We classify the family of positive constant curvature surfaces in $\mathbb{R}^3$ whose extrinsic conformal structure is biholomorphic to a planar circular domain, and whose Gauss map is a diffeomorphism onto a finitely punctured sphere. We give applications to the generalized Minkowski problem, the existence of harmonic diffeomorphisms between certain domains of $\mathbb{S}^2$, the existence of capillary surfaces in $\mathbb{R}^3$, and the space of solutions to a Hessian equation of Monge-Ampère type. Joint work with Antonio Alarcón.