Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. xx (2004), 2–7

Volume, energy and spacelike energy of vector fields on Lorentzian manifolds

A. Hurtado[1]

[1]*Departamento de Geometría y Topología, Universidad de Valencia, 46100 Burjassot, Valencia, Spain*
e-mail: ana.maria.hurtado@uv.es

Abstract

In this paper we will extend the results concerning the spacelike energy of unit timelike vector fields obtained in [2] to the volume and to the energy functionals.

Keywords: Energy; volume; critical point; reference frame; Robertson-walker spacetime; Gödel universe; Hopf vector fields.

2000 Mathematics Subject Classification: 53C50; 53C43; 53C25; 53C80.

1. Introduction

A smooth vector field V on a semi-Riemannian manifold (M, g) can be seen as a map into its tangent bundle endowed with the Sasaki metric g^S, defined by g.

When g is positive definite the energy and the volume can be defined in the space of smooth vector fields in a natural way. The energy of the map V is given, up to constant factors, by $\int_M \|\nabla V\|^2 \mathrm{dv}$ and the volume is defined as the volume of the submanifold $V(M)$ of (TM, g^S).

Many authors have studied the condition for a vector field to be a critical point of these functionals and the existence of minimizers among unit vector fields. Some of these results can be seen in the references of [3] and [4].

If we consider a Lorentzian manifold, the situation is not similar even if we restrict our attention to unit timelike vector fields. The energy is not bounded bellow, so the study of minimizers has no sense and the volume is not always defined. As a consequence, a new functional called spacelike energy, is introduced in [2] on the space of unit timelike vector fields. It is given by the integral of the square norm of the projection of the covariant derivative of the vector field onto its orthonormal complement.

The aim of this paper is to study to what extent the results obtained in [2] are still valid for the volume and for the energy.

The paper is organized as follows. In section 2 we give the definitions, the characterization of critical point and the expression of the second variation of the functionals. In section 3 we exhibit several examples of critical points. Moreover, we analyze the critical character of distinguished observers in spacetimes such as GRW and the classical Gödel universe.

2. Volume, energy and spacelike energy of vector fields

Given a semi-Riemannian manifold (M, g), the *Sasaki metric* g^S on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows :

$$g^S(\zeta_1, \zeta_2) = g(\pi_* \circ \zeta_1, \pi_* \circ \zeta_2) + g(\kappa \circ \zeta_1, \kappa \circ \zeta_2),$$

where $\pi : TM \to M$ is the projection and κ is the connection map of ∇.

Definition 2.1 *The energy of a vector field V, is given by*

$$E(V) = \frac{n+1}{2} + \frac{1}{2}\int_M \|\nabla V\|^2 \mathrm{dv}.$$

The relevant part of the energy, $B(V) = \int_M b(V)\mathrm{dv}$ where $b(V) = \frac{1}{2}\|\nabla V\|^2$, when considered as a functional on the manifold of unit vector fields, is sometimes called the total bending of the vector field. The first and second variation of B have been widely studied by Wiegmink [6]. The covariant version of these results, as it appears in [4], involves the $1-$form $\omega_V(X) = g(X, \nabla^*\nabla V)$ where $\nabla^*\nabla V$ is the rough Laplacian.

It is easy to see that the similar results also holds for a reference frame (unit timelike vector field) on a Lorentzian manifold. More precisely,

Proposition 2.2 *Given a reference frame Z on a compact Lorentzian manifold (M, g) then*

1. *Z is a critical point of the total bending if and only if $\omega_Z(X) = 0$ for all vector field X orthogonal to Z, where $\omega_Z(X) = g(X, \sum_i \varepsilon_i(\nabla_{E_i}(\nabla Z))(E_i))$.*

2. If Z is a critical point and X is orthogonal to Z then

$$(HessB)_Z(X) = \int_M (-\|X\|^2 \omega_Z(Z) + \|\nabla X\|^2)\mathrm{dv}.$$

Definition 2.3 *The volume of a unit vector field V on a Riemannian manifold is defined as the volume of the submanifold $V(M)$ of (T^1M, g^S). Since $(V^*g^S)(X,Y) = g(X,Y) + g(\nabla_X V, \nabla_Y V)$*

$$F(V) = \int_M f(V)\mathrm{dv}_g = \int_M \sqrt{\det L_V}\mathrm{dv}_g,$$

where $L_V = \mathrm{Id} + (\nabla V)^t \circ \nabla V$.

In contrast with the energy, the volume of a reference frame Z, is not always defined on a Lorentzian manifold, since the $2-$covariant field Z^*g^S can be degenerated. Due to this, we study the volume restricted to unit timelike vector fields for which Z^*g^S is a Lorentzian metric on M. We will denote this set of vector fields by $\Gamma^-(T^{-1}M)$ and it is an open subset of the set of smooth references frames.

With the same method used in [5] and [6] we have obtained

Proposition 2.4 *Let M be a compact Lorentzian manifold and let Z be a reference frame such that $Z \in \Gamma^-(T^{-1}M)$, then*

a) Z is a critical point of the volume if and only if $\widehat{\omega}_Z(X) = 0$ for all $X \in Z^\perp$, where $\widehat{\omega}_Z = \sum_i (\nabla_{E_i}\widehat{K}_Z)^i$ and $\widehat{K}_Z = f(Z)L_Z^{-1} \circ (\nabla Z)^t$.

b) If Z is a critical point and $X \in Z^\perp$

$$\begin{aligned}(HessF)_Z(X) = \int_M \Big(&- \|X\|^2 \widehat{\omega}_Z(Z) - \mathrm{tr}(L_Z^{-1} \circ (\nabla X)^t \circ \nabla Z \circ \widehat{K}_Z \circ \nabla X) \\ &+ \frac{1}{f(Z)}(\sigma_2(\widehat{K}_Z \circ \nabla X)) + f(Z)\mathrm{tr}(L_Z^{-1} \circ (\nabla X)^t \circ \nabla X)\Big)\mathrm{dv},\end{aligned}$$

where $\sigma_2(C) = \mathrm{tr}^2(C) - \mathrm{tr}(C^2)$.

Remark. Let us point out that if we compare these results with those obtained in [5] and [4] for Riemannian metrics, the only difference is the minus sign of the first term of the expression of the Hessian.

The spacelike energy density of Z is defined as

$$\widetilde{b}(Z) = \frac{1}{2}\|A_Z \circ P_Z\|^2 = \frac{1}{2}\sum_{i=1}^n g(\nabla_{E_i} Z, \nabla_{E_i} Z),$$

where $A_Z = -\nabla Z$, $P_Z(X) = X + g(X,Z)Z$ and $\{E_i, Z\}_{i=1}^n$ is an adapted orthonormal local frame.

Definition 2.5 *The spacelike energy is given by*

$$\widetilde{B}(Z) = \int_M \widetilde{b}(Z)\mathrm{dv}.$$

The condition for a reference frame to be spatially harmonic (critical point of the spacelike energy) and the second variation at critical points have been computed in [2]. We summarize here these results.

Proposition 2.6 *Let Z be a reference frame on a compact Lorentzian manifold.*

a) Z is spatially harmonic if and only if the $1-$form $\widetilde{\omega}_Z$ annihilates $Z^{\perp}$, where $\widetilde{\omega}_Z = -\sum_i (\nabla_{E_i}\widetilde{K}_Z)^i + g(\widetilde{K}_Z(\nabla_Z Z))$ and $\widetilde{K}_Z = (\nabla Z \circ P_Z)^t$.

b) If Z is spatially harmonic and $X \in Z^{\perp}$, we have

$$\begin{aligned}(Hess\widetilde{B})_Z(X) &= \int_M (\|\nabla X\|^2 + 2g(\nabla_X X, \nabla_Z Z) + \|\nabla_X Z + \nabla_Z X\|^2)\mathrm{dv} \\ &+ \int_M \|X\|^2(\|\nabla_Z Z\|^2 + \widetilde{\omega}_Z(Z))\mathrm{dv}.\end{aligned}$$

As for the energy and the volume, the condition of critical point obtained is tensorial, so we can define critical points even if the manifold is not compact and the functional is not defined.

3. Examples

The easiest examples of spatially harmonic reference frames are those of null spacelike energy. If we write the spacelike energy in terms of the kinematical quantities of the reference frame, then

$$\widetilde{B}(Z) = \frac{1}{2}\int_M (\|\Omega\|^2 + \|\sigma\|^2 + \frac{1}{n}\Theta^2)\mathrm{dv},$$

where Ω, σ and Θ are the rotation, the shear and the expansion respectively.

So, the spacelike energy vanishes when the reference frame is rigid and irrotational. As a consequence, we have the following proposition.

Proposition 3.1 ([2]) *In a static spacetime, the infimum of the spacelike energy is zero and it is attained by the static observer.*

In what concerns energy and volume, computing the Euler-Lagrange equations for this type of vector fields we have shown that

Proposition 3.2 *Let Z be a rigid and irrotational reference frame.*

a) *Z is a critical point of the energy if and only if, $\nabla_Z\nabla_Z Z = \|\nabla_Z Z\|^2 Z$.*

b) *$Z \in \Gamma^-(T^{-1}M)$ is minimal if and only if*

$$Z(\frac{1}{f(Z)})g(X, \nabla_Z Z) + \frac{1}{f(Z)}g(X, \nabla_Z\nabla_Z Z) = 0 \quad \text{for all } X \in Z^\perp.$$

One of the most important cosmological models are the Robertson-Walker spacetimes and the so-called generalized Robertson-Walker spacetimes (see [1] for more details). In [2], using the results obtained for Lorentzian manifolds admitting a closed and conformal timelike vector field, it has been shown that

Proposition 3.3 *The comoving observer ∂_t is spatially harmonic. Furthermore, if M is assumed to be compact and satisfying the null convergence condition, ∂_t is an absolute minimizer of the spacelike energy.*

Following similar arguments, we have shown the following proposition.

Proposition 3.4 *In a GRW spacetime the comoving observer is a minimal immersion.*

Another example concerns the classical Gödel universe that is $\mathbb{R}^4$ endowed with the metric $ds^2 = dx_1^2 + dx_2^2 - \frac{1}{2}e^{2\alpha x_1}dy^2 - 2e^{\alpha x_1}dydt - dt^2$, where α is a positive constant. In this coordinate system we have two distinguished timelike vector fields ∂_t and ∂_y.

Proposition 3.5 ([2]) *In the Gödel universe we have*

1. *The reference frame ∂_t is a critical point of the energy and it is also spatially harmonic.*
2. *The reference frame $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ is not spatially harmonic.*

In contrast with part 2 of the above proposition, we have shown that

Proposition 3.6 *The reference frame $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ is a critical point of the energy.*

In what concerns the volume functional, it is easy to see that ∂_t belongs to $\Gamma^-(T^{-1}M)$ and if $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$, then $\det L_Z = (1 - \frac{\alpha^2}{2})(1 + \frac{\alpha^2}{2})$, so $Z \in \Gamma^-(T^{-1}M)$ if and only if $\alpha^2 < 2$.

Proposition 3.7 *In the Gödel Universe* *a)* $\partial_t \in \Gamma^-(T^{-1}M)$ *and it is a minimal immersion.*

b) $Z = \sqrt{2}e^{-\alpha x_1}\partial_y \in \Gamma^-(T^{-1}M)$ *if and only if* $\alpha^2 < 2$ *and it is not a critical point of the volume.*

To finish the paper, we would like to remark that another interesting examples are the Hopf vector fields in Lorentzian Berger spheres, that are a particular case of generalized Taub-NUT spacetimes. The study of the spacelike energy in these spaces can be seen in [2], while the study of the energy and the volume has been widely developed in [3].

Acknowledgments

This work has been partially supported by DGI Grant No. BFM2001-3548 and by Ministerio de Educación y Cultura Research Grant. The author wants to thank O. Gil-Medrano for her collaboration in the preparation of this paper.

References

[1] L.J. ALÍAS, A. ROMERO, M. SÁNCHEZ, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, *Gen. Rel. and Grav.* **27** (1995), 71-84.

[2] O. GIL-MEDRANO, A. HURTADO, Spacelike energy of timelike unit vector fields on a Lorentzian manifold, *J. Geom. and Phys.*, **51** (2004), 82-100.

[3] O. GIL-MEDRANO, A. HURTADO, Volume, energy and generalized energy of unit vector fields on Berger's spheres. Stability of Hopf vector fields, *Preprint*, (2004) .

[4] O. GIL-MEDRANO, E. LLINARES-FUSTER, Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields, *Math. Ann.*, **320** (2001), 531-545.

[5] O. GIL-MEDRANO, E. LLINARES-FUSTER, Minimal unit vector fields , *Tohoku Math. J.*, **54** (2002), 71-84.

[6] G. WIEGMINK, Total bending of vector fields on Riemannian manifolds, *Math. Ann.* **303** (1995), 325-344.