an orthonormal moving frame in a neighbourhood of $p \in S$, parallel at p for the metric σ, that is, $\sigma(E_i, E_j) = \varepsilon_i \delta_{ij}$, $\nabla^\sigma_{E_i(p)} E_j = 0$, where $\varepsilon_i = \pm 1$ and δ_{ij} the Kronecker delta.

Using that $\langle \nabla_{E_i} N, E_j\rangle = -\sigma(E_i, E_j) = -\varepsilon_i \delta_{ij}$ we can calculate $\langle \Delta^\sigma N, E_j\rangle$ at p:

$$\begin{aligned} \langle \Delta^\sigma N, E_j\rangle &= \sum_{i=1}^n \varepsilon_i \langle E_i(E_i(N)), E_j\rangle = \sum_{i=1}^n \varepsilon_i \langle \nabla_{E_i}\nabla_{E_i} N, E_j\rangle \\ &= \sum_{i=1}^n \varepsilon_i (E_i\langle \nabla_{E_i} N, E_j\rangle - \langle \nabla_{E_i} N, \nabla_{E_i} E_j\rangle) \\ &= \sum_{i=1}^n \varepsilon_i \langle -\nabla_{E_i} N, \nabla_{E_i} E_j\rangle. \end{aligned} \tag{2}$$

Moreover, if $G = (g_{kl}) = (\langle E_k, E_l\rangle)$ and $G^{-1} = \left(g^{lk}\right)$ is its inverse matrix, then

$$-\nabla_{E_i} N = \sum_{l=1}^n \varepsilon_i g^{il} E_l. \tag{3}$$

Since the Lie bracket $[E_i, E_j](p) = 0$, from (2), (3) and Koszul formula, we obtain

$$\begin{aligned} \langle \Delta^\sigma N, E_j\rangle &= \sum_{i,l=1}^n g^{il} \langle \nabla_{E_i} E_j, E_l\rangle = \frac{1}{2}\sum_{i,l=1}^n g^{il}\,(E_i\langle E_j, E_l\rangle + E_j\langle E_i, E_l\rangle \\ &\qquad -E_l\langle E_i, E_j\rangle) = \frac{1}{2}\sum_{i,l=1}^n g^{il} E_j(g_{il}) = \frac{1}{2}\,\mathrm{trace}(G^{-1}E_j(G)) \\ &= \frac{1}{2} E_j(\log(\det(G))), \end{aligned}$$

where det denotes the usual determinant.

As the curvature K satisfies $|K|\det(G)=1$

$$\langle \Delta^{\sigma} N, E_j\rangle = -\frac{1}{2}E_j(\log|K|). \tag{4}$$

Thus, it is clear that the tangent part of $\Delta^{\sigma} N$ vanishes if, and only if, K is constant.

On the other hand, using (3)

$$\begin{aligned} \langle \Delta^{\sigma} N, N\rangle &= \sum_{i=1}^{n}\varepsilon_i\langle E_i(E_i(N)),N\rangle = \sum_{i=1}^{n}\varepsilon_i\langle \nabla_{E_i}\nabla_{E_i}N,N\rangle \\ &= \sum_{i=1}^{n}\varepsilon_i(E_i\langle \nabla_{E_i}N,N\rangle - \langle \nabla_{E_i}N,\nabla_{E_i}N\rangle) \\ &= \sum_{i,l=1}^{n} g^{il}\langle E_l,\nabla_{E_i}N\rangle = -\sum_{i,l=1}^{n} g^{il}\varepsilon_i\delta_{il} = -\sum_{i=1}^{n} g^{ii}\varepsilon_i \\ &= -nH. \end{aligned} \tag{5}$$

In that way, from (4) and (5), K is constant if and only if $\Delta^{\sigma} N = -n\,H\,N$ and we conclude the first assertion of the Lemma.

Now, the Codazzi equation gives

$$\langle \nabla_{E_i}N, \nabla_{E_j}E_k\rangle = \langle \nabla_{E_j}N,\nabla_{E_i}E_k\rangle, \qquad i,j,k=1,\ldots,n,$$

and from (2), (3) and (4) we obtain:

$$\begin{aligned} \langle \Delta^{\sigma} x, E_j\rangle &= \sum_{i=1}^{n}\varepsilon_i\langle E_i(E_i(x)),E_j\rangle = \sum_{i=1}^{n}\varepsilon_i\langle \nabla_{E_i}\nabla_{E_i}x,E_j\rangle \\ &= \sum_{i=1}^{n}\varepsilon_i\langle \nabla_{E_i}E_i,E_j\rangle = \sum_{i,k=1}^{n}\varepsilon_i\varepsilon_k g_{jk}\langle \nabla_{E_i}E_i,-\nabla_{E_k}N\rangle \\ &= \sum_{i,k=1}^{n}\varepsilon_i\varepsilon_k g_{jk}\langle \nabla_{E_k}E_i,-\nabla_{E_i}N\rangle = \sum_{k=1}^{n}\varepsilon_k g_{jk}\langle \Delta^{\sigma}N,E_k\rangle \\ &= -\frac{1}{2}\sum_{k=1}^{n}\varepsilon_k g_{jk}E_k(\log|K|). \end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned} \langle \Delta^{\sigma} x, N\rangle &= \sum_{i=1}^{n}\varepsilon_i\langle E_i(E_i(x)),N\rangle = \sum_{i=1}^{n}\varepsilon_i\langle \nabla_{E_i}\nabla_{E_i}x,N\rangle \\ &= \sum_{i=1}^{n}\varepsilon_i\langle \nabla_{E_i}E_i,N\rangle = \sum_{i=1}^{n}\varepsilon_i\langle E_i,-\nabla_{E_i}N\rangle = n. \end{aligned} \tag{7}$$

Since the matrix $(\varepsilon_k g_{jk})$ has non-zero determinant, from (6) and (7), K is constant if and only if $\Delta^{\sigma} x = n\,N$. ∎

As a consequence of the above Lemma we obtain (see [2], [3]).

Corollary 1. *An orientable hypersurface in $\mathbb{R}^{n+1}$ with non-degenerate second fundamental form has constant curvature if and only if its Gauss map is harmonic for σ.*

Lemma 1 and Alexandrov reflection principle let us get estimates of the maximum height at which K-hypersurfaces can rise above a hyperplane. We also characterize the spherical caps as the unique graphs that reach those bounds.

Let S be a compact n-manifold with a connected boundary ∂S and consider $x : S \longrightarrow \mathbb{R}^{n+1}$ a K-hypersurface such that,

$$\Gamma = x(\partial S) \subset P = \{p \in \mathbb{R}^{n+1} \mid \langle p, a\rangle = 0,\ |a| = 1\}.$$

THEOREM 1. *If x is an embedding, then the maximum height at which $x(S)$ can rise above P is $2/\sqrt[n]{K}$.*

PROOF. Up to an isometry, we can assume that $a = (0, \cdots, 1)$. and $x(S)$ lies in $P^+ = \{p \in \mathbb{R}^{n+1} \mid \langle p, a\rangle \geq 0\}$. Since $x(S)$ is compact, there exists a point where every principal curvature has the same sign. Thus, σ is definite.

By using, in a standard way, the Alexandrov reflection principle respect to parallel hyperplanes to P coming down from the highest point, $x(S)$ must be a graph, at least until the hyperplane is halfway down to P. Thus, it is sufficient to check that the bound $1/\sqrt[n]{K}$ is satisfied if $x(S)$ is a graph.

We suppose $x(S)$ is a graph and choose the inner normal N. Then, σ is positive definite and

$$\Delta^\sigma \left(\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \right) \ = \ n \left(\sqrt[n]{K} - H \right) \langle N, a\rangle. \tag{8}$$

Since $H \geq \sqrt[n]{K}$, see [8], and $\langle N, a\rangle \leq 0$

$$\Delta^\sigma \left(\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \right) \ \geq \ 0 \quad \text{on } S. \tag{9}$$

Now, bearing in mind that $\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \leq 0$ on the boundary, we have $\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \leq 0$ on S and the inequality follows. ∎

THEOREM 2. *If $x(S)$ is a graph on a compact domain in P and the Euclidean gradient of height function, $\langle x, a\rangle$, is bounded along ∂S (that is, there exists a real constant m such that $|\nabla\langle x, a\rangle| \leq m \leq 1$ on ∂S), then $\langle x, a\rangle \leq (1 - \sqrt{1 - m^2})/\sqrt[n]{K}$.*

Moreover, the equality holds if and only if $x(S)$ is a spherical cap.

PROOF. As before we can assume that $a = (0, \cdots, 1)$ and $x(S)$ lies in P^+. Consider N the inner normal, then $\langle N, a\rangle = -\sqrt{1 - |\nabla\langle x, a\rangle|^2}$. Thus, on the boundary of S

$$\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \ = \ \langle N, a\rangle \ \leq \ -\sqrt{1 - m^2}$$

and then, from (9), we have, $\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle \leq -\sqrt{1 - m^2}$ on S, that is,

$$\langle x, a\rangle \ \leq \ \frac{-\langle N, a\rangle - \sqrt{1 - m^2}}{\sqrt[n]{K}} \ \leq \ \frac{1 - \sqrt{1 - m^2}}{\sqrt[n]{K}}. \tag{10}$$

Moreover, if the equality hods, then there exists an interior point on the domain where $\sqrt[n]{K}\langle x, a\rangle + \langle N, a\rangle = -\sqrt{1 - m^2}$. Using again (9) and the maximum principle the equality holds everywhere. Therefore, from (8), $H = \sqrt[n]{K}$ and S is a spherical cap, see [8]. ∎

3. Curvature Estimates.

Let S be a compact n-manifold with connected boundary ∂S and $x : S \longrightarrow \mathbb{R}^{n+1}$ an immersion such that the image of the boundary of S lies in the hyperplane $P = \{p \in$

$\mathbb{R}^{n+1} \mid \langle p, a\rangle = 0,\ |a| = 1\}$. Then, the number

$$\overline{A} \;=\; \frac{1}{n}\int_{\partial S}\langle x\times dx\times \overset{n-1)}{\cdots} \times dx\,,\, a\rangle$$

is called the algebraic area of $x(\partial S)$. Moreover, the above number does not depend on the parametrization of the immersion and if $x(\partial S)$ is embedded then $|\overline{A}|$ is the enclosed volume by $x(\partial S)$ in P.

Recall from the Introduction that if x is a K-hypersurface, then the curvature of $x_{|\partial S}$ in $P \equiv \mathbb{R}^n$ does not vanish at any point. Moreover, if $n \geq 3$, x is an embedding.

Now, we obtain a necessary condition for a connected $(n-1)$-manifold lying in a hyperplane P to be the boundary of a compact K-hypersurface.

Theorem 3. *If $x : S \longrightarrow \mathbb{R}^{n+1}$ is a K-hypersurface, then*

$$n\,K\,|\overline{A}| \;\leq\; \int_{\partial S} K_{\partial S}\,dA = vol(\mathbb{S}^{n-1})deg(\eta),$$

where η is the Gauss map of $x : \partial S \longrightarrow P$ and $K, K_{\partial S} > 0$ denote the curvature of S and ∂S in $\mathbb{R}^{n+1}$ and P, respectively.

Moreover, the equality holds if and only if $x(S)$ is a hemisphere.

Proof. Choose N and η such that, K and $K_{\partial S}$ are positive. It is clear that

$$dN\times \overset{n)}{\cdots} \times dN \;=\; K\,dx\times \overset{n)}{\cdots} \times dx$$

and using that K is constant, we have

$$d\left(N\times dN\times \overset{n-1)}{\cdots} \times dN\right) \;=\; d\left(K\,x\times dx\times \overset{n-1)}{\cdots} \times dx\right).$$

Then, from Stoker's theorem we obtain

$$n\,K\,|\overline{A}| \;=\; K\int_{\partial S}\langle x\times dx\times \overset{n-1)}{\cdots} \times dx, a\rangle\,dA \;=\; \int_{\partial S}\langle N\times dN\times \overset{n-1)}{\cdots} \times dN, a\rangle\,dA,$$

where a is a unit normal vector to P such that the above integrate is non-negative.

On the other hand, there exists a real function θ such that $N = \cos\theta\,\eta + \sin\theta\,a$. Thus, $dN = d\theta\,(-\sin\theta\,\eta + \cos\theta\,a) + \cos\theta\,d\eta$ and

$$\langle N\times dN\times \overset{n-1)}{\cdots} \times dN, a\rangle \;=\; (\cos\theta)^n\,\langle \eta\times d\eta\times \overset{n-1)}{\cdots} \times d\eta, a\rangle.$$

Therefore,

$$n\,K\,|\overline{A}| \;=\; \int_{\partial S}(\cos\theta)^n\,K_{\partial S}\,dA \;\leq\; \int_{\partial S}K_{\partial S}\,dA = vol(\mathbb{S}^{n-1})deg(\eta). \tag{11}$$

Moreover, if the equality holds then $\cos\theta = 1$ along ∂S and $N = \eta$, that is, $\langle N, a\rangle = 0$ on ∂S. Hence, $x(S)$ meets P ortogonally and $x(S)$ must be a graph on a convex domain in P if $n \geq 3$.

In this way, for $n \geq 3$, using Alexandrov reflection principle in any direction, v, perpendicular to a, $x(S)$ must be symmetric respect to a hyperplane with normal vector v, see [11]. Therefore, $x(S)$ is a revolution hypersurface.

Since equality holds $x(\partial S)$ must be a sphere of radius $1/\sqrt[n]{K}$ and using again Alexandrov reflection principle for graphs with the same boundary, $x(S)$ must be a hemisphere.

For $n=2$, since $\langle N,a\rangle = 0$, $x(\partial S)$ is a line of curvature and its geodesic curvature vanishes identically. Thus, from Gauss–Bonnet theorem

$$2\pi\chi(S) \;=\; \int_S K \;>\; 0.$$

Therefore, the Euler characteristic of S, $\chi(S)$ is positive, that is, $\chi(S)=1$ and using Lemma 2 in [5] $x(S)$ is a hemisphere. ∎

Remark 1. In the conditions of the above theorem,

1. If S is a surface in $\mathbb{R}^3$, that is, $n=2$, then
$$K\,|\overline{A}| \;\leq\; \pi\,|i(\partial S)|$$
where $i(\partial S)$ is the rotation index of the curve $x(\partial S)$.

2. If $n \geq 3$,
$$nK\,|\overline{A}| \;\leq\; vol(\mathbb{S}^{n-1}).$$

Now, we study compact graphs with non-zero curvature (non necessarily constant).

Theorem 4. *If $x(S)$ is a graph with non-zero positive curvature on a compact domain in the hyperplane P and the Euclidean gradient of the height function is bounded by a real constant m along ∂S (that is, $|\nabla\langle x,a\rangle| \leq m \leq 1$ on ∂S), then*

$$n\,K_0\,|\overline{A}| \;\leq\; m^n \int_{\partial S} K_{\partial S}\,dA$$

where $K, K_{\partial S} > 0$ denote the curvature of $x(S)$ and $x(\partial S)$ in $\mathbb{R}^{n+1}$ and P, respectively, and K_0 is the minimum of K on $x(S)$.

Moreover, the equality holds if and only if $x(S)$ is a spherical cap.

Proof. We can consider $x(S) \subset P^+$. By taking the inner normals N, η along S and ∂S, respectively, and using the Stoker's theorem

$$\begin{aligned} n\,K_0\,|\overline{A}| &= K_0 \int_{\partial S} \langle x \times dx \times \overset{n-1)}{\cdots} \times dx, a\rangle\, dA \;\leq\; \int_{\partial S} K\,\langle x \times dx \times \overset{n-1)}{\cdots} \times dx, a\rangle\, dA \\ &= \int_S K\, d\Big(\langle x \times dx \times \overset{n-1)}{\cdots} \times dx, a\rangle\Big)\, dA \;=\; \int_S d\langle N \times dN \times \overset{n-1)}{\cdots} \times dN, a\rangle\, dA \\ &= \int_{\partial S} \langle N \times dN \times \overset{n-1)}{\cdots} \times dN, a\rangle\, dA. \end{aligned}$$

Arguing as in the above theorem $N = \cos\theta\,\eta + \sin\theta\,a$ along ∂S and

$$n\,K_0\,|\overline{A}| \;\leq\; \int_{\partial S} \cos^n\theta\, K_{\partial S}\, dA.$$

Since $\sin\theta = \langle N,a\rangle = -\sqrt{1-|\nabla\langle x,a\rangle|^2}$ on ∂S and $\cos\theta > 0$ then $\cos\theta = |\nabla\langle x,a\rangle| \leq m$ and the theorem follows.

If the equality holds $K = K_0$ on S and $|\nabla\langle x,a\rangle| = m$ on ∂S. Thus, using Alexandrov reflection principle, as in the above theorem, $x(S)$ must be a spherical cap. ∎

4. Volume estimates. In this section, we give a estimation about the enclosed volume by a graph with constant curvature and planar boundary.

With the same notation as in §3 we have,

THEOREM 5. *If $x(S)$ is a K-hypersurface such that it is a graph on a compact domain in the hyperplane P with bounded Euclidean gradient of the height function along ∂S (that is, there exists a real constant m such that $|\nabla\langle x,a\rangle| \leq m \leq 1$ on ∂S), then*

(a) for $n = 2$

$$V \;\leq\; \frac{2-\sqrt{1-m^2}\,(2+m^2)}{3\sqrt{K^3}}\,\pi,$$

(b) for $n \geq 3$

$$nV \;\leq\; \frac{vol(\mathbb{S}^{n-1})}{\sqrt[n]{K^{n+1}}}\int_{\sqrt{1-m^2}}^{1}\sqrt{(1-t^2)^n}\,dt,$$

where V is the enclosed volume by S and P.

Moreover, the equality holds if and only if S is a spherical cap.

PROOF. We can assume $x(S) \subset P^+$ and $a = (0,\cdots,1)$. Then,

$$V \;=\; \int_0^h |\overline{A_t}|\,dt$$

where h is the maximum height of the graph above P and $\overline{A_t}$ is the algebraic area of $B_t = S \cap \{\langle x,a\rangle = t\}$.

Since from (11),

$$n\,K\,V \;=\; \int_0^h \left(\int_{B_t} |\nabla\langle x,a\rangle|^n\, K_{B_t}\, dA_t\right) dt$$

curvature of B_t in $\{\langle x,a\rangle = t\}$.

Then, from (10)

$$1-|\nabla\langle x,a\rangle|^2 \;=\; \langle N,a\rangle^2 \;\geq\; \left(\sqrt{1-m^2}+\sqrt[n]{K}\,\langle x,a\rangle\right)^2,$$

and we have

$$\begin{aligned} n\,K\,V \;&\leq\; \int_0^h \left(\int_{B_t}\left(1-\left(\sqrt{1-m^2}+\sqrt[n]{K}\,t\right)^2\right)^{\frac{n}{2}} K_{B_t}\,dA\right) dt \\ &=\; \int_0^h \left(1-\left(\sqrt{1-m^2}+\sqrt[n]{K}\,t\right)^2\right)^{\frac{n}{2}} \left(\int_{B_t} K_{B_t}\,dA\right) dt. \end{aligned}$$

But, $\int_{B_t} K_{B_t}\,dA = vol(\mathbb{S}^{n-1})$ does not depend on t, consequently,

$$n\,K\,V \;\leq\; \int_0^h \left(1-\left(\sqrt{1-m^2}+\sqrt[n]{K}\,t\right)^2\right)^{\frac{n}{2}} dt\, vol(\mathbb{S}^{n-1})$$

and the theorem follows as in Theorem 4. ■

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