Spacelike energy of timelike unit vector fields on a Lorentzian manifold

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Abstract

On a Lorentzian manifold, we define a new functional on the space of unit timelike vector fields given by the L_2 norm of the restriction of the covariant derivative of the vector field to its orthogonal complement. This spacelike energy is related with the energy of the vector field as a map on the tangent bundle endowed with the Kaluza-Klein metric, but it is more adapted to the situation. We compute the first and second variation of the functional and we exhibit several examples of critical points on cosmological models as Generalized Robertson-Walker spaces and Gödel universe, on Einstein and contact manifolds and on Lorentzian Berger's spheres. For these critical points we have also studied to what extend they are stable or even absolute minimizers.

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1 Introduction

A smooth vector field V on a semiriemannian manifold (M, g) can be seen as a map into its tangent bundle endowed with the Kaluza-Klein metric defined by g. The energy of the map V is given, up to constant factors, by $\int_M \|\nabla V\|^2 \mathrm{dv}_g$. Unit vector fields that are critical points for variations among unit vector fields, have been identified as those for which $\nabla^*\nabla V$ is colineal to V, where $\nabla^*\nabla$ is the rough Laplacian.

If g is positive definite, the energy vanishes only for parallel vector fields and it can be seen as a measure of the failing of a vector field to be parallel. This kind of vector fields, when they exist are the absolute minimizers. For many natural manifolds admitting smooth unit vector fields but not parallel ones, the value of the infimum and the regularity of minimizers is now an open problem. In the last years many authors have studied all this questions, as can be seen in the references of [7] and [8].

If we consider a Lorentzian manifold and the energy of a unit timelike vector field, the Euler-Lagrange equation involves, in that case, the rough D'Alembertian which is not an elliptic operator. But more important, since the functional is not bounded bellow, to study minimizers has no sense. This has leaded us to define a new functional more adapted to the situation, that will be called *spacelike energy.* For a reference frame Z, it is given (up to constant factors) by the integral of the square norm of the restriction of ∇Z to $Z^{\perp}$.

We have computed the Euler-Lagrange equation of this new variational problem showing that critical points are characterized as those reference frames for which $\widetilde{D}Z$ is colineal to Z where $\widetilde{D}$ is a differential operator, which is second order and elliptic on space coordinates, and only first order on time coordinates. We will say that such a vector field is spatially harmonic. We have also computed the second variation at critical points. These are the contents of section 3.

Section 4 is devoted to the study of different examples of reference frames as static reference frames and projective vector fields. In that case criticality can be described in terms of the Ricci tensor and in particular:

a) *Every affine reference frame on an Einstein manifold is spatially harmonic.*

b) *If the characteristic vector field of a Lorentzian K-contact manifold (and in particular, of a Lorentzian Sasakian manifold) is timelike then it is spatially harmonic.*

In section 5 we have considered the well known Robertson-Walker cosmological model and the comoving reference frame. We have shown that: *In a GRW, the comoving reference frame ∂_t is a spatially harmonic reference frame. Furthermore, if the manifold is assumed to be compact and satisfying the null convergence condition, the comoving reference frame is an absolute minimizer of the spacelike energy.* This result has been obtained as a particular case of the corresponding result for the Lorentzian manifolds endowed with a timelike vector field which is closed and conformal.

Section 6 is devoted to the study of the classical Gödel Universe, that is defined as $\mathbb{R}^4$ with the metric

$$\langle\ .\ ,\ .\ \rangle_L = dx_1^2 + dx_2^2 - \frac{1}{2}e^{2\alpha x_1}dy^2 - 2e^{\alpha x_1}dydt - dt^2$$

where α is a positive constant. We show that ∂_t is spatially harmonic and that it has the same energy that another non critical reference frame; consequently, it can not be an absolute minimizer. In fact, by computing the Hessian, we see that ∂_t is unstable and it is not even a local minimum.

To finish the paper we study the Hopf vector fields defined on the Lorentzian Berger's spheres. These metrics g_μ, with $\mu < 0$, on the sphere S^{2n+1} are obtained as the canonical variation of the submersion defined by the Hopf fibration $\pi : (S^{2n+1}, g) \longrightarrow \mathbb{C}P^n$ where g is the usual metric. As can be seen in [9] Hopf vector fields are critical for the energy and consequently, since they are geodesic, they are also spatially harmonic; moreover their energy and spacelike energy coincide. We have shown also in [9] that they are unstable for the energy when $n = 1$ but the stability in higher dimensions is an open question. Nevertheless, the second variation of both functionals at Hopf vector fields is different and, in contrast with the usual energy, the problem for the spacelike energy is completely solved because we show in section 7 that *Hopf vector fields on Lorentzian Berger's spheres are stable critical points of the spacelike energy.*

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2 Preliminaries

The energy density of a map $\varphi : (M, g) \to (N, h)$ from a semiriemannian manifold to another is defined as $e(\varphi) = \frac{1}{2}\mathrm{tr}(L_\varphi)$, where L_φ is the $(1,1)$ tensor field completely determined by $(\varphi^* h)(X, Y) = g(L_\varphi(X), Y)$. If $\{E_i\}$ is a g-orthonormal local frame and $\varepsilon_i = g(E_i, E_i)$ then

$$\mathrm{tr}(L_\varphi) = \sum_{i=1}^{n+1} \varepsilon_i (h \circ \varphi)(\varphi_*(E_i), \varphi_*(E_i)).$$

The energy of φ is then defined by

$$E(\varphi) = \int_M e(\varphi) \mathrm{dv}_g,$$

where dv_g represents the density on M, or the volume element for oriented M defined by the metric.

It is well known that the Euler-Lagrange equations give rise to the definition of *tension* of a map that is a vector field along the map whose vanishing defines harmonic maps. In a g-orthonormal local frame as above, the tension is expressed in terms of the Levi-Civita connections ∇^g and ∇^h as

$$\tau_g(\varphi) = \sum_{i=1}^{n+1} \varepsilon_i \Big(\nabla^h_{E_i} \varphi_*(E_i) - \varphi_*(\nabla^g_{E_i} E_i) \Big).$$

If we consider the tangent bundle $\pi : TM \to M$ and a semiriemannian metric g on M, we can construct a natural metric on TM as follows: at each point $v \in TM$, we consider on the vertical subspace of $T_v(TM)$ the inner product g (up to the usual identification with T_pM, where $p = \pi(v)$), we take the horizontal subspace determined by the Levi-Civita connection as a suplementary of the vertical and we declare them to be orthogonal; finally, we define the inner product of horizontal vectors as the product of their projections, with the metric g. The so constructed metric g^S is sometimes referred as the *Sasaki* or *Kaluza-Klein metric.*

Definition 2.1. *For a vector field V we have $(V^* g^S)(X, Y) = g(X, Y) + g(\nabla_X V, \nabla_Y V)$ and consequently $L_V = \mathrm{Id} + (\nabla V)^t (\nabla V)$. So, the energy of the map $V : (M, g) \to (TM, g^S)$, that is known as the energy of the vector field, is given by*

$$E(V) = \frac{n+1}{2} + \frac{1}{2} \int_M \|\nabla V\|^2 \mathrm{dv}.$$

Moreover the tension is (see [7])

$$\tau_g(V) = \Big(\sum_i \varepsilon_i R((\nabla V)(E_i), V, E_i)\Big)^{hor} + \Big(\sum_i \varepsilon_i (\nabla_{E_i}(\nabla V))(E_i)\Big)^{ver}$$

where for a vector field X we have represented by X^{ver} its vertical lift and by X^{hor} its horizontal lift and $\nabla^*\nabla V = \sum_i \varepsilon_i(\nabla_{E_i}(\nabla V))(E_i)$ is the rough Laplacian.

Critical points of the energy are those vector fields with vanishing rough Laplacian. If the manifold is compact, and the metric is positive definite, this means that the vector field should be parallel.

In a Riemannian manifold, the condition for a unit vector field to be a critical point for variations among unit vector fields has been obtained by direct computation of the Euler-Lagrange equation. The second variation at a critical point has also been computed.

The relevant part of the energy, $B(V) = \int_M b(V)\mathrm{dv}$ where $b(V) = \frac{1}{2}\|\nabla V\|^2$, when considered as a functional on the manifold of unit vector fields, is sometimes called the total bending of the vector field.

Proposition 2.2. *([14]) Given a unit vector field V on a compact Riemannian manifold (M, g) then*

1. *V is a critical point of the total bending if and only if $\nabla^*\nabla V$ is colineal to V.*

2. *If V is a critical point and X is orthogonal to V then*

$$(HessB)_V(X) = \int_M (\|\nabla X\|^2 - \|\nabla V\|^2\|X\|^2)\mathrm{dv}.$$

The covariant version of proposition above, as it appears in [10] has been very useful for the study of particular examples and also to compute the second variation by a different method. Let ω_V be the 1-form $\omega_V(X) = g(X, \nabla^*\nabla V)$ associated to $\nabla^*\nabla V$ by the metric.

Proposition 2.3. *Given a unit vector field V on a Riemannian manifold (M, g) then*

1. *V is a critical point of the total bending if and only if $\omega_V(X) = 0$ for all vector field X orthogonal to V.*

2. *If V is a critical point and X is orthogonal to V then*

$$(HessB)_V(X) = \int_M (\|\nabla X\|^2 + \|X\|^2\omega_V(X))\mathrm{dv}.$$

It is easy to see that the similar results also holds for a unit timelike vector field on a Lorentzian manifold that is: Z is a critical point of the energy if and only if the rough D'Alembertian, $\sum_i \varepsilon_i(\nabla_{E_i}(\nabla Z))(E_i)$, is colineal to Z.

Let (M, g) be a Lorentzian manifold and let Z be a reference frame (unit timelike vector field) on M, the spacelike energy density of Z will be defined as:

$$\widetilde{b}(Z) = \frac{1}{2}\|A_Z \circ P_Z\|^2,$$

where $A_Z = -\nabla Z$ and $P_Z(X) = X + g(X, Z)Z$. In the sequel we will denote $A'_Z = A_Z \circ P_Z$, the restriction of $-\nabla Z$ to $Z^\perp$. The spacelike energy density can be also written as

$$\widetilde{b}(Z) = \frac{1}{2}(\mathrm{tr}(A_Z^t \circ A_Z) + g(\nabla_Z Z, \nabla_Z Z)) = \frac{1}{2}\sum_{i=1}^{n} g(\nabla_{E_i} Z, \nabla_{E_i} Z),$$

where $\{E_i, Z\}_{i=1}^n$ is an adapted orthonormal local frame.

Definition 2.4. *The spacelike energy of a reference frame Z is defined as*

$$\widetilde{B}(Z) = \int_M \widetilde{b}(Z)\mathrm{dv}.$$

For compact M, the spacelike energy is finite for every vector field. This energy is always nonnegative and it vanishes if and only if $A'_Z = 0$, that is to say if and only if the reference frame is rigid and irrotational. In particular, for static space-times the infimum of spacelike energy is zero and it is attained.

In the positive definite case, the energy of unit vector fields is bounded on terms of the Ricci tensor as follows:

Proposition 2.5. *[5] Let V be a unit vector field on a compact manifold M of dimension $n + 1$.*

1. *If $n \geq 2$, then*
$$B(V) \geq \frac{1}{2n-2}\int_M \mathrm{Ric}(V, V)\mathrm{dv}. \qquad (1)$$
2. *If $n \geq 3$, then the equality in (1) holds if and only if V is totally geodesic, the n-dimensional distribution generated by $V^\perp$ is integrable and defines a Riemannian totally umbilical foliation.*

Following similar arguments we can show that

Proposition 2.6. *Let Z be a reference frame on a compact Lorentzian manifold of dimension $n + 1$.*

1. *If $n \geq 2$, then*
$$\widetilde{B}(Z) \geq \frac{1}{2n-2}\int_M \mathrm{Ric}(Z, Z)\mathrm{dv}.$$
2. *If $n \geq 3$, then the equality above holds if and only if the n-dimensional distribution generated by $Z^\perp$ is integrable and defines a totally umbilical foliation.*

Proof. Let $\{E_i, Z\}_{i=1}^n$ be an adapted orthonormal local frame and let us denote by $h_{ij} = g(\nabla_{E_i} Z, E_j)$, then $\widetilde{b}(Z) = \frac{1}{2}\sum_{i,j} h_{ij}^2$ that can be written as (see [5])

$$\begin{aligned}
\widetilde{b}(Z) &= \frac{1}{2n-2}\sum_{i<j}(h_{ii}-h_{jj})^2 + \frac{1}{n-1}\sum_{i<j}(h_{ii}h_{jj} - h_{ij}h_{ji}) \\
&+ \frac{1}{2n-2}\sum_{i<j}(h_{ij}+h_{ji})^2 + \frac{n-2}{2n-2}\sum_{i\neq j} h_{ij}^2 \geq \frac{1}{n-1}\sum_{i<j}(h_{ii}h_{jj} - h_{ij}h_{ji}) \\
&= \frac{1}{n-1}\sigma_2,
\end{aligned}$$

where σ_2 is the second mean curvature of the distribution defined by $Z^\perp$. Using that $\int_M (\mathrm{Ric}(Z,Z) - 2\sigma_2)\mathrm{dv} = 0$ (see [12]) we have

$$\widetilde{B}(Z) \geq \frac{1}{2n-2}\int_M \mathrm{Ric}(Z,Z)\mathrm{dv},$$

with equality if and only if $h_{ij} = 0$ for $i \neq j$ and $h_{ii} = h_{jj}$ for all i, j. □

Remark 2.7. On a Lorentzian manifold the inequality of proposition 2.5 does not hold because $\|\nabla Z\|^2 = -\|\nabla_Z Z\|^2 + \sum_{i,j} h_{ij}^2$ and then $\|\nabla Z\|^2$ can not be bounded by $\sum_{i,j} h_{ij}^2$.

3 First and second Variation

Let us compute the first and second variation of this new functional.

Proposition 3.1. *Let Z be a reference frame on a Lorentzian manifold M. Then for all vector field X orthogonal to Z,*

$$(d\widetilde{B})_Z(X) = \int_M (\mathrm{tr}((\nabla Z)^t \circ \nabla X) + g(\nabla_X Z, \nabla_Z Z) + g(\nabla_Z X, \nabla_Z Z))\mathrm{dv}.$$

Proof. Let $Z : I \longrightarrow \Gamma(T^{-1}M)$ be a curve of unit timelike vector fields for some open interval I containing 0 such that $Z(0) = Z$ and $Z'(0) = X$.

$$\widetilde{b} \circ Z(t) = \frac{1}{2}(\mathrm{tr}(A^t_{Z(t)} \circ A_{Z(t)}) + g(\nabla_{Z(t)} Z(t), \nabla_{Z(t)} Z(t))).$$

Then

$$(\widetilde{b} \circ Z)'(t) = \mathrm{tr}((\nabla Z(t))^t \circ \nabla Z'(t)) + g(\nabla_{Z'(t)} Z(t) + \nabla_{Z(t)} Z'(t), \nabla_{Z(t)} Z(t)). \quad (2)$$

Therefore,

$$(\widetilde{b} \circ Z)'(0) = \mathrm{tr}((\nabla Z)^t \circ \nabla X) + g(\nabla_X Z, \nabla_Z Z) + g(\nabla_Z X, \nabla_Z Z)$$

from where the result follows. □

If $\{E_i, Z\}$ is an adapted orthonormal local frame the differential of $\widetilde{B}$ at Z can be written as:

$$(d\widetilde{B})_Z(X) = \int_M (\sum_i g(\nabla_{E_i}X, \nabla_{E_i}Z) + g(\nabla_X Z, \nabla_Z Z))\mathrm{dv}.$$

To write the differential of $\widetilde{B}$, and therefore the condition of critical point, in a simpler form we will use the following lemma:

Lemma 3.2. *Given K a $(1,1)$-tensor field and X a vector field, we have:*

$$(C_1^1 \nabla K)(X) = -\mathrm{tr}(K \circ \nabla X) - \delta\alpha,$$

where C_1^1 is the tensor contraction, δ represents the divergence operator of g and $\alpha(Y) = g(K(X), Y)$.

Corollary 3.3. *Let Z be a reference frame on a compact Lorentzian manifold, then for all X orthogonal to Z we have*

$$\begin{aligned}(d\widetilde{B})_Z(X) &= \int_M (-(C_1^1\nabla\widetilde{K})(X) + g(\nabla_X Z, \nabla_Z Z))\mathrm{dv} \\ &= \int_M (-(C_1^1\nabla\widetilde{K}) + g(\widetilde{K}(\nabla_Z Z))(X)\mathrm{dv} \\ &= \int_M \widetilde{\omega}_Z(X)\mathrm{dv},\end{aligned}$$

where $\widetilde{\omega}_Z = -C_1^1\nabla\widetilde{K} + g(\widetilde{K}(\nabla_Z Z))$ and $\widetilde{K} = (\nabla Z \circ P_Z)^t$.

As for the Riemannian case, we can conclude the following

Proposition 3.4. *A reference frame Z on a compact Lorentzian manifold is a critical point of the spacelike energy if and only if the 1-form $\widetilde{\omega}_Z$ annihilates $Z^\perp$.*

Since the condition of critical point that we have obtained is a tensorial condition, we can define critical points even if the functional is not defined when M is not compact. In this case we have

Proposition 3.5. *A unit timelike vector field Z verifies $\widetilde{\omega}_Z(Z^\perp) = 0$ if and only if for every open subset U with compact closure the functional $\widetilde{B}^U$ defined by*

$$\widetilde{B}_U(Z) = \int_U \widetilde{b}(Z)\mathrm{dv},$$

verifies $(d\widetilde{B}_U)_Z(X) = 0$ for all $X \in Z^\perp$ with support in U.

Let us analize the relationship between the condition of critical point of the spacelike energy and the usual one. As in the Riemannian case, if $\widetilde{X}_Z$ is the vector field associated by the metric to $\widetilde{\omega}_Z$, we have that Z is a critical point of the spacelike energy if and only if $\widetilde{X}_Z$ is colineal to Z.

It is easy to see that

$$\widetilde{X}_Z = -\sum_{i=1}^{n+1} \varepsilon_i (\nabla_{E_i}(\nabla Z \circ P_Z))(E_i) + (\nabla Z \circ P_Z)^t (\nabla_Z Z),$$

that in an adapted orthonormal local frame can be written as

$$\begin{aligned}\widetilde{X}_Z &= -\sum_{i=1}^{n} (\nabla_{E_i}\nabla_{E_i} Z - \nabla_{\nabla_{E_i}E_i} Z) - \mathrm{div}(Z)\nabla_Z Z - ((\nabla Z) - (\nabla Z)^t)(\nabla_Z Z) \\ &= -\nabla^*\nabla Z - \nabla_Z\nabla_Z Z - \mathrm{div}(Z)\nabla_Z Z + (\nabla Z)^t(\nabla_Z Z).\end{aligned}$$

So, if Z is geodesic then Z is a critical point of $\widetilde{B}$ if and only if Z is a critical point of the usual energy. In contrast with the rough Laplacian, the differential operator $\widetilde{D}$ given by $\widetilde{D}Z = \widetilde{X}_Z$ is second order but elliptic on space coordinates.

We can now give the following

Definition 3.6. *A reference frame on a Lorentzian manifold is said spatially harmonic if and only if it is a critical point of the spacelike energy, or equivalent if $\widetilde{D}Z$ is colineal to Z.*

Let us compute the second variation of the spacelike energy.

Proposition 3.7. *Given Z a spatially harmonic reference frame on a compact Lorentzian manifold and $X \in Z^{\perp}$, we have*

$$\begin{aligned}(Hess\widetilde{B})_Z(X) &= \int_M (\|\nabla X\|^2 + 2g(\nabla_X X, \nabla_Z Z) + \|\nabla_X Z + \nabla_Z X\|^2)\mathrm{dv} \\ &+ \int_M \|X\|^2(\|\nabla_Z Z\|^2 - (C_1^1\nabla\widetilde{K})(Z))\mathrm{dv}.\end{aligned}$$

Proof. Let $Z : I \longrightarrow \Gamma(T^{-1}M)$ be a curve as in proposition 3.1 such that $Z(0) = Z$, $Z'(0) = X$, using (2)

$$\begin{aligned}(\widetilde{b}\circ Z)''(0) &= \mathrm{tr}((\nabla X)^t \circ \nabla X + (\nabla Z)^t \circ \nabla Z''(0)) + g(\nabla_{Z''(0)}Z + \nabla_Z Z''(0), \nabla_Z Z) \\ &+ 2g(\nabla_X X, \nabla_Z Z) + g(\nabla_X Z + \nabla_Z X, \nabla_X Z + \nabla_Z X) \\ &= \|\nabla X\|^2 + \mathrm{tr}((\nabla Z \circ P_Z)^t \circ \nabla Z''(0)) + 2g(\nabla_X X, \nabla_Z Z) \\ &+ g(\nabla_{Z''(0)}Z, \nabla_Z Z) + \|\nabla_X Z + \nabla_Z X\|^2.\end{aligned}$$

Now, from lemma 3.2 we obtain after integration

$$\begin{aligned}\int_M (\widetilde{b}\circ Z)''(0)\mathrm{dv} &= \int_M (\|\nabla X\|^2 + 2g(\nabla_X X, \nabla_Z Z) + \|\nabla_X Z + \nabla_Z X\|^2)\mathrm{dv} \\ &+ \int_M (g(\nabla_{Z''(0)}Z, \nabla_Z Z) - (C_1^1\nabla\widetilde{K})(Z''(0)))\mathrm{dv}.\end{aligned}$$

Now, $Z''(0) = P_Z(Z''(0)) + \|X\|^2 Z$ and since $\widetilde{\omega}_Z(Z''(0)) = g(\nabla_{P_Z(Z''(0))}Z, \nabla_Z Z) - C_1^1\nabla\widetilde{K}(Z''(0))$ then the criticality of Z implies that

$$\begin{aligned} g(\nabla_{Z''(0)}, \nabla_Z Z) - C_1^1\nabla\widetilde{K}(Z''(0)) &= \widetilde{\omega}_Z(Z''(0)) + \|X\|^2\|\nabla_Z Z\|^2 \\ &= \|X\|^2(\|\nabla_Z Z\|^2 + \widetilde{\omega}_Z(Z)) \\ &= \|X\|^2(\|\nabla_Z Z\|^2 - (C_1^1\nabla\widetilde{K})(Z)) \end{aligned}$$

from where the result holds. □

As for the first variation, when the manifold is not compact the stability can be defined as follows

Definition 3.8. *Let $Z \in \Gamma(T^{-1}M)$ be a critical point of the spacelike energy. We say that Z is stable if for every open subset U with compact closure,*

$$(Hess\widetilde{B}_U)_Z(X) \geq 0$$

for all $X \in Z^\perp$ with support in U, where $\widetilde{B}_U$ is the restriction of the functional to the open subset U.

4 Examples

As we mentioned in the preliminaries, the easiest examples of spatially harmonic reference frames are those of null spacelike energy. In order to give a physical interpretation of this condition, let us recall the decomposition of $-A'_Z$ in its symmetric S and skew-symmetric Ω parts, called the deformation and the rotation of the reference frame Z respectively. Now, if we decompose S as $S = \sigma + \frac{\Theta}{n}P_Z$, where σ is trace-free, then $-A'_Z$ can be written as

$$-A'_Z = \Omega + \sigma + \frac{\Theta}{n}P_Z.$$

In this case, Θ is called the expansion and σ the shear of the reference frame Z.

Using this decomposition the spacelike energy takes the form

$$\widetilde{B}(Z) = \frac{1}{2}\int_M (\|\Omega\|^2 + \|\sigma\|^2 + \frac{1}{n}\Theta^2)\mathrm{dv}.$$

Consequently, $\widetilde{B}(Z)$ is zero if and only if $S = 0$ and $\Omega = 0$, that is, if and only if Z is rigid and irrotational. As a particular case of this type of reference frames we have the static reference frames that are defined as follows:

Definition 4.1. *A vector field Z is stationary if and only if there exists a positive function f on M, such that fZ is a Killing vector field. A vector field Z is static if and only if it is stationary and irrotational.*

The condition for a Killing vector field Z to be a critical point of the energy can be written in terms of the Ricci tensor. It's natural to study when a Killing reference frame is spatially harmonic. Since, in contrast with the Riemannian case, a Lorentzian manifold can admit affine unit vector fields that are not Killing, we are going to study how the criticality condition can be expressed under the weaker hypothesis of Z being a projective reference frame. The interest of these vector fields in general relativity can be seen in [12].

Definition 4.2. *Let Z be a vector field on a Lorentzian manifold. We will say that Z is projective if and only if there exists a $1-$form μ on M such that*

$$(\mathcal{L}_Z\nabla)(X,Y) = \mu(X)Y + \mu(Y)X \quad \forall X, Y \in \Gamma(TM).$$

If $\mu = 0$, Z is called affine.

Proposition 4.3. *If Z is projective, then we have*

$$\nabla_U\nabla_V Z - \nabla_{\nabla_U V} Z = R(U,Z)V + \mu(U)V + \mu(V)U$$

for all $U, V \in \Gamma(TM)$.

By developing the defining condition we get that $2\mu(Z) = g(\nabla_Z Z, \nabla_Z Z)$ and $\mu(X) = g(\nabla_X Z, \nabla_Z Z)$ for $X \in Z^\perp$.

Let X be a vector field orthogonal to Z, it is easy to see that

$$(C_1^1\nabla\widetilde{K})(X) = (C_1^1\nabla(\nabla Z)^t)(X) + g(X, \nabla_Z\nabla_Z Z) + \operatorname{div}(Z)g(X, \nabla_Z Z).$$

And if Z is projective then in a orthonormal adapted local frame we have

$$\begin{aligned}(C_1^1\nabla\widetilde{K})(X) &= g(X, \nabla_{\nabla_Z Z} Z) + \sum_i g(R(E_i, Z)E_i + 2\mu(E_i)E_i, X) + \operatorname{div}(Z)g(X, \nabla_Z Z) \\ &= g(X, \nabla_{\nabla_Z Z} Z) - \operatorname{Ric}(Z, X) + 2\mu(X) + \operatorname{div}(Z)g(X, \nabla_Z Z),\end{aligned}$$

and

$$\widetilde{\omega}_Z(X) = -g(X, \nabla_{\nabla_Z Z} Z) + \operatorname{Ric}(Z, X) - \mu(X) - \operatorname{div}(Z)g(X, \nabla_Z Z).$$

Therefore Z is spatially harmonic if and only if

$$-g(X, \nabla_{\nabla_Z Z} Z) + Ric(Z, X) - \mu(X) - \operatorname{div}(Z)g(X, \nabla_Z Z) = 0 \quad \forall X \in Z^\perp.$$

Now, if we assume Z to be affine, that is $\mu = 0$, we can prove that $\nabla_Z Z = 0$ and then the condition to be spatially harmonic (and then a critical point of the usual energy since Z is geodesic) can be expressed as

$$\operatorname{Ric}(X, Z) = 0 \quad \forall X \in Z^\perp.$$

Consequently we have:

Proposition 4.4. *Let Z be a projective reference frame. Then* *a)* *Z is spatially harmonic if and only if*

$$-g(X, \nabla_{\nabla_Z Z} Z) + \mathrm{Ric}(Z, X) - \mu(X) - \mathrm{div}(Z) g(X, \nabla_Z Z) = 0 \quad \forall X \in Z^{\perp}.$$

b) *If Z is affine then it is a critical point of the usual energy (and then spatially harmonic) if and only if* $\mathrm{Ric}(X, Z) = 0 \quad \forall X \in Z^{\perp}$.

c) *Let M be an Einstein manifold with* $\mathrm{Ric} = \lambda g$, $\lambda \leq 0$ *and Z be an affine reference frame. Then Z is a critical point of the usual energy. Moreover, since it is geodesic, it is also spatially harmonic.*

Remark 4.5. For Einstein manifolds, only negative values of λ are admissible since for $\lambda > 0$ unit timelike projective vector fields don't exist.

A particular case of a Lorentzian manifold admitting unit timelike affine (in fact Killing) vector fields is that of a Sasakian manifold with Lorentzian metric (see [13], [6]), that is defined as follows :

Definition 4.6. *Given φ, ξ and η tensor fields of type* $(1,1)$, $(1,0)$ *and* $(0,1)$, *respectively,* (φ, ξ, η) *is called and almost contact structure on M if the followings are satisfied :*

1. $\eta(\xi) = 1$.
2. $\eta(\phi(X)) = 0, \qquad X \in \Gamma(TM)$.
3. $\phi^2(X) = -X + \eta(X)\xi, \qquad X \in \Gamma(TM)$.

Definition 4.7. $(\phi, \xi, \eta, g, \varepsilon)$ *is called an almost contact metric structure on M, if* (ϕ, ξ, η) *is an almost contact structure on M and g is a semiriemannian metric on M such that*

1. $g(\xi, \xi) = \varepsilon \qquad \varepsilon = 1$ *or* -1.
2. $\eta(X) = \varepsilon g(\xi, X), \qquad X \in \Gamma(TM)$.
3. $g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y) \qquad X,\ Y \in \Gamma(TM)$.

Moreover, if $d\eta(X, Y) = g(\phi(X), Y)$ *for all* $X,\ Y \in \Gamma(TM)$ *then* $(\phi, \xi, \eta, g, \varepsilon)$ *is called a contact metric structure.*

Definition 4.8. *A contact metric structure on M is said to be normal if*

$$(\nabla_X \phi) Y = \varepsilon \eta(Y) X - g(X, Y)\xi, \quad X, Y \in \Gamma(TM).$$

In this case we call M a Sasakian manifold.

It is easy to see that the characteristic field of a Sasakian manifold is a Killing vector field. So it can be seen as a particular case of a $K-$contact manifold.

Definition 4.9. *A contact metric structure on M is said to be a $K-$contact structure if the characteristic field is Killing.*

In the Riemannian case it is known (see [3], pg. 92) that the Ricci tensor of a $K-$contact manifold verifies that $\mathrm{Ric}(\xi, X) = 0$ for all vector field orthogonal to ξ. It is easy to see that the same proof also works in the Lorentzian case, and then

Corollary 4.10. *If the characteristic field ξ of a Lorentzian $K-$contact manifold (and then of a Sasakian manifold) is timelike then it is a critical point of the usual energy. Furthermore, since it is geodesic, it is also spatially harmonic.*

5 Generalized Robertson-Walker Space-times

Definition 5.1. *A vector field X on a Lorentzian manifold is said to be closed and conformal if there exists a function $\phi \in C^\infty(M)$ such that*

$$\nabla_u X = \phi u \quad \textit{for every} \quad u \in TM.$$

Let M^{n+1} be a space-time endowed with a timelike vector field X that is closed and conformal. To study the spacelike harmonicity of the vector field $\nu = \frac{X}{\sqrt{-|X|^2}}$ for such a space-time we need the following:

Proposition 5.2. *[11] Let M^{n+1}, $n \geq 1$, be a Lorentzian manifold endowed with a timelike vector field which is closed and conformal. Then, we have*

a) The n-dimensional distribution tangent to $X^\perp$ is integrable and the functions $|X|^2$, $\mathrm{div} X$ and $X(\phi)$ are constant on the leaves of the corresponding foliation.

b) The unit timelike vector field defined by $\nu = \frac{X}{\sqrt{-|X|^2}}$ on M^{n+1} satisfies

$$\nabla_\nu \nu = 0, \quad \nabla_Y \nu = \frac{\phi}{\sqrt{-|X|^2}} Y \quad \textit{if} \quad \langle Y, \nu \rangle = 0.$$

Proposition 5.3. *The unit timelike vector field defined by $\nu = \frac{X}{\sqrt{-|X|^2}}$ is a critical point of the usual energy. Moreover, since ν is geodesic it is also spatially harmonic.*

Proof. Let us compute the rough Laplacian using an adapted orthonormal local frame $\{E_i, \nu\}_{i=1}^n$

$$\begin{aligned}
\nabla^*\nabla\nu &= -\sum_{i=1}^n (\nabla_{E_i}\nabla_{E_i}\nu - \nabla_{\nabla_{E_i}E_i}\nu) \\
&= -\frac{\phi}{\sqrt{-|X|^2}} \sum_{i=1}^n (\nabla_{E_i}E_i - P_\nu(\nabla_{E_i}E_i)) \\
&= -\frac{\phi^2}{|X|^2} n\nu.
\end{aligned}$$

□

If we now assume M to be compact and such that the Ricci curvature is nonnegative on null directions, that is, if M satisfies the null convergence condition, then these observers are not only critical points of the spacelike energy, we can show using proposition 2.6 that in fact they are absolute minimizers of the functional.

Proposition 5.4. *Let M^{n+1} be a Lorentzian manifold equipped with a closed and conformal timelike vector field X satisfying the null convergence condition, then the unit vector field $\nu = \frac{X}{\sqrt{-|X|^2}}$ is an absolute minimizer of the spacelike energy.*

Proof. Let Z be a reference frame on M, by proposition 2.6

$$\widetilde{B}(Z) \geq \frac{1}{2(n-1)} \int_M \mathrm{Ric}(Z,Z)\mathrm{dv} \quad \text{and } \widetilde{B}(\nu) = \frac{1}{2(n-1)} \int_M \mathrm{Ric}(\nu,\nu)\mathrm{dv}.$$

To get the result we only need to use that, under the hypothesis on M, we have

$$\mathrm{Ric}(Z,Z) \geq \mathrm{Ric}(\nu,\nu) \quad \text{for all} \quad Z \quad \text{such that } |Z|^2 = -1$$

as can be seen in [11]. □

Among the space-times admitting a closed and conformal timelike vector field, we find one of the most important cosmological models: the Robertson-Walker space-times and the so called generalized Robertson-Walker space-times [1]. In fact, it has been shown in [11], that any such a space-time is locally isometric to a Lorentzian warped product with a (negative definite) 1−dimensional factor.

Definition 5.5. *A generalized Robertson-Walker (GRW) space-time is a warped product $B \times_f F$ where $(B, g_B) = (I, -dt^2)$ with $I \subseteq \mathbb{R}$ an open interval, (F, g_F) a Riemannian manifold and $f : I \longrightarrow (0,\infty)$ a positive function. The reference frames defined by $Z = \partial_t$ are called comoving reference frames.*
If (F, g_F) is a model space $(S^{n-1}(1), \mathbb{R}^{n-1}, \mathbb{H}^{n-1}(-1))$, then the corresponding space-time is called a Robertson-Walker space-time.

As a particular case of the result obtained above, we have the following

Proposition 5.6. *Let M be a GRW space-time, then the comoving reference frame ∂_t is a spatially harmonic reference frame. Furthermore, if M is assumed to be compact and satisfying the null convergence condition, the comoving reference frame is an absolute minimizer of the spacelike energy.*

6 Gödel Universe

Another interesting space-time in General Relativity is the classical Gödel Universe, which is an exact solution of Einstein's field equations in which the matter takes the form of a rotating pressure-free perfect fluid. This model is $\mathbb{R}^4$ endowed with the metric,

$$\langle\ .\ ,\ .\ \rangle_L = dx_1^2 + dx_2^2 - \frac{1}{2}e^{2\alpha x_1}dy^2 - 2e^{\alpha x_1}dydt - dt^2$$

where α is a positive constant.
If we compute the Christoffel symbols of this metric we obtain,

Lemma 6.1.

$$\begin{aligned}
\nabla_{\partial_t}\partial_t &= 0, \quad \nabla_{\partial_y}\partial_t = \frac{\alpha}{2}e^{\alpha x_1}\partial_{x_1}, \quad \nabla_{\partial_{x_1}}\partial_t = \alpha(\partial_t - e^{-\alpha x_1}\partial_y),\\
\nabla_{\partial_{x_2}}\partial_t &= 0, \quad \nabla_{\partial_y}\partial_y = \frac{\alpha}{2}e^{2\alpha x_1}\partial_{x_1}, \quad \nabla_{\partial_{x_1}}\partial_y = \frac{\alpha}{2}e^{\alpha x_1}\partial_t,\\
\nabla_{\partial_{x_2}}\partial_y &= 0, \quad \nabla_{\partial_{x_i}}\partial_{x_j} = 0.
\end{aligned}$$

Let us denote by $\partial_{\tilde{y}} = \sqrt{2}(e^{-\alpha x_1}\partial_y - \partial_t)$. The Levi-Civita connection in the orthonormal frame $\{\partial_{x_1}, \partial_{x_2}, \partial_{\tilde{y}}, \partial_t\}$ is given by

Lemma 6.2.

$$\begin{aligned}
\nabla_{\partial_t}\partial_{\tilde{y}} &= \sqrt{2}e^{-\alpha x_1}\nabla_{\partial_t}\partial_y = \frac{\alpha}{\sqrt{2}}\partial_{x_1} = \nabla_{\partial_{\tilde{y}}}\partial_t,\\
\nabla_{\partial_y}\partial_{\tilde{y}} &= \sqrt{2}e^{-\alpha x_1}\nabla_{\partial_y}\partial_y - \sqrt{2}\nabla_{\partial_y}\partial_t = 0 = \nabla_{\partial_{\tilde{y}}}\partial_y,\\
\nabla_{\partial_{x_1}}\partial_{\tilde{y}} &= -\sqrt{2}\alpha e^{-\alpha x_1}\partial_y + \frac{\alpha}{\sqrt{2}}\partial_t - \sqrt{2}\alpha(\partial_t - e^{-\alpha x_1}\partial_y) = -\frac{\alpha}{\sqrt{2}}\partial_t,\\
\nabla_{\partial_{\tilde{y}}}\partial_{x_1} &= \frac{\alpha}{\sqrt{2}}\partial_t + \alpha\partial_{\tilde{y}},\\
\nabla_{\partial_{\tilde{y}}}\partial_{\tilde{y}} &= -\alpha\partial_{x_1}.
\end{aligned}$$

Proposition 6.3. *In the Gödel universe we have*

1. *The reference frame ∂_t is a critical point of the usual energy. Moreover, since ∂_t is geodesic it is also spatially harmonic.*
2. *The reference frame $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ is not spatially harmonic but $\widetilde{B}(Z) = \widetilde{B}(\partial_t)$.*

Proof. If we compute the rough Laplacian of ∂_t

$$\begin{aligned}
\nabla^*\nabla\partial_t &= \nabla_{\nabla_{\partial_{\tilde{y}}}\partial_{\tilde{y}}}\partial_t - \nabla_{\partial_{\tilde{y}}}\nabla_{\partial_{\tilde{y}}}\partial_t - \nabla_{\partial_{x_1}}\nabla_{\partial_{x_1}}\partial_t\\
&= -\alpha\nabla_{\partial_{x_1}}\partial_t - \frac{\alpha}{\sqrt{2}}\nabla_{\partial_{\tilde{y}}}\partial_{x_1} + \frac{\alpha}{\sqrt{2}}\nabla_{\partial_{x_1}}\partial_{\tilde{y}}\\
&= -\alpha^2\partial_t,
\end{aligned}$$

that is colineal to ∂_t.
Let us show that $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ does not satisfy the Euler-Lagrange equations.

Since

$$\widetilde{X}_Z = -\sum_{i=1}^{4}\varepsilon_i(\nabla_{E_i}(\nabla Z \circ P_Z))(E_i) + (\nabla_Z \circ P_Z)^t(\nabla_Z Z),$$

and

$$\begin{aligned}
(\nabla Z \circ P_Z)(\partial_t) &= (\nabla Z)(\partial_t - 2e^{-\alpha x_1}\partial_y) \\
&= \frac{\alpha}{\sqrt{2}}\partial_{x_1} - \sqrt{2}\alpha\partial_{x_1} = -\frac{\alpha}{\sqrt{2}}\partial_{x_1}, \\
(\nabla Z \circ P_Z)(\partial_{\tilde{y}}) &= (\nabla Z)(\partial_{\tilde{y}} + \sqrt{2}e^{-\alpha x_1}\partial_y) = \alpha\partial_{x_1}, \\
(\nabla Z \circ P_Z)(\partial_{x_1}) &= (\nabla Z)(\partial_{x_1}) = -\sqrt{2}\alpha e^{-\alpha x_1}\partial_y + \frac{\alpha}{\sqrt{2}}\partial_t, \\
(\nabla Z \circ P_Z)(\partial_{x_2}) &= 0
\end{aligned}$$

from where,

$$\begin{aligned}
(\nabla Z \circ P_Z)^t(\nabla_Z Z) &= (\nabla Z \circ P_Z)^t(\alpha\partial_{x_1}) = -g(\alpha\partial_{x_1}, -\frac{\alpha}{\sqrt{2}}\partial_{x_1})\partial_t + g(\alpha\partial_{x_1}, \alpha\partial_{x_1})\partial_{\tilde{y}} \\
&= \frac{\alpha^2}{\sqrt{2}}\partial_t + \alpha^2\partial_{\tilde{y}},
\end{aligned}$$

we have that

$$\begin{aligned}
\widetilde{X}_Z &= \nabla_{\partial_t}(-\frac{\alpha}{\sqrt{2}}\partial_{x_1}) - \nabla_{\partial_{\tilde{y}}}(\alpha\partial_{x_1}) - \nabla_{\partial_{x_1}}(-\sqrt{2}\alpha e^{-\alpha x_1}\partial_y + \frac{\alpha}{\sqrt{2}}\partial_t) \\
&\quad +(\nabla_Z \circ P_Z)(\nabla_{\partial_{\tilde{y}}}\partial_{\tilde{y}}) + \frac{\alpha^2}{\sqrt{2}}\partial_t + \alpha^2\partial_{\tilde{y}} \\
&= \alpha^2\sqrt{2}(e^{-\alpha x_1}\partial_y - \partial_t).
\end{aligned}$$

Therefore, Z is not a critical point of the spacelike energy. Nevertheless, Z and ∂_t have the same spacelike energy, since

$$\begin{aligned}
2\widetilde{b}(\sqrt{2}e^{-\alpha x_1}\partial_y) &= -\|\nabla_{\partial_t}(\sqrt{2}e^{-\alpha x_1}\partial_y)\|^2 + \|\nabla_{\partial_{\tilde{y}}}(\sqrt{2}e^{-\alpha x_1}\partial_y)\|^2 + \|\nabla_{\partial_{x_1}}(\sqrt{2}e^{-\alpha x_1}\partial_y)\|^2 \\
&\quad +\|\nabla_{\partial_{x_2}}(\sqrt{2}e^{-\alpha x_1}\partial_y)\|^2 + \|\nabla_{\sqrt{2}e^{-\alpha x_1}\partial_y}(\sqrt{2}e^{-\alpha x_1}\partial_y)\|^2 \\
&= -2e^{-2\alpha x_1}\|\nabla_{\partial_t}\partial_y\|^2 + 2\alpha^2\|\frac{1}{2}\partial_t - e^{-\alpha x_1}\partial_y\|^2 + 4e^{-4\alpha x_1}\|\nabla_{\partial_t}\partial_t\|^2 \\
&= -\frac{\alpha^2}{2} + \frac{\alpha^2}{2} + \alpha^2,
\end{aligned}$$

and

$$\begin{aligned}
2\widetilde{b}(\partial_t) &= \|\nabla_{\partial_{\tilde{y}}}\partial_t\|^2 + \|\nabla_{\partial_{x_1}}\partial_t\|^2 + \|\nabla_{\partial_{x_2}}\partial_t\|^2 \\
&= \alpha^2.
\end{aligned}$$

□

Consequently, although ∂_t is a critical point it can not be an absolute minimizer. In fact, it is unstable as we can see by the following argument,

Proposition 6.4. *∂_t is unstable.*

Proof. To prove the instability of ∂_t, we have to show that there exists an open subset U with compact closure and a vector field X orthogonal to ∂_t with support in U, such that

$$(Hess\widetilde{B}_U)_{\partial_t}(X) < 0.$$

So, given $\delta \in \mathbb{R}^+$ let U be the open ball centered at $(2\delta, 0, 0, 0)$ of radius 3δ and $X = f_\delta \partial_{x_2}$, where $f_\delta = e^{-\frac{\alpha}{2}x_1} h_\delta$ and h_δ is the test function

$$h_\delta(r) = \begin{cases} 1 & r \leq \delta \\ e^{\frac{\delta}{r-2\delta}}(e^{\frac{\delta}{\delta-r}} + e^{\frac{\delta}{r-2\delta}})^{-1} & \delta < r < 2\delta \\ 0 & r \geq 2\delta, \end{cases}$$

with r being the distance to the point $(2\delta, 0, 0, 0)$.
Then, using proposition 3.7 and lemma 6.1

$$\begin{aligned}(Hess\widetilde{B}_U)_{\partial_t}(f_\delta\partial_{x_2}) &= \int_{B(2\delta)-B(\delta)} (\|\nabla(f_\delta\partial_{x_2})\|^2 + \|\nabla_{\partial_t}(f_\delta\partial_{x_2})\|^2 - f_\delta^2\alpha^2)\mathrm{dv} \\ &\quad - \int_{B(\delta)} (e^{-\alpha x_1}\alpha^2 - \frac{\alpha^2}{4}e^{-\alpha x_1})\mathrm{dv}.\end{aligned}$$

If we denote by h'_δ the first derivative with respect to $r = \sqrt{(x_1 - 2\delta)^2 + x_2^2 + y^2 + t^2}$ then

$$\begin{aligned}\partial_t(f_\delta) &= e^{-\frac{\alpha}{2}x_1} h'_\delta(r)\frac{t}{r}, \\ \partial_y(f_\delta) &= e^{-\frac{\alpha}{2}x_1} h'_\delta(r)\frac{y}{r}, \\ \partial_{x_1}(f_\delta) &= e^{-\frac{\alpha}{2}x_1} h'_\delta(r)\frac{x_1 - 2\delta}{r} - \frac{\alpha}{2}e^{-\frac{\alpha}{2}x_1}h_\delta(r), \\ \partial_{x_2}(f_\delta) &= e^{-\frac{\alpha}{2}x_1} h'_\delta(r)\frac{x_2}{r},\end{aligned}$$

where

$$h'_\delta(r) = \begin{cases} 0 & r \leq \delta \\ -\dfrac{e^{\frac{\delta}{r-2\delta}} e^{\frac{\delta}{\delta-r}} \left(\frac{\delta}{(\delta-r)^2} + \frac{\delta}{(r-2\delta)^2}\right)}{\left(e^{\frac{\delta}{r-2\delta}} + e^{\frac{\delta}{\delta-r}}\right)^2} & \delta < r < 2\delta \\ 0 & r \geq 2\delta. \end{cases}$$

Since $\mathrm{dv} = \frac{e^{\alpha x_1}}{\sqrt{2}}\mathrm{dv}_0$ then

$$\begin{aligned}\sqrt{2} \quad & \int_{B(2\delta)-B(\delta)} (\|\nabla(f_\delta\partial_{x_2})\|^2 + \|\nabla_{\partial_t}(f_\delta\partial_{x_2})\|^2 - f_\delta^2\alpha^2)\mathrm{dv} = \\ & \int_{B(2\delta)-B(\delta)} e^{\alpha x_1}((\partial_{\widetilde{y}}(f_\delta))^2 + (\partial_{x_1}(f_\delta))^2 + (\partial_{x_2}(f_\delta))^2 - f_\delta^2\alpha^2)\mathrm{dv}_0 = \\ & \int_{B(2\delta)-B(\delta)} (2(h'_\delta(r))^2(e^{-2\alpha x_1}\frac{y^2}{r^2} + \frac{t^2}{r^2} - 2e^{-\alpha x_1}\frac{yt}{r^2}) + \frac{\alpha^2}{4}h_\delta^2(r) + (h'_\delta(r))^2\frac{(x_1 - 2\delta)^2}{r^2} \\ & -\alpha h_\delta(r)h'_\delta(r)\frac{x_1 - 2\delta}{r} + (h'_\delta(r))^2\frac{x_2^2}{r^2} - \alpha^2 h_\delta^2(r))\mathrm{dv}_0.\end{aligned}$$

Using that $|h'_\delta(r)| < \frac{2}{\delta}$, $x_i^2 r^{-2} < 4$ and $e^{-kx_1} < 1$ for $k > 0$. Then

$$\sqrt{2}\int_{B(2\delta)-B(\delta)} (\|\nabla(f_\delta \partial_{x_2})\|^2 + \|\nabla_{\partial_t}(f_\delta \partial_{x_2})\|^2 - f_\delta^2\alpha^2)\mathrm{dv} \leq (\frac{116}{\delta^2} + \frac{4\alpha}{\delta})\mathrm{vol}(B(2\delta) - B(\delta)).$$

And

$$\sqrt{2}(Hess\widetilde{B}_U)_{\partial_t}(f_\delta \partial_{x_2}) \leq (\frac{116}{\delta^2} + \frac{4\alpha}{\delta})(\mathrm{vol}(B(2\delta)) - \mathrm{vol}(B(\delta))) - \frac{3}{4}\alpha^2 \mathrm{vol}(B(\delta)),$$

where vol means the volume in the Euclidean metric.

Consequently, since the positive term is of order $O(\delta^3)$ and the negative of order $O(\delta^4)$ then, to get the result we only have to choose δ big enough. $\square$

7 Hopf vector fields on Lorentzian Berger's spheres

It is well known that Hopf fibration $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$ determines a foliation of S^{2n+1} by great circles and that a unit vector field can be choosen as a generator of this distribution. It is given by $V = JN$ where N represents the unit normal to the sphere and J the usual complex structure on $\mathbb{R}^{2n+2}$. V is the standard Hopf vector field. In S^{2n+1} we can consider the canonical variation g_μ, with $\mu \neq 0$, of the usual metric g

$$\begin{aligned} g_\mu|_{V^\perp} &= g|_{V^\perp}, \\ g_\mu|_V &= \mu g|_V, \\ g_\mu(V, |V^\perp) &= 0. \end{aligned}$$

For $n = 1$ and $\mu > 0$ these metrics on the sphere are known as Berger's metrics (see [2] pg. 252). For all $\mu \neq 0$ the map $\pi : (S^{2n+1}, g_\mu) \longrightarrow \mathbb{C}P^n$ is a semiriemannian submersion with totally geodesic fibers. The distribution determined by the fibers admits as a unit generator $V^\mu = \frac{1}{\sqrt{|\mu|}}JN$ which is timelike for negative μ and we will call also Hopf vector field. As can be seen in [9] Hopf vector fields are critical for the usual energy of unit vector fields and consequently, since they are geodesic, they are also spatially harmonic and $\tilde{B}(V^\mu) = B(V^\mu)$. Nevertheless the second variation of both functionals at V^μ is different, in fact:

$$\begin{aligned} (Hess\tilde{B})_{V^\mu}(A) &= \int_{S^{2n+1}} (\|\nabla^\mu A\|^2 + \|\nabla^\mu_A V^\mu + \nabla^\mu_{V^\mu} A\|^2 - \|A\|^2(C_1^1\nabla^\mu \widetilde{K})(V^\mu))\mathrm{dv}_\mu \\ &= \int_{S^{2n+1}} (\|\nabla^\mu A\|^2 + \|\nabla^\mu_A V^\mu + \nabla^\mu_{V^\mu} A\|^2 - 2n\mu\|A\|^2)\mathrm{dv}_\mu \\ &= \int_{S^{2n+1}} \|\nabla^\mu_A V^\mu + \nabla^\mu_{V^\mu} A\|^2 \mathrm{dv}_\mu + (HessB)_{V^\mu}(A). \end{aligned} \quad (3)$$

Where ∇^μ is the Levi-Civita connection of g_μ that is related to ∇ by

$$\begin{aligned} \nabla^\mu_V X &= \nabla_V X + (\lambda - 1)\nabla_X V, \\ \nabla^\mu_X V &= \lambda \nabla_X V, \\ \nabla^\lambda_X Y &= \nabla_X Y \quad X, Y \in V^\perp. \end{aligned}$$

We have shown in [9] that V^μ is unstable for B when $n=1$ but the stability in higher dimensions is an open question. In contrast, the problem for the spacelike energy is completely understood.

Proposition 7.1. *Hopf vector fields on Lorentzian Berger's spheres are stable critical points of the spacelike energy.*

Proof. Let $A : S^{2n+1} \to (JN)^\perp \subset \mathbb{C}^{n+1}$, we set :

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}d\theta \in (JN)_p^\perp$$

so that the Fourier serie of A is :

$$A(p) = \sum_{l\in\mathbb{Z}} A_l(p).$$

Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$, we have :

$$\nabla_{JN}A = \bar{\nabla}_{JN}A = \sum_{l\in\mathbb{Z}} ilA_l$$

and, if $\mathcal{C}(p)$ denotes the fiber of the Hopf fibration passing through p, :

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0,$$

if $l \neq q$. As in [4] we can show that if $l \neq q$ then

$$(Hess\tilde{B})_{V^\mu}(A) = \sum_{l\in\mathbb{Z}} (Hess\tilde{B})_{V^\mu}(A_l).$$

Now,

$$\|\nabla^\mu A\|^2 = -\|\nabla^\mu_{V^\mu}A\|^2 + \mu\|A\|^2 + \sum_{i,j=1}^{2n}(g(\nabla^\mu_{E_i}A, E_j))^2,$$

then

$$(Hess\tilde{B})_{V^\mu}(A) = \int_{S^{2n+1}}(-2n\mu\|A\|^2 + \sum_{i,j=1}^{2n}(g(\nabla^\mu_{E_i}A, E_j))^2 + 2\frac{\mu}{\sqrt{-\mu}}g_\mu(\nabla^\mu_{V^\mu}A, JA))\mathrm{dv}_\mu.$$

Since $g_\mu(\nabla^\mu_{V^\mu}A_l, JA_l) = \frac{l+\mu-1}{\sqrt{-\mu}}\|A_l\|^2$, and $2(1-\mu-l) \geq 0$ for $l \in \mathbb{Z}^-$ then $(Hess\tilde{B})_{V^\mu}(A_l) \geq 0$. Let us see now what happens for positive l. In [9], it has been shown that

$$(HessB)_{V^\mu}(A) \geq \int_{S^{2n+1}}((\mu(1-4n) + (2n+2) - \mu n^2)\|A\|^2 - \|\nabla^\mu_{V^\mu}A - n\sqrt{-\mu}JA\|^2)\mathrm{dv}_\mu,$$

from where we obtain using (3) that

$$\begin{aligned}(Hess\tilde{B})_{V^\mu}(A) &\geq \int_{S^{2n+1}}((\mu(1-4n)+2n+2-\mu n^2)\|A\|^2 + \\ &-\|\nabla^\mu_{V^\mu}A - n\sqrt{-\mu}JA\|^2 + \|\nabla^\mu_{V^\mu}A - \sqrt{-\mu}JA\|^2)\mathrm{dv}_\mu \\ &= \int_{S^{2n+1}}((\mu(1-4n)+2n+2-\mu n^2)\|A\|^2 + \\ &+\frac{1}{\mu}\|\nabla_V A + (\mu-1+n\mu)JA\|^2 - \frac{1}{\mu}\|\nabla_V A + (2\mu-1)JA\|^2\mathrm{dv}_\mu.\end{aligned}$$

Consequently,

$$\begin{aligned}(Hess\tilde{B})_{V^\mu}(A_l) &\geq \int_{S^{2n+1}}(\mu(1-4n)+2n+2-\mu n^2)\|A_l\|^2\mathrm{dv}_\mu \\ &+ \int_{S^{2n+1}}\frac{1}{\mu}((l+\mu-1+n\mu)^2-(l+2\mu-1)^2)\|A_l\|^2\mathrm{dv}_\mu \\ &= \int_{S^{2n+1}}(\mu(-2n-2)+2nl-2l+4)\mathrm{dv}_\mu \geq 0\end{aligned}$$

for all positive l and so V^μ is stable.

□

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