Volume, energy and generalized energy of unit vector fields on Berger spheres. Stability of Hopf vector fields

Olga Gil-Medrano* and Ana Hurtado*

Abstract

We study to what extent the known results concerning the behaviour of Hopf vector fields, with respect to volume, energy and generalized energy functionals, on the round sphere are still valid for the metrics obtained by performing the canonical variation of the Hopf fibration.

1 Introduction

Let $V : M \to TM$ be a smooth vector field on a manifold. For a given Riemannian metric g on M the tangent manifold can be endowed with a natural metric g^S known as the Sasaki metric. The volume of V is the volume of $V(M)$ considered as a submanifold of (TM, g^S). Analogously we can define the energy of V as the energy of the map $V : (M, g) \to (TM, g^S)$, and more generally, if $\tilde{g}$ is another metric on M, the energy of $V : (M, \tilde{g}) \to (TM, g^S)$ that we will call the generalized energy $E_{\tilde{g}}$. On each manifold, these functionals have a lower bound and then, a natural question arises, namely that of determining the infimum of their values when acting on vector fields such that $g(V, V) = 1$ and finding the minimizers, or at least a minimizing sequence. It is easy to see that if M admits unit parallel vector fields, these should be exactly the minimizers and so, volume and energy can be seen as a measure of how much the vector field deviates from being parallel.

The geometrically simplest manifolds admitting unit vector fields but not parallel ones are odd-dimensional round spheres, and Hopf vector fields on them are very special unit vector fields. They are tangent to the fibres of the Hopf fibration $\pi : S^{2m+1} \to \mathbb{C}P^m$. When both manifolds are endowed with their usual metrics, this map is a Riemannian submersion with totally geodesic fibres whose tangent space is generated by the unit vector field $V = JN$, where N is the unit normal to the sphere and J is the usual complex structure of $\mathbb{R}^{2m+2}$. It is usual to call also a Hopf vector field any vector field obtained as the image of N by any complex structure; they can be characterized as the unit Killing vector fields of the sphere.

*Partially supported by DGI (Spain) and FEDER Project MTM 2004-06015-C02-01 and by a grant AVCiTGRUPOS03/169. The second author is supported by a Research Grant from *Ministerio de Educación y Cultura"*

2000 Mathematics Subject Classification: 53C20, 58E15, 53C25, 58E20.

Keywords and phrases: Energy and volume functionals; unit vector fields on spheres; critical points; Hopf fibration; Berger metrics.

In [10], Gluck and Ziller showed that Hopf vector fields on the 3-dimensional round sphere are the absolute minimizers of the volume and the analogous result for the energy was shown by Brito in [3]. For spheres of higher dimension, they are unstable critical points of the energy (see [14],[15],[8]) and critical points of the volume, which is equivalent to define a minimal immersion in the unit tangent bundle, as has been shown by the first author and E. Llinares-Fuster in [9].

The results quoted above are independent on the radius of the sphere. Nevertheless, in what concerns the stability as critical points of the volume it has been shown, by the first author with Llinares-Fuster ([8]) and with Borrelli ([2]), that for $m > 1$ they are unstable if and only if the curvature is lower than $2m - 3$. The infimum of the volume of unit vector fields (as well as the regularity and properties of minimizers) shows to be a very sensitive geometrical invariant that enables us to detect a variation of the metric by homotheties.

In order to better understand these phenomena we study in this paper the behaviour of the Hopf vector field with respect to the volume and the energy when we consider another variation of the standard metric on the sphere, which is a little more complicated but also very natural: the canonical variation of the Riemannian submersion given by the Hopf fibration. The metrics so constructed are known as Berger metrics, they consist in a 1-parameter variation g_μ for $\mu > 0$. In the last section, we will consider also the Lorentzian Berger metrics, i. e. when $\mu < 0$. This paper is organized as follows:

We devote section 2 to recall the definitions and to state the results we will need in the sequel, as well as to show that for all $\mu \neq 0$, the unit Hopf vector field V^μ defines a harmonic map $V^\mu : (S^{2m+1}, g_\lambda) \to (T^1S^{2m+1}, g_\mu^S)$ for all $\lambda \neq 0$ and consequently it is a critical point for the generalized energy E_{g_λ}. Moreover, V^μ defines a minimal immersion.

In section 3, we study the special case of the 3-dimensional sphere and we have shown that *the unit Hopf vector field on (S^3, g_μ) is the only absolute minimizer of the energy, and of the volume, if and only if $\mu \leq 1$. For $\mu > 1$, we will show that it is not even a local minimum, since it is unstable.*

So, the minimizing properties of Hopf vector fields on the round S^3 can be extended to Berger 3-spheres if $\mu \leq 1$, but not otherwise. It is worthwhile to recall here that with these metrics, the sphere can be isometrically immersed as a geodesic sphere in the complex projective space and that, in contrast, for $\mu > 1$ it can be identified with a geodesic sphere of the complex hyperbolic space.

For higher dimensional spheres, we have determined the values of μ for which the Hopf vector field is stable as a critical point of the energy and as a critical point of the volume. More precisely *the Hopf vector field on (S^{2m+1}, g_μ), with $m > 1$, is energy stable if and only if $(2m-2)\mu^2 \leq 1$ and it is volume stable if and only if $(2m-2)\mu^3 - \mu \leq 1$*. This is done in section 5, by using the methods developed in [8] and [2] for the round sphere and the various expressions of the Hessians computed in section 4.

We have used the same ideas to study the subset $\mathcal{E}$ of $\mathbb{R}^+ \times \mathbb{R}^+$ of pairs (μ, λ) such that V^μ is stable as a critical point of the generalized energy E_{g_λ}. Although a complete description of $\mathcal{E}$ is still an open question, we can show, for example, that if $(2m-1)\mu \leq 2$ then $(\mu, \lambda) \in \mathcal{E}$, for all $\lambda > 0$, and that if $(2m-1)\mu > 2$ and $\mu \leq 3/2$ then $(\mu, \lambda) \in \mathcal{E}$ if and only if $((2m-1)\mu - 2)\lambda \leq (\mu - 1)^2$. As a consequence, Hopf vector fields of the round sphere S^{2m+1}, with $m > 1$, are unstable as critical points of the generalized energy E_{g_λ}, for all $\lambda > 0$.

Section 6 is devoted to the study of the behaviour of Hopf vector fields on Lorentzian Berger spheres with respect to energy and volume functionals. To obtain the corresponding expressions for the Hessians is a straightforward question: one should only pay attention to the timelike character of the Hopf vector field in these metrics. We have shown that *on (S^{2m+1}, g_μ), with $\mu < 0$, if $(2m-2)\mu^2 < 1$ the unit Hopf vector field is an unstable critical point of the energy and if $(2-2m)\mu^3 + (4m-4)\mu^2 + \mu < 1$ it is an unstable critical point of the volume.* In contrast, neither the stability results nor the minimizing properties for the 3-dimensional case have Lorentzian analogous; in fact we have shown that *on (S^3, g_μ), for all $\mu < 0$, the unit Hopf vector field is unstable.* These kind of difficulties we met when trying to determine the stability are not exclusive of Berger spheres and moreover, since on Lorentzian manifolds the energy of unit timelike vector fields is not bounded below, it is not natural to talk about absolute minimizers. These facts led us to define in [7] a new functional on the space of unit timelike vector fields of a Lorentz manifold, that we called spacelike energy, and is given by the integral of the square norm of the projection of the covariant derivative of the vector field onto its orthogonal complement. In [7] we have shown that Hopf vector fields are stable critical points of the spacelike energy. We finish this paper by showing that *on any Lorentzian Berger 3-sphere, the Hopf vector field is, up to sign, the only minimizer of the spacelike energy.*

2 Definitions and first results

2.1 Energy and volume of vector fields.

Given a Riemannian manifold (M, g), the Sasaki metric g^S on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows:

$$g^S(\zeta_1, \zeta_2) = g(\pi_* \circ \zeta_1, \pi_* \circ \zeta_2) + g(\kappa \circ \zeta_1, \kappa \circ \zeta_2),$$

where $\pi : TM \to M$ is the projection and κ is the connection map of ∇. We will consider also its restriction to the tangent sphere bundle, obtaining the Riemannian manifold (T^1M, g^S).

As in [6], for each metric $\tilde{g}$ on M we can define the generalized energy of the vector field V, denoted $E_{\tilde{g}}(V)$, as the energy of the map $V : (M, \tilde{g}) \to (TM, g^S)$ that is given by

$$E_{\tilde{g}}(V) = \frac{1}{2} \int_M \operatorname{tr} L_{(\tilde{g},V)} \; dv_{\tilde{g}},$$

where $L_{(\tilde{g},V)}$ is the endomorphism determined by $V^* g^S(X, Y) = \tilde{g}(L_{(\tilde{g},V)}(X), Y)$. This energy can also be written as

$$E_{\tilde{g}}(V) = \frac{1}{2} \int_M \sqrt{\det P_{\tilde{g}}} \; \operatorname{tr}(P_{\tilde{g}}^{-1} \circ L_V) \; dv_g \tag{2.1}$$

where $P_{\tilde{g}}$ and L_V are defined by $\tilde{g}(X, Y) = g(P_{\tilde{g}}(X), Y)$ and $V^* g^S(X, Y) = g(L_V(X), Y)$, respectively. By the definition of the Sasaki metric, $L_V = \operatorname{Id} + (\nabla V)^t \circ \nabla V$. In particular, for $\tilde{g} = g$

$$E_g(V) = \frac{1}{2} \int_M \operatorname{tr} L_V \; dv_g = \frac{n}{2} vol(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv_g. \tag{2.2}$$

This functional is known as the energy and will be represented by E. Its relevant part, $B(V) = \frac{1}{2}\int_M \|\nabla V\|^2 dv_g$, is known as the total bending of V and its restriction to unit vector fields has been thorougly studied by Wiegmink in [14], (see also [15]).

On the other hand, the volume of a vector field V is defined as the volume of the submanifold $V(M)$ of (TM, g^S). It is given by

$$F(V) = \int_M \sqrt{\det L_V}\, dv_g. \tag{2.3}$$

Since for $\tilde{g} = V^* g^S$ we have $P_{\tilde{g}} = L_V$, then (2.1) and (2.3) give

$$F(V) = \frac{2}{n} E_{V^* g^S}(V).$$

The first variation of the generalized energy has been computed in [6]. It has also been shown there that V is a critical point of F if and only if V is a critical point of $E_{V^* g^S}$ and that, on a compact M, a critical vector field of any of these generalized energies should be parallel. This is one of the reasons why it is usual to restrict the functionals to the submanifold of unit vector fields and so, critical points are those V which are stationary for variations consisting on unit vector fields, or equivalently with variational field orthogonal to V.

For now on, we are going to consider the restriction of these functionals to the submanifold of unit vector fields.

The following proposition shown in [6] generalizes the characterization of critical points of the total bending in [14] and of the volume in [9].

Proposition 2.1. ([6]) *Let (M, g) be a Riemannian manifold, a unit vector field V is a critical point of $E_{\tilde{g}}$ if and only if*

$$\omega_{(V,\tilde{g})}(V^\perp) = \{0\},$$

with $\omega_{(V,\tilde{g})} = C_1^1 \nabla K_{(V,\tilde{g})}$ *and* $K_{(V,\tilde{g})} = \sqrt{\det P_{\tilde{g}}}\; P_{\tilde{g}}^{-1} \circ (\nabla V)^t$.

Remark 2.2. For a $(1,1)$-tensor field K, if $\{E_i\}$ is a g-orthonormal local frame, we have

$$C_1^1 \nabla K(X) = \sum_i g((\nabla_{E_i} K)X, E_i).$$

As a particular case of Proposition 2.1, for $\tilde{g} = g$, a unit vector field V is a critical point of the energy (or of the total bending) if and only if

$$\omega_{(V,g)}(V^\perp) = \{0\}, \quad \text{with} \quad \omega_{(V,g)} = C_1^1 \nabla (\nabla V)^t.$$

Furthermore, if we put $\tilde{g} = V^* g^S$, we obtain that critical points of the volume are characterized by the condition

$$\omega_V(V^\perp) = \{0\}, \quad \text{where} \quad \omega_V = C_1^1 \nabla K_V \text{ and } \quad K_V = \sqrt{\det L_V}\, L_V^{-1} \circ (\nabla V)^t.$$

In [9] it has been proved that a unit vector field is a critical point of F if and only if it defines a minimal immersion in (T^1M, g^S). Nevertheless, as it has been shown in [6], for a critical point V of $E_{\tilde{g}}$ to determine a harmonic map of $(M, \tilde{g})$ in (T^1M, g^S), V has to satisfy the condition

$$\sum_i R((\nabla V)\tilde{E}_i, V, \tilde{E}_i) + \sum_i (\nabla_{\tilde{E}_i} \tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) = 0, \tag{2.4}$$

where $\{\tilde{E}_i\}$ is a $\tilde{g}$-orthonormal frame and R represents the curvature operator of the metric g, that is

$$R(X,Y,Z) = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

Theorem 2.3. ([8]) *Let V be a unit vector field on the Riemannian manifold (M,g).*
a) If V is a critical point of $E_{\tilde{g}}$, the Hessian of $E_{\tilde{g}}$ at V acting on $A \in V^{\perp}$ is given by

$$(HessE_{\tilde{g}})_V(A) = \int_M \|A\|^2 \omega_{(V,\tilde{g})}(V)\ dv_g + \int_M \sqrt{\det P_{\tilde{g}}}\ \operatorname{tr}\left(P_{\tilde{g}}^{-1} \circ (\nabla A)^t \circ \nabla A\right) dv_g.$$

b) If V is a critical point of the energy, the Hessian of E at V acting on $A \in V^{\perp}$ is given by

$$(HessE)_V(A) = \int_M \|A\|^2 \omega_{(V,g)}(V)\ dv_g + \int_M \|\nabla A\|^2 dv_g.$$

c) For a unit vector field V defining a minimal immersion, the Hessian of F at V acting on $A \in V^{\perp}$ is given by

$$\begin{aligned}
(HessF)_V(A) \ = \ & \int_M \|A\|^2 \omega_V(V)\ dv_g + \int_M \frac{2}{\sqrt{\det L_V}}\ \sigma_2(K_V \circ \nabla A) dv_g \\
- \ & \int_M \operatorname{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla V \circ K_V \circ \nabla A\right) dv_g \\
+ \ & \int_M \sqrt{\det L_V}\ \operatorname{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla A\right) dv_g,
\end{aligned}$$

where σ_2 is the second elementary symmetric polynomial function. In particular, $\sigma_2(K_V \circ \nabla A) = 1/2(\operatorname{tr}(K_V \circ \nabla A))^2 - \operatorname{tr}(K_V \circ \nabla A)^2$.

Remark 2.4. The Hessian of the volume at a vector field V defining a minimal immersion can be simplified if V is assumed to be a Killing vector field. Using Lemma 9 of [8] we obtain

$$\begin{aligned}
(HessF)_V(A) \ = \ & \int_M \|A\|^2 \omega_V(V)\ dv_g + \int_M \frac{2}{\sqrt{\det L_V}}\ \sigma_2(K_V \circ \nabla A) dv_g \\
+ \ & \int_M \sqrt{\det L_V}\ \operatorname{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ L_V^{-1} \circ \nabla A\right) dv_g. \qquad (2.5)
\end{aligned}$$

2.2 Berger spheres.

Hopf vector fields on odd-dimensional spheres are tangent to the fibres of the Hopf fibration $\pi : (S^{2m+1}, g) \to (\mathbb{C}P^m, \overline{g})$, where g is the usual metric of curvature 1 and $\overline{g}$ is the Fubini-Study metric with sectional curvatures between 1 and 4. This map is a Riemannian submersion with totally geodesic fibres whose tangent space is generated by the unit vector field $V = JN$, where N is the unit outward normal to the sphere and J is the usual complex structure of $\mathbb{R}^{2m+2}$; in other words, $V(p) = ip$.

The canonical variation of the submersion is the one-parameter family of metrics (S^{2m+1}, g_μ), $\mu \neq 0$, defined by

$$g_\mu|_{V^\perp} = g|_{V^\perp}, \qquad g_\mu(V,V) = \mu g(V,V), \qquad g_\mu(V, V^\perp) = 0, \qquad (2.6)$$

where $V^\perp$ denotes the orthogonal with respect to metric g of the 1-dimensional distribution generated by V. When $\mu > 0$ the new metric is Riemannian and if $\mu < 0$ the metric is Lorentzian and V is timelike.

For all $\mu \neq 0$, the map $\pi : (S^{2m+1}, g_\mu) \to (\mathbb{C}P^m, \overline{g})$ is a semi-Riemannian submersion with totally geodesic fibres. (S^3, g_μ), with $\mu > 0$, is known as a Berger sphere. We will use the same name for all dimensions and we will call $V^\mu = \frac{1}{\sqrt{|\mu|}} V$ the Hopf vector field. It is a unit Killing vector field with geodesic flow.

We denote by $\bar{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{2m+2}$. The Levi-Civita connection ∇ on (S^{2m+1}, g) is $\nabla_X Y = \bar{\nabla}_X Y - < \bar{\nabla}_X Y, N > N$ and $\bar{\nabla}_X V = J\bar{\nabla}_X N = JX$. Therefore $\nabla_V V = 0$ and if $< X, V >= 0$ then $\nabla_X V = JX$.

Using Koszul formula, one obtains the relation of ∇^μ, the Levi-Civita connection of the metric g_μ, with ∇

$$\nabla^\mu_V X = \nabla_V X + (\mu - 1)\nabla_X V, \qquad \nabla^\mu_X V = \mu \nabla_X V, \qquad \nabla^\mu_X Y = \nabla_X Y, \tag{2.7}$$

for all X, Y in $V^\perp$.

By straightforward computations it can be seen that the sectional curvature K_μ of (S^{2m+1}, g_μ) takes the value

$$K_\mu(\sigma) = 1 + (1 - \mu) g(X, JY)^2,$$

if $\sigma \subset V^\perp$ and $\{X, Y\}$ is an orthonormal basis and it takes the value $K_\mu(\sigma) = \mu$, if the plane σ contains de vector V^μ. Consequently, the Ricci tensor has the form

$$\begin{aligned} Ric_\mu(V^\mu, V^\mu) = 2m|\mu|, &\qquad Ric_\mu(X, V^\mu) = 0, \\ Ric_\mu(X, Y) &= 2(1 - \mu + m) g(X, Y), \end{aligned} \tag{2.8}$$

for all X, Y in $V^\perp$, and the scalar curvature is given by

$$S_\mu = 2m(2 + 2m - \mu).$$

It has been shown in [6] that, for all $\lambda > 0$, the map $V : (S^{2m+1}, g_\lambda) \to (T^1(S^{2m+1}), g^S)$ is harmonic. More generally we have

Proposition 2.5. *For all $\mu, \lambda \neq 0$, the map $V^\mu : (S^{2m+1}, g_\lambda) \to (T^1(S^{2m+1}), g^S_\mu)$ is harmonic.*

Proof. According to Proposition 2.1 and condition (2.4), we need to show that

$$\omega_{(V^\mu, g_\lambda)}(X) = 0 \qquad \text{for all } X \in V^\perp \tag{2.9}$$

and

$$\sum_i R_\mu((\nabla^\mu V^\mu)\tilde{E}_i, V^\mu, \tilde{E}_i) + \sum_i (\nabla^\mu_{\tilde{E}_i} \tilde{E}_i - \nabla^\lambda_{\tilde{E}_i} \tilde{E}_i) = 0, \tag{2.10}$$

where $\{\tilde{E}_i\}$ is a g_λ-orthonormal frame with $\tilde{E}_{2m+1} = V^\lambda$.

Using (2.7), for $i = 1, \ldots 2m$, we have

$$\nabla^\mu_{\tilde{E}_i} \tilde{E}_i - \nabla^\lambda_{\tilde{E}_i} \tilde{E}_i = 0$$

and it is easy to see that $R_\mu(X, V^\mu, Y) = \mu g(X,Y)V^\mu$, for all $X, Y \in V^\perp$, and then

$$R_\mu((\nabla^\mu V^\mu)\tilde{E}_i, V^\mu, \tilde{E}_i) = \mu g((\nabla^\mu V^\mu)\tilde{E}_i, \tilde{E}_i)V^\mu = 0.$$

For the last equality we use that V^μ is a Killling vector field. Since it is also geodesic we get (2.10).

The endomorphism P_{g_λ} relating the metrics g_μ and g_λ is the identity on $V^\perp$ and $P_{g_\lambda}(V) = (\lambda/\mu)V$. On the other hand, for $X \in V^\perp$,

$$(\nabla^\mu V^\mu)(X) = \frac{\mu}{\sqrt{|\mu|}}\; JX.$$

Then $K_{(V^\mu, g_\lambda)}(V^\mu) = 0$ and

$$K_{(V^\mu, g_\lambda)}(X) = -\frac{\mu}{|\mu|}\sqrt{|\lambda|}\; JX.$$

Therefore, when either $Y \in V^\perp$ or $Y = V$,

$$(\nabla^\mu_Y K_{(V^\mu, g_\lambda)})X = \frac{\mu}{|\mu|}\sqrt{|\lambda|}\; g(X,Y)V,$$

and then

$$g_\mu((\nabla^\mu_Y K_{(V^\mu, g_\lambda)})X, Y) = 0,$$

from where we get (2.9).

□

Since $(V^\mu)^* g^S_\mu = (1 + |\mu|)g_\lambda$ where $\lambda = \mu/(1 + |\mu|)$, as a consequence of the Proposition above, we have the following

Corollary 2.6. *For all $\mu \neq 0$, the Hopf vector field V^μ is a critical point of the generalized energy E_{g_λ}, for all $\lambda \neq 0$, and it defines a minimal immersion.*

Remark 2.7. Although we have stated Proposition 2.1 and condition (2.4) only for Riemannian metrics, it is easy to see that for Lorentzian metrics, the analogous result also holds, up to the sign of the terms involving V^λ, which does not appear in this case because V^λ is geodesic.

Let us end this section by the description of the holomorphic and anti-holomorphic derivatives. Since a vector field on S^{2m+1} can be seen as a map on $\mathbb{C}^{m+1}$, apart from the covariant derivatives ∇^μ we will use other differential operators that take into account the complex structure. Although it turns out that these operators are independent of μ, and so the description is identical to the corresponding one in [2], we find it convenient to reproduce it here.

Let $W : \mathcal{U} \subset \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a vector field. We put $D^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W$ and $\bar{D}^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W$. Recall that W is *holomorphic* (resp.*anti-holomorphic*) if for all X, $D^{\mathbb{C}}_X W = 0$ (resp. $\bar{D}^{\mathbb{C}}_X W = 0$).

Let $V^\perp$ be the distribution $Span(x, Jx)^\perp$ on $\mathbb{C}^{m+1} \setminus \{0\}$ and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to V^\perp$ be the natural projections $\{x\} \times \mathbb{C}^{m+1} \to V^\perp_x$. We denote by $\|\pi \circ D^{\mathbb{C}} W\|_{V^\perp}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$ that is :

$$\|\pi \circ D^{\mathbb{C}} W\|^2_{V^\perp} = \sum_{i=1}^{2m} \|\pi \circ D^{\mathbb{C}}_{E_i} W\|^2$$

where $E_1, ..., E_{2m}$ is a local orthonormal frame of $V^\perp$. Similarly :

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{V^\perp} = \sum_{i=1}^{2m} \|\pi \circ \bar{D}^{\mathbb{C}}_{E_i} W\|^2,$$

but in that case

$$\pi \circ \bar{D}^{\mathbb{C}} W_{|V^\perp} = \bar{D}^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$$

so that :

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{V^\perp} = \|\bar{D}^{\mathbb{C}} W\|^2_{V^\perp}.$$

We compute $\|\pi \circ D^{\mathbb{C}} A\|^2_{V^\perp}$ and $\|\bar{D}^{\mathbb{C}} A\|^2_{V^\perp}$ in terms of the matrix B of ∇A in a local frame, i. e. $B_i^j =< \nabla_{E_i} A, E_j >$, obtaining :

$$\begin{aligned} \frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{V^\perp} &= \sum_{i,j=1}^{m} (B_{i*}^{j*} - B_i^j)^2 + (B_{i*}^j + B_i^{j*})^2 \qquad (2.11) \\ &= \sum_{i,j=1}^{2m} (B_i^j)^2 + 2\sum_{i,j=1}^{m} (B_{i*}^j B_i^{j*} - B_{i*}^{j*} B_i^j), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\|\bar{D}^{\mathbb{C}} A\|^2_{V^\perp} &= \sum_{i,j=1}^{m} (B_{i*}^{j*} + B_i^j)^2 + (B_{i*}^j - B_i^{j*})^2 \qquad (2.12) \\ &= \sum_{i,j=1}^{2m} (B_i^j)^2 - 2\sum_{i,j=1}^{m} (B_{i*}^j B_i^{j*} - B_{i*}^{j*} B_i^j). \end{aligned}$$

In the sequel, when not otherwise stated, we will assume that the parameters μ and λ are positive; the study of Lorentzian Berger metrics will be performed in last section.

3 The special case of S^3

The aim of this section is to show that the unit Hopf vector field on (S^3, g_μ) is the only absolute minimizer of the energy, and of the volume, if and only if $\mu \leq 1$. For $\mu > 1$, we will show that it is not even a local minimum.

Since $S^3 \subset \mathbb{R}^4 = \mathbb{H}$, we can define on S^3 a global g-orthonormal frame $\{V = J_0 N, E_1 = J_1 N, E_2 = J_2 N\}$ where $\{J_0, J_1, J_2\}$ denote the three standard complex structures defining the quaternionic structure of $\mathbb{R}^4$. Then $\{V^\mu, E_1, E_2\}$ is a g_μ-orthonormal frame.

Lemma 3.1. *Let X be a unit vector field which is an element of the 2-dimensional space generated by $\{E_1, E_2\}$ and let W be of the form $W = \cos(t)V^\mu + \sin(t)X$, then:*

$$\|\nabla^\mu W\|^2 = 2\mu + 4\sin^2(t)\frac{1-\mu}{\mu}.$$

In particular, $\|\nabla^\mu X\|^2 = 2\mu + 4\frac{1-\mu}{\mu}$ and $\|\nabla^\mu V^\mu\|^2 = 2\mu$. Moreover

$$\det L_W = (1+\mu)^2 + 4\sin^2(t)(1+\mu)\frac{1-\mu}{\mu}.$$

In particular $\det L_X = (1+\mu)^2 + 4(1+\mu)\frac{1-\mu}{\mu}$ *and* $\det L_{V^\mu} = (1+\mu)^2$.

Proof. Since,

$$\begin{aligned} \nabla_V V = \nabla_{E_1} E_1 = \nabla_{E_2} E_2 = 0 \qquad & \nabla_{E_1} E_2 = -\nabla_{E_2} E_1 = -V \\ \nabla_V E_1 = -\nabla_{E_1} V = -E_2 \qquad & \nabla_{E_2} V = -\nabla_V E_2 = -E_1, \end{aligned}$$

then, using (2.7)

$$\begin{aligned} \nabla^\mu_{V^\mu} V^\mu = \nabla^\mu_{E_1} E_1 = \nabla^\mu_{E_2} E_2 = 0 \qquad & \nabla^\mu_{E_1} E_2 = -\nabla^\mu_{E_2} E_1 = -\sqrt{\mu}\ V^\mu \\ \nabla^\mu_{V^\mu} E_1 = \frac{\mu-2}{\sqrt{\mu}} E_2 \qquad & \nabla^\mu_{E_1} V^\mu = \sqrt{\mu}\ E_2 \\ \nabla^\mu_{V^\mu} E_2 = -\frac{\mu-2}{\sqrt{\mu}} E_1 \qquad & \nabla^\mu_{E_2} V^\mu = -\sqrt{\mu}\ E_1. \end{aligned}$$

For a unit vector field $X = a_1E_1 + a_2E_2$ with $a_i \in C^\infty(S^3)$, if we take $Y = -a_2E_1 + a_1E_2$, then

$$\begin{aligned} \nabla^\mu_X X &= X(a_1)E_1 + X(a_2)E_2, \\ \nabla^\mu_{V^\mu} X &= V^\mu(a_1)E_1 + V^\mu(a_2)E_2 + \frac{\mu-2}{\sqrt{\mu}} Y, \\ \nabla^\mu_Y X &= Y(a_1)E_1 + Y(a_2)E_2 + \sqrt{\mu}\ V^\mu. \end{aligned} \tag{3.1}$$

If we assume that the functions a_1 and a_2 are constant then

$$\nabla^\mu_X X = 0, \quad \nabla^\mu_{V^\mu} X = \frac{\mu-2}{\sqrt{\mu}} Y \quad \text{and} \quad \nabla^\mu_Y X = \sqrt{\mu}\ V^\mu.$$

Therefore if $W = \cos(t)V^\mu + \sin(t)X$, since $\nabla^\mu W = \cos(t)\nabla^\mu V^\mu + \sin(t)\nabla^\mu X$, it is not difficult to see that

$$\begin{aligned} L_W(X) &= (1+\mu\cos^2(t))X + (\mu-2)\sin(t)\cos(t)V^\mu \\ L_W(V^\mu) &= (\mu-2)\sin(t)\cos(t)X + \Big(1 + \sin^2(t)\frac{(\mu-2)^2}{\mu}\Big)V^\mu, \\ L_W(Y) &= (1+\mu)Y, \end{aligned}$$

from where

$$\operatorname{tr} L_W = 3 + \|\nabla^\mu W\|^2 = 3 + 2\mu + 4\sin^2(t)\frac{1-\mu}{\mu},$$

and

$$\det L_W = (1+\mu)^2 + 4\sin^2(t)(1+\mu)\frac{1-\mu}{\mu}.$$

$\square$

For $\mu = 1$, all the elements of the unit sphere of the 3-dimensional vector space generated by $\{V^\mu, E_1, E_2\}$ are also called Hopf vector fields, they can be characterized as the unit Killing vector fields. It is known ([3] and [10]) that Hopf vector fields have all the same volume and the same energy and that they are the only minimizers of both functionals. For $\mu \neq 1$, the situation is quite different as it will be described in the next result.

Theorem 3.2. *Let (S^3, g_μ) be the three-dimensional Berger sphere.*

a) If $\mu < 1$, V^μ is, up to sign, the only minimizer of the energy and of the volume of unit vector fields; the minima of the functionals are $E(V^\mu) = (\frac{3}{2} + \mu)vol(S^3, g_\mu)$ and $F(V^\mu) = (1+\mu)vol(S^3, g_\mu)$,respectively.

b) If $\mu > 1$, for all unit vector fields A in the 2-dimensional space generated by $\{E_1, E_2\}$ we have

$$E(A) = (\frac{3}{2} + \mu + 2\frac{1-\mu}{\mu})vol(S^3, g_\mu) < E(V^\mu)$$

and

$$F(A) = \sqrt{(1+\mu)^2 + 4(1+\mu)\frac{1-\mu}{\mu}}vol(S^3, g_\mu) < F(V^\mu).$$

In fact V^μ is not even a local minimum. Moreover, for all unit vector field X

$$E(X) > \Big(\frac{7}{2} - \mu\Big)vol(S^3, g_\mu) \quad \text{and} \quad F(X) > (3-\mu)vol(S^3, g_\mu).$$

Proof. For any three dimensional compact manifold, the energy and the volume of unit vector fields are related with the integral of the Ricci tensor, as it has been shown in [3]. In this particular case the inequalities are written as

$$\begin{aligned} E(X) \geq &\ \tfrac{3}{2}vol(S^3, g_\mu) + \tfrac{1}{2}\textstyle\int_{S^3} Ric_\mu(X,X)dv_\mu, \\ F(X) \geq &\ \ vol(S^3, g_\mu) + \tfrac{1}{2}\textstyle\int_{S^3} Ric_\mu(X,X)dv_\mu. \end{aligned} \tag{3.2}$$

In both cases, the equality holds if and only if $\nabla^\mu_X X = 0$, $h_{11} = h_{22}$ and $h_{12} = -h_{21}$, where $h_{ij} = g_\mu(\nabla^\mu_{\overline{E}_i} X, \overline{E}_j)$ and $\{X, \overline{E}_1, \overline{E}_2\}$ is a g_μ-orthonormal frame.

Using 2.8 we have that if $\mu < 1$, then

$$Ric_\mu(X,X) \geq Ric_\mu(V^\mu, V^\mu) = 2\mu$$

for all unit X, with equality if and only if $X = \pm V^\mu$ and therefore $E(X) \geq E(V^\mu)$ and $F(X) \geq F(V^\mu)$. Hopf vector field is then, up to sign, the only minimizer and we have shown a).

The first sentence of b) is a direct consequence of Lemma 3.1. To see that, for $\mu > 1$, the Hopf vector field is not a local minimum, we only need to consider the curve of unit vector fields $W(t) = \cos(t)V^\mu + sin(t)A$ where $A = a_1E_1 + a_2E_2$,with $a_i \in \mathbb{R}$, is a unit vector field. In Lemma 3.1 we have computed the value of the functions $E(t) = E(W(t))$ and $F(t) = F(W(t))$ from where we observe that for $t = 0$ both

functions reach their maximum.
On the other hand, 2.8 gives us that if $\mu > 1$ then

$$Ric_\mu(X,X) \geq Ric_\mu(A,A) = 2(2-\mu),$$

for all unit X and all unit $A \in V^\perp$, with equality if and only if $X \in V^\perp$. Consequently, if we use (3.2)

$$E(X) \geq \left(\frac{7}{2} - \mu\right) vol(S^3, g_\mu) \quad \text{and} \quad F(X) \geq (3-\mu) vol(S^3, g_\mu), \tag{3.3}$$

with equality if and only if $X \in V^\perp$, $\nabla^\mu_X X = 0$, $h_{11} = h_{22}$ and $h_{12} = -h_{21}$.

Let us assume that a unit vector field X satisfies the four conditions above. Firstly $X = a_1 E_1 + a_2 E_2$ with $a_i \in C^\infty(S^3)$ and if we take $Y = -a_2 E_1 + a_1 E_2$ then by (3.1) the other three conditions become

$$\begin{aligned} X(a_1) = X(a_2) &= 0, &(3.4)\\ -a_2 Y(a_1) + a_1 Y(a_2) &= 0, &(3.5)\\ a_2 V^\mu(a_1) - a_1 V^\mu(a_2) &= \frac{2\mu - 2}{\sqrt{\mu}}. &(3.6) \end{aligned}$$

If a_2 vanishes identically, then a_1 should be constant and (3.6) give us a contradiction. So, the open set where $a_2 \neq 0$ is not empty and (3.4) and (3.5) then imply that on it we have

$$X\left(\frac{a_1}{a_2}\right) = Y\left(\frac{a_1}{a_2}\right) = 0.$$

But, by the choice of X and Y, this is equivalent to

$$E_1\left(\frac{a_1}{a_2}\right) = E_2\left(\frac{a_1}{a_2}\right) = 0,$$

that, using the relation between V^μ and $[E_1, E_2]$, implies $V^\mu(a_1/a_2) = 0$, which is again in contradiction with (3.6). Therefore, the lower bounds in (3.3) are never reached. □

Remark 3.3. 1.– Since on any 3-dimensional manifold, the functional E (resp. F) is bounded below by $\frac{3}{2}$ times the volume (resp. by the volume) of the manifold, the lower bounds appearing in the Theorem above are relevant only for $\mu < 2$.

2.– Part a) of the Theorem is a particular case of a result of [11] concerning unit Killing vector fields on a 3-dimensional compact manifold.

A relation between the energy and the integral of the Ricci tensor similar to the one quoted in (3.2) is valid for any compact manifold (see [3]) and then we have ,

Proposition 3.4. *For all unit vector field X on (S^{2m+1}, g_μ),*

$$E(X) \geq \frac{2m+1}{2} vol(S^{2m+1}, g_\mu) + \frac{1}{2(2m-1)} \int_{S^{2m+1}} Ric_\mu(X,X) dv_\mu,$$

with equality if and only if $\nabla^\mu_X X = 0$, and the distribution $X^\perp$ determines a foliation with umbilical leaves.

Since $Ric_\mu(V^\mu, V^\mu) = 2m\mu$ and $Ric_\mu(A, A) = 2(1 - \mu + m)\ \|A\|^2$ for all $A \in V^\perp$, if $\mu < 1$

$$E(X) \geq \Big(\frac{2m+1}{2} + \frac{m\mu}{2m-1}\Big) vol(S^{2m+1}, g_\mu).$$

Moreover, if $m \neq 1$, equality never holds because this will imply $X = V^\mu$ and

$$E(V^\mu) = \frac{2m+1+2m\mu}{2} vol(S^{2m+1}, g_\mu).$$

If $\mu > 1$ then

$$E(X) \geq \Big(\frac{2m+1}{2} + \frac{1-\mu+m}{2m-1}\Big) vol(S^{2m+1}, g_\mu),$$

with equality if and only if $X \in V^\perp$, $\nabla^\mu_X X = 0$, and the distribution $X^\perp$ determines a foliation with umbilical leaves. As for the 3-dimensional sphere, this lower bound is relevant for $1 < \mu < m + 1$.

In the case of the round sphere, $\mu = 1$, the bound $(\frac{2m+1}{2} + \frac{m}{2m-1}) vol(S^{2m+1})$ is the value of the energy of radial vector fields defined on the complementary of two antipodal points. Moreover, it has been shown in [1] that it is the infimum of the energy. In contrast with this situation, for $\mu > 1$ we do not know if any unit vector field exists, having this energy, even if we allow singularities.

In what concerns the volume, the difference between the case $\mu = 1$ and the general one is deeper. In fact, it has been shown in [4] that for the round spheres the volume of radial vector fields is also a lower bound of F but the proof is based on an inequality relating the volume of a unit vector field with the curvature of the manifold that is only valid for constant curvature spaces.

4 Second variation of the generalized energy and of the volume at Hopf vector fields

For fixed $\mu \neq 0$, V^μ is a critical point of E and F and also it is a critical point of E_{g_λ}, for all $\lambda \neq 0$. Then we can compute the Hessians of these functionals at V^μ. For simplicity we give the proof only for positive values of the parameter.

Proposition 4.1. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), for each vector field A orthogonal to V^μ we have:*

$$a) (Hess E_{g_\lambda})_{V^\mu}(A) = \int_{S^{2m+1}} \Big(- 2m\sqrt{\lambda\mu}\ \|A\|^2 + \sqrt{\lambda/\mu}\ \|\nabla^\mu A\|^2 + (\sqrt{\mu/\lambda} - \sqrt{\lambda/\mu}\ )\ \|\nabla^\mu_{V^\mu} A\|^2\Big) dv_\mu.$$

$$b)\ (Hess E)_{V^\mu}(A) = \int_{S^{2m+1}} \Big(- 2m\mu\|A\|^2 + \|\nabla^\mu A\|^2\Big) dv_\mu.$$

$$c)\ (Hess F)_{V^\mu}(A) = (1+\mu)^{m-2} \int_{S^{2m+1}} \Big(\mu(-2m\mu + 2(1-\mu))\|A\|^2 + \|\nabla^\mu A\|^2 + \mu\|\nabla^\mu_{V^\mu} A + \sqrt{\mu} J A\|^2\Big) dv_\mu.$$

Proof. We only need to compute the elements appearing in Theorem 2.3 for this particular case. Since $L_{g_\lambda} = \sqrt{\lambda/\mu}(g_\lambda^{-1} g_\mu)$ and $\nabla^\mu V^\mu = \sqrt{\mu} J$, by direct computation we obtain

$$\omega_{(V^\mu, g_\lambda)}(V^\mu) = -2m\sqrt{\lambda\mu},$$

and

$$\begin{aligned}\mathrm{tr}(L_{g_\lambda} \circ (\nabla^\mu A)^t \circ (\nabla^\mu A)) &= \sqrt{\lambda/\mu}\sum_{i=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, \nabla^\mu_{E_i} A) + \sqrt{\lambda/\mu} g_\mu(\nabla^\mu_{V^\lambda} A, \nabla^\mu_{V^\lambda} A) \\ &= \sqrt{\lambda/\mu}\sum_{i=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, \nabla^\mu_{E_i} A) + \sqrt{\mu/\lambda} g_\mu(\nabla^\mu_{V^\mu} A, \nabla^\mu_{V^\mu} A),\end{aligned}$$

from where a) and b) hold.

Since $\nabla^\mu V^\mu = \sqrt{\mu} J$, on $(V^\mu)^\perp$ and V^μ is geodesic, we have that $L_{V^\mu}(V^\mu) = V^\mu$ and $L_{V^\mu} = (1+\mu)\mathrm{Id}$ on $(V^\mu)^\perp$. Then, $f(V^\mu) = (1+\mu)^m$ and $K_{V^\mu} = -(1+\mu)^{m-1}\nabla^\mu V^\mu$. By direct computation we obtain

$$\omega_{V^\mu}(V^\mu) = -2m\mu(1+\mu)^{m-1},$$

and

$$(K_{V^\mu} \circ \nabla^\mu A)(X) = -\sqrt{\mu}(1+\mu)^{m-1}(\nabla^\mu_X JA + g(X, A)V),$$

from where

$$\frac{2}{f(V^\lambda)}\sigma_2(K_{V^\mu} \circ \nabla^\mu A) = 2\mu(1+\mu)^{m-2}\Big(\sigma_2(\nabla^\mu JA) - \sqrt{\mu} g(\nabla^\mu_{V^\mu} JA, A)\Big).$$

Using the fact that, on any Riemannian manifold, $2\sigma_2(\nabla X)$ and $Ric(X, X)$ differ in a divergence (see for example [13], p. 170) and the value of the Ricci tensor of g_μ (2.8) we have

$$\begin{aligned}&\int_{S^{2m+1}} \frac{1}{f(V^\lambda)} \quad \sigma_2(K_{V^\mu} \circ \nabla^\mu A) dv_\mu \\ &\qquad = \mu(1+\mu)^{m-2}\int_{S^{2m+1}} \Big((m-\mu+1)\|A\|^2 - \sqrt{\mu} g(\nabla^\mu_{V^\mu} JA, A)\Big) dv_\mu.\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{tr}(L^{-1}_{V^\mu} \circ (\nabla^\mu A)^t \circ L^{-1}_{V^\mu} \circ \nabla^\mu A) &= (1+\mu)^{-2}\sum_{i,j=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, E_j)^2 \\ &\quad +(1+\mu)^{-1}(\mu\|JA\|^2 + \|\nabla^\mu_{V^\mu} A\|^2) \\ &= (1+\mu)^{-2}\Big(\|\nabla^\mu A\|^2 + \mu^2\|JA\|^2 + \mu\|\nabla^\mu_{V^\mu} A\|^2\Big).\end{aligned}$$

For the last equality we have used that

$$\|\nabla^\mu A\|^2 = \sum_{i,j=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, E_j)^2 + \mu\|JA\|^2 + \|\nabla^\mu_{V^\mu} A\|^2.$$

Since V^μ is a Killing vector field, we can use (2.5) to compute the Hessian and then we get c). $\square$

In order to study the stability of the Hopf vector field it will be useful to find new expressions of the Hessians. We will proceed following closely the arguments used in [2], for the volume functional in the case of the round spheres. There, the key was to relate the integral of $\|\nabla A\|^2$ with the integral of $\|\pi \circ D^C A\|^2_{V^\perp}$ and that of $\|\overline{D}^C A\|^2_{V^\perp}$.

Firstly, since

$$\sum_{i,j=1}^{m} (B^j_{i*} B^{j*}_i - B^{j*}_{i*} B^j_i) = -\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A),$$

(2.11) and (2.12) can be written as

$$\begin{aligned}
\|\nabla^\mu A\|^2 &= \frac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + \mu\|A\|^2 + 2\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A),\\
\|\nabla^\mu A\|^2 &= \frac{1}{2}\|\overline{D}^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + \mu\|A\|^2 - 2\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A).
\end{aligned}$$

The second step is the following lemma, the proof of which is very similar to the corresponding one in [2] and will be omitted

Lemma 4.2. *For all $\mu \neq 0$ we have*

$$a)\ 2m\sqrt{|\mu|}V^\mu = -\sum_{i=1}^{m}[E_i, JE_i] + \sum_{i=1}^{m} \mathrm{div}^\mu(JE_i)E_i - \sum_{i=1}^{m} \mathrm{div}^\mu(E_i)JE_i,$$

and

$$\begin{aligned}
b)\ m\sqrt{|\mu|}\int_{S^{2m+1}} g_\mu(\nabla^\mu_{V^\mu} A, JA)dv_\mu &= (m\mu - m - 1)\int_{S^{2m+1}} \|A\|^2 dv_\mu\\
&\quad + \int_{S^{2m+1}} \sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A)dv_\mu.
\end{aligned}$$

Now, as a consequence, we have

Lemma 4.3. *For all $\mu > 0$,*

$$\begin{aligned}
a)\ \int_{S^{2m+1}} \|\nabla^\mu A\|^2 dv_\mu &= \int_{S^{2m+1}} \Big(\frac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + (2 + \mu + 2m(1-\mu))\|A\|^2\\
&\quad + 2m\sqrt{\mu}\ g_\mu(\nabla^\mu_{V^\mu} A, JA)\Big)dv_\mu\\
b)\ \int_{S^{2m+1}} \|\nabla^\mu A\|^2 dv_\mu &= \int_{S^{2m+1}} \Big(\frac{1}{2}\|\overline{D}^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + (\mu - 2 + 2m(\mu-1))\|A\|^2\\
&\quad - 2m\sqrt{\mu}\ g_\mu(\nabla^\mu_{V^\mu} A, JA)\Big)dv_\mu.
\end{aligned}$$

If we use these values on the corresponding expressions of Proposition 4.1, we obtain

Proposition 4.4. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), for each vector field A orthogonal to V^μ we have:*

$$
\begin{aligned}
a1)(HessE_{g_\lambda})_{V^\mu}(A) = \quad & \int_{S^{2m+1}} \Big((\sqrt{\lambda\mu}(1-4m) + \sqrt{\lambda/\mu}(2m+2-\lambda m^2))\ \|A\|^2 \\
& + \sqrt{\mu/\lambda}\ \|\nabla^\mu_{V^\mu} A + \frac{\lambda m}{\sqrt{\mu}} JA\|^2 + \frac{1}{2}\sqrt{\lambda/\mu}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big) dv_\mu. \\
a2)(HessE_{g_\lambda})_{V^\mu}(A) = \quad & \int_{S^{2m+1}} \Big((\sqrt{\lambda\mu} - \sqrt{\lambda/\mu}(2m+2+\lambda m^2))\ \|A\|^2 \\
& + \sqrt{\mu/\lambda}\ \|\nabla^\mu_{V^\mu} A - \frac{\lambda m}{\sqrt{\mu}} JA\|^2 + \frac{1}{2}\sqrt{\lambda/\mu}\ \|\overline{D}^C A\|^2_{V^\perp} \Big) dv_\mu. \\
b1)\ (HessE)_{V^\mu}(A) = \quad & \int_{S^{2m+1}} \Big((2m+2-\mu(m^2+4m-1))\ \|A\|^2 \\
& + \|\nabla^\mu_{V^\mu} A + m\sqrt{\mu} JA\|^2 + \frac{1}{2}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big) dv_\mu. \\
b2)\ (HessE)_{V^\mu}(A) = \quad & \int_{S^{2m+1}} \Big((-2m-2-\mu(m^2-1))\ \|A\|^2 \\
& + \ \|\nabla^\mu_{V^\mu} A - m\sqrt{\mu} JA\|^2 + \frac{1}{2}\ \|\overline{D}^C A\|^2_{V^\perp} \Big) dv_\mu. \\
c1)\ (HessF)_{V^\mu}(A) = \quad & (1+\mu)^{m-2} \int_{S^{2m+1}} \Big(f_1(m,\mu)\ \|A\|^2 \\
& + (1+\mu)\ \|\nabla^\mu_{V^\mu} A + \frac{\sqrt{\mu}(m+\mu)}{1+\mu} JA\|^2 + \frac{1}{2}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big) dv_\mu. \\
c2)\ (HessF)_{V^\mu}(A) = \quad & (1+\mu)^{m-2} \int_{S^{2m+1}} \Big(f_2(m,\mu)\ \|A\|^2 \\
& + (1+\mu)\ \|\nabla^\mu_{V^\mu} A + \frac{\sqrt{\mu}(\mu-m)}{1+\mu} JA\|^2 + \frac{1}{2}\ \|\overline{D}^C A\|^2_{V^\perp} \Big) dv_\mu.
\end{aligned}
$$

where

$$f_1(m,\mu) = \mu(3-\mu-2m-2m\mu) + (2m+2) - \frac{\mu(m+\mu)^2}{1+\mu}$$

and

$$f_2(m,\mu) = \mu(3-\mu+2m-2m\mu) - (2m+2) - \frac{\mu(\mu-m)^2}{1+\mu}.$$

5 Stability of Hopf vector fields on S^{2m+1} with $m > 1$

The instability results for the round spheres have been obtained by showing that the Hessian is negative when acting on the vector fields $A_a = a - \langle a, V\rangle V - \langle a, N\rangle N = a - \bar{f}_a V - f_a N$ for all $a \in \mathbb{R}^{2m+2}$, $a \neq 0$. A geometrical description of these vector fields can be seen in [8], as well as the following

Lemma 5.1.

$$\int_{S^{2m+1}} \bar{f}_a^2 dv = \int_{S^{2m+1}} f_a^2 dv = \frac{|a|^2}{2m+2} vol(S^{2m+1})$$

If we use Proposition 4.1 to compute the value of the Hessian acting on these particular vector fields we obtain

Lemma 5.2. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ). For each $a \in \mathbb{R}^{2m+2}$, $a \neq 0$ we have:*

$$\begin{aligned}
a)(HessE_{g_\lambda})_{V^\mu}(A_a) &= \frac{\sqrt{\lambda}m}{m+1}|a|^2\Big((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\lambda}\Big)vol(S^{2m+1}). \\
b)\ (HessE)_{V^\mu}(A_a) &= \frac{\sqrt{\mu}m}{m+1}|a|^2\Big((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\mu}\Big)vol(S^{2m+1}). \\
c)\ (HessF)_{V^\mu}(A_a) &= (1+\mu)^{m-2}\frac{\sqrt{\mu}m}{m+1}|a|^2 f(m,\mu)vol(S^{2m+1}).
\end{aligned}$$

where $f(m,\mu) = \Big((1-2m)\mu(1+\mu) + 2m\mu + 2 + (1+\mu)\frac{(\mu-1)^2}{\mu}\Big)$.

Proof. We need to compute all the elements appearing in the formulae given in Proposition 4.1. Since A_a is orthogonal to V we have, as in the case $\mu = 1$ computed in [8],

$$\|A\|^2 = |a|^2 - \bar{f}_a^2 - f_a^2 \quad \text{and} \quad \sum_{i,j=1}^{2m}(B_i^j)^2 = 2m(\bar{f}_a^2 + f_a^2).$$

But now

$$\nabla^\mu_{V^\mu} A = -\frac{\mu-1}{\sqrt{\mu}}\sum_{j=1}^{m}(E_j(\bar{f}_a)E_j + E_{j*}(\bar{f}_a)E_{j*}),$$

and then

$$\begin{aligned}
\|\nabla^\mu_{V^\mu} A\|^2 &= \frac{(\mu-1)^2}{\mu}\sum_{j=1}^{m}((E_j(\bar{f}_a))^2 + (E_{j*}(\bar{f}_a))^2) = \frac{(\mu-1)^2}{\mu}(|a|^2 - \bar{f}_a^2 - f_a^2), \\
g(\nabla^\mu_{V^\mu} A, JA) &= \frac{\mu-1}{\sqrt{\mu}}(|a|^2 - \bar{f}_a^2 - f_a^2).
\end{aligned}$$

The integrands of the Hessians are obtained by straightforward computation and then we use Lemma 5.1 to conclude. ☐

It is an immediate consequence of a) that, if $m > 1$, Hopf vector fields of the round sphere are unstable when considered as critical points of all the energy functionals E_{g_λ}, thus generalizing the corresponding result for the usual energy. But Lemma 5.2 give us the instability of V^μ in many other cases, that we summarize in

Proposition 5.3. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ) with $m > 1$,*

a) If $(2m-1)\mu > 2$, and $((2m-1)\mu - 2)\lambda > (\mu-1)^2$, then V^μ is an unstable critical point of the energy E_{g_λ}.

b) If $(2m-2)\mu^2 > 1$, then V^μ is energy unstable.

c) If $(2m-2)\mu^3 - \mu > 1$, then V^μ is volume unstable.

In all cases the index is at least $2m+2$.

Proof. To show b) and c) we only need to use Lemma 5.2 to write, respectively, the conditions

$$(HessE)_{V^\mu}(A_a) < 0 \qquad \text{and} \qquad (HessF)_{V^\mu}(A_a) < 0.$$

Analogously, from a) of Lemma 5.2, we obtain that

$$(HessE_{g_\lambda})_{V^\mu}(A_a) < 0$$

if $(1-2m)\mu + 2 + \frac{(\mu-1)^2}{\lambda} < 0$, which is equivalent to the condition stated in a).

□

We are going to show that in what concerns volume and energy the sufficient condition for instability is also necessary. For the other functionals the situation is more complicated and is still open for some values of (μ, λ).

In order to obtain the stability results, it is convenient to see a vector field A on S^{2m+1}, orthogonal to the Hopf vector field, as a map $A : S^{2m+1} \longrightarrow V^\perp \subset \mathbb{C}^{m+1}$ where $V^\perp$ represents the distribution $V_x^\perp = Span\{x, Jx\}^\perp$. For such a map A, we write

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}d\theta \in V_p^\perp$$

so that

$$A(p) = \sum_{l\in Z} A_l(p)$$

is the Fourier series of A. Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$ then

$$\nabla_V A = \bar{\nabla}_V A = \sum_{l\in Z} ilA_l = \sum_{l\in Z} lJA_l$$

and

$$\|\nabla^\mu_{V^\mu} A_l + \alpha JA_l\|^2 = \frac{1}{\mu}(l - 1 + \mu + \alpha\sqrt{\mu})^2\ \|A_l\|^2.$$

If $\mathcal{C}(p)$ denotes the fibre of the Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbb{C}P^m$ passing through p, and for $l \neq q$,

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0.$$

By the construction of the Berger metrics, this fact is independent of μ and so, the essential following Lemma, shown in [2] for the volume functional in the case $\mu = 1$, remains valid

Lemma 5.4.

$$\begin{aligned}
a)(HessE_{g_\lambda})_{V^\mu}(A) &= \sum_{l\in Z}(HessE_{g_\lambda})_{V^\mu}(A_l).\\
b)\ (HessE)_{V^\mu}(A) &= \sum_{l\in Z}(HessE)_{V^\mu}(A_l).\\
c)\ (HessF)_{V^\mu}(A) &= \sum_{l\in Z}(HessF)_{V^\mu}(A_l).
\end{aligned}$$

We can now show the following

Theorem 5.5. *On (S^{2m+1}, g_μ), with $m > 1$, the Hopf unit vector field V^μ is stable as a critical point of the energy if and only if $(2m-2)\mu^2 \leq 1$ and it is stable as a critical point of the volume if and only if $(2m-2)\mu^3 - \mu \leq 1$.*

Proof. We only need to show that under the hypothesis on μ, the corresponding Hessians are non negative, when acting on any vector field A orthogonal to V^μ.

By Proposition 4.4 part b1),

$$(HessE)_{V^\mu}(A_l) \geq e_1(m, \mu, l) \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$e_1(m,\mu,l) = \mu(1-m^2-4m)+2m+2+\frac{1}{\mu}(l-1+\mu(m+1))^2 = \mu(2-2m)+2l(m+1)+\frac{1}{\mu}(l-1)^2.$$

Therefore, if $(2m-2)\mu^2 \leq 1$,

$$e_1(m,\mu,l) \geq 2l(m+1) + \sqrt{2m-2}((l-1)^2 - 1).$$

Consequently, $(HessE)_{V^\mu}(A_l) \geq 0$ for all $l \geq 0$. If we use now Proposition 4.4 part b2),

$$(HessE)_{V^\mu}(A_l) \geq e_2(m, \mu, l) \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$e_2(m,\mu,l) = \mu(1-m^2)-2m-2+\frac{1}{\mu}(l-1+\mu(1-m))^2 = \mu(2-2m)+2l(1-m)-4+\frac{1}{\mu}(l-1)^2.$$

If we assume again $(2m-2)\mu^2 \leq 1$, we obtain

$$e_2(m,\mu,l) \geq 2l(1-m) - 4 + \sqrt{2m-2}((l-1)^2 - 1).$$

Since $\sqrt{2m-2}((l-1)^2-1) \geq 4$, for all $l < 0$, we have $(HessE)_{V^\mu}(A_l) \geq 0$.
Lemma 5.4 part b) gives us that V^μ is energy stable. The corresponding result for the volume can be established in a similar way.
By Proposition 4.4 part c1),

$$(HessF)_{V^\mu}(A_l) \geq (1+\mu)^{m-2} f_1(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$f_1(m,\mu,l) = f_1(m,\mu) + \frac{(1+\mu)}{\mu}\Big(l - 1 + \mu + \frac{\mu(m+\mu)}{1+\mu}\Big)^2.$$

Developing the righthand side, we have

$$f_1(m,\mu,l) = \mu^2(2-2m) + \mu 4l + 2l(m+1) + (l-1)^2 + \frac{1}{\mu}(l-1)^2.$$

In particular, $\mu f_1(m,\mu,0) = \mu^3(2-2m) + \mu + 1$ and the condition $(2m-2)\mu^3 - \mu \leq 1$ then implies that $(HessF)_{V^\mu}(A_0) \geq 0$.

Let us point out that if μ verifies the condition above, then it should also verify $\mu \leq 1$ and then for $l \geq 1$ we have $f_1(m,\mu,l) > (2-2m) + 2(m+1)$ and then $(HessF)_{V^\mu}(A_l) > 0$.
Let us use now Proposition part c2),

$$(HessF)_{V^\mu}(A_l) \geq (1+\mu)^{m-2} f_2(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$f_2(m,\mu,l) = f_2(m,\mu) + \frac{(1+\mu)}{\mu}\Big(l-1+\mu+\frac{\mu(\mu-m)}{1+\mu}\Big)^2.$$

Developing the righthand side, we have

$$f_2(m,\mu,l) = \mu^2(2-2m) + \mu 4l + l^2 - 3 - 2ml + \frac{1}{\mu}(l-1)^2.$$

So, under the hypothesis, $f_2(m,\mu,l) \geq -1 + \mu 4l + l^2 - 3 - 2ml + \frac{1}{\mu}(l^2 - 2l)$ and, since $\mu \leq 1$, for all $l < 0$ we have $f_2(m,\mu,l) \geq 2l + 2l^2 - 4 - 2ml \geq 0$ and then $(HessF)_{V^\mu}(A_l) \geq 0$. □

For the generalized energy we can use the same arguments to obtain the stability on Hopf vector fields under some conditions.

Proposition 5.6. *On (S^{2m+1}, g_μ), with $m > 1$, the Hopf unit vector field V^μ is stable as a critical point of the energy E_{g_λ} in the following cases:*

- *a) If $(2m-1)\mu \leq 2$, for all $\lambda > 0$.*
- *b) If $\frac{2}{2m-1} < \mu \leq \frac{3}{2}$, for $\lambda \leq \frac{(\mu-1)^2}{(2m-1)\mu-2}$.*
- *c) If $\mu \geq 2m+2$, for $\lambda \leq \frac{\mu-2m-2}{m^2}$.*
- *d) If $m > 2$ and $\frac{3}{2} < \mu$, for λ such that $\frac{2\mu-3}{2m-4} \leq \lambda \leq \frac{(\mu-1)^2}{(2m-1)\mu-2}$.*

Proof. Let us show that under the hypothesis, the corresponding Hessian is non negative, when acting on any vector field A orthogonal to V^μ.

That this is the case for (μ, λ) as in c) is a direct consequence of Proposition 4.4 part a2). If we use instead part a1),

$$(HessE_{g_\lambda})_{V^\mu}(A_l) \geq e_1(m,\mu,\lambda,l)\sqrt{\lambda/\mu} \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$e_1(m,\mu,\lambda,l) = \mu(1-4m) + 2m + 2 - \lambda m^2 + \frac{1}{\lambda}(l-1+\mu+\lambda m)^2.$$

In particular, if we assume a), b) or d)

$$e_1(m,\mu,\lambda,0) = \frac{(\mu-1)^2}{\lambda} + \mu(1-2m) + 2 \geq 0.$$

Moreover

$$e_1(m,\mu,\lambda,l) = e_1(m,\mu,\lambda,0) + \frac{1}{\lambda}(l^2 + 2l(-1+\mu+\lambda m)) \geq 0,$$

provided $l > 1$ or $l = 1$ and $\mu \geq 1$. But

$$e_1(m, \mu, \lambda, 1) = \frac{\mu^2}{\lambda} + \mu + 2 + 2m(1 - \mu) \geq 0,$$

when $\mu \leq 1$. Consequently, $(HessE_{g_\lambda})_{V^\mu}(A_l) \geq 0$ for all $l \geq 0$. If we use now Proposition 4.4 part a2),

$$(HessE_{g_\lambda})_{V^\mu}(A_l) \geq e_2(m, \mu, \lambda, l)\sqrt{\lambda/\mu} \int_{S^{2m+1}} \|A_l\|^2 dv_\mu,$$

with

$$e_2(m, \mu, \lambda, l) = \mu - (2m + 2 + \lambda m^2) + \frac{1}{\lambda}(l - 1 + \mu - \lambda m)^2.$$

Under the hypothesis a) b) or d),

$$e_2(m, \mu, \lambda, l) \geq -4 - 2lm + \frac{1}{\lambda}(l^2 - 2l + 2l\mu).$$

Therefore, if $\mu \leq 3/2$, then $e_2(m, \mu, \lambda, l) \geq 0$ for all $l < 0$.

We get the same result if we assume that $m > 2$ and $\lambda \geq \frac{2\mu - 3}{2m - 4}$, since

$$\frac{2\mu - 3}{2m - 4} \geq \frac{-2l\mu - l^2 + 2l}{-2lm - 4},$$

for all $l \leq -1$. Lemma 5.4 part a) gives us that V^μ is stable. $\square$

The proposition above, jointly with Proposition 5.3, solves completely the problem of the stability of Hopf vector fields as critical points of the generalized energy for $\mu \leq 3/2$. For other values of μ we have only a partial answer. It is also worthwhile to point out that, depending on μ, the set of values of λ for which the condition d) is fulfilled can be empty.

6 The Lorentzian case

In this section we will consider the sphere endowed with a Berger metric g_μ with $\mu < 0$. Then $\|V^\mu\|^2 = -1$ and it is a critical point of the energy restricted to unit timelike vector fields.

Using the definition of the Sasaki metric in terms of horizontal and vertical lifts, it is easy to see that g^S_μ is a metric of index 2. The restriction of it to the bundle of vectors of square -1, $T^{-1}S^{2m+1}$, has index 1. So $(T^{-1}S^{2m+1}, g^S_\mu)$ is a Lorentzian manifold. These facts are true for any Lorentz manifold (M, g).

In contrast with the energy, that is defined for all vector fields, the volume of a unit timelike vector field V will be defined only if V is an element of the open subset consisting in the sections of $T^{-1}M$ such that $V^* g^S$ is non degenerated.

Now, since g^S is Lorentzian, this subset has exactly two connected components corresponding to unit timelike vector fields for which $V^* g^S$ is Riemannian and those for which $V^* g^S$ is Lorentzian. Variational calculus has to be done separately in each component.

In particular, Hopf vector fields on Berger Lorentzian spheres induce Lorentzian metrics $(V^\mu)^* g^S_\mu$ on the sphere and V^μ is critical for the volume restricted to the open set of unit timelike vector fields having this property, that we will denote by $\Gamma^-(T^{-1}S^{2m+1})$.

Using that, on a Lorentzian manifold, if V is a unit timelike vector field and $\{V, E_i\}_{i=1}^{2m}$ is an adapted orthonormal local frame then the vector fields E_i are spacelike for all $1 \leq i \leq 2m$ and that all vector field X can be written as $X = -g(X,V)V + \sum_i g(X,E_i)E_i$, we have

Proposition 6.1 ([12]). *Let V be a unit timelike vetor field on the compact Lorentzian manifold (M,g).*
a) If V is a critical point of the energy, the Hessian of E at V acting on $A \in V^\perp$ is given by

$$(HessE)_V(A) = -\int_M \|A\|^2 \omega_{(V,g)}(V)\ dv_g + \int_M \|\nabla A\|^2 dv_g.$$

b) For a unit timelike vector field $V \in \Gamma^-(T^{-1}M)$ defining a minimal immersion, the Hessian of F at V acting on $A \in V^\perp$ is given by

$$\begin{aligned}(HessF)_V(A) \;&=\; - \int_M \|A\|^2\omega_V(V)\ dv_g + \int_M \frac{2}{\sqrt{\det L_V}}\ \sigma_2(K_V \circ \nabla A) dv_g \\ &\quad - \int_M \operatorname{tr}\Big(L_V^{-1} \circ (\nabla A)^t \circ \nabla V \circ K_V \circ \nabla A\Big) dv_g \\ &\quad + \int_M \sqrt{\det L_V}\ \operatorname{tr}\Big(L_V^{-1} \circ (\nabla A)^t \circ \nabla A\Big) dv_g.\end{aligned}$$

In a similar way to that described in Proposition 4.1, we can show, by straightforward computation,

Proposition 6.2. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), where $\mu < 0$, for each vector field A orthogonal to V^μ we have:*

$$\begin{aligned} &a)\ (HessE)_{V^\mu}(A) = \int_{S^{2m+1}} \Big(-2m\mu\|A\|^2 + \|\nabla^\mu A\|^2\Big) dv_\mu. \\ &b)\ (HessF)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} \Big(\mu(2m\mu + 2\mu - 4m - 2)\|A\|^2 + \|\nabla^\mu A\|^2 \\ &\qquad + \mu\|\nabla^\mu_{V^\mu} A - \sqrt{-\mu} JA\|^2\Big) dv_\mu. \end{aligned}$$

Using these expressions to compute the Hessian in the direction of the vector fields A_a, as in Lemma 5.2, we obtain

Lemma 6.3. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. For each $a \in \mathbb{R}^{2m+2}$, $a \neq 0$ we have:*

$$\begin{aligned} a)\ (HessE)_{V^\mu}(A_a) &= \frac{\sqrt{-\mu}m}{m+1}|a|^2\Big((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\mu}\Big) vol(S^{2m+1}). \\ b)\ (HessF)_{V^\mu}(A_a) &= (1-\mu)^{m-2}\frac{\sqrt{-\mu}m}{m+1}|a|^2 f(m,\mu) vol(S^{2m+1}). \end{aligned}$$

where $f(m,\mu) = \Big((2m-1)\mu^2 + (1-4m)\mu + 2 + (1-\mu)\frac{(\mu-1)^2}{\mu}\Big)$.

From here, an immediate consequence is the following

Proposition 6.4. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. If $(2m-2)\mu^2 < 1$, it is energy unstable and if $(2-2m)\mu^3 + (4m-4)\mu^2 + \mu < 1$, then it is volume unstable . In particular, on (S^3, g_μ) the Hopf vector field is unstable, for all $\mu < 0$.*

The alternative expressions of the Hessian (see Proposition 4.4), used to show stability results in the Riemannian case, can be extended without difficulties to include negative values of μ.

Proposition 6.5. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. For each vector field A orthogonal to V^μ we have:*

a1) $$(HessE)_{V^\mu}(A) = \int_{S^{2m+1}} \Big((2m+2-\mu(m^2+4m-1))\ \|A\|^2 - \|\nabla^\mu_{V^\mu} A - m\sqrt{-\mu} JA\|^2 + \frac{1}{2}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big) dv_\mu.$$

a2) $$(HessE)_{V^\mu}(A) = \int_{S^{2m+1}} \Big((-2m-2-\mu(m^2-1))\ \|A\|^2 - \ \|\nabla^\mu_{V^\mu} A + m\sqrt{-\mu} JA\|^2 + \frac{1}{2}\ \|\overline{D}^C A\|^2_{V^\perp} \Big) dv_\mu.$$

b1) $$(HessF)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} \Big(f_1(m,\mu)\ \|A\|^2 - (1-\mu)\ \|\nabla^\mu_{V^\mu} A - \frac{\sqrt{-\mu}(m-\mu)}{1-\mu} JA\|^2 + \frac{1}{2}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big) dv_\mu.$$

b2) $$(HessF)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} \Big(f_2(m,\mu)\ \|A\|^2 - (1-\mu)\ \|\nabla^\mu_{V^\mu} A + \frac{\sqrt{-\mu}(\mu+m)}{1-\mu} JA\|^2 + \frac{1}{2}\ \|\overline{D}^C A\|^2_{V^\perp} \Big) dv_\mu.$$

where

$$f_1(m,\mu) = \mu(-1+\mu-6m+2m\mu) + (2m+2) - \frac{\mu(m-\mu)^2}{1-\mu}$$

and

$$f_2(m,\mu) = \mu(1+\mu+2m\mu) - (2m+2)(\mu+1) - \frac{\mu(\mu+m)^2}{1-\mu}.$$

Nevertheless, the arguments used in Theorem 5.5 do not allow us to conclude and thus the stability question is open.

All these facts led us to consider in [7] a new functional $\tilde{B}$, better adapted to the Lorentzian situation, that we called the spacelike energy. It is defined on the manifold of unit timelike vector fields and it is related to the energy by

$$\tilde{B}(X) = E(X) - \int_{S^{2m+1}} \Big(\frac{2m+1}{2} - \|\nabla^\mu_X X\|^2 \Big) dv_\mu.$$

Since the Hopf vector field is geodesic, $\tilde{B}(V^\mu) = E(V^\mu) - \frac{2m+1}{2} vol(S^{2m+1}, g_\mu) = B(V^\mu)$. We have shown in [7] that it is also a critical point of the spacelike energy but, in contrast to Proposition 6.4, *for any odd-dimensional sphere, endowed with a Lorentzian Berger metric, the Hopf vector field is stable as a critical point of the* *spacelike energy.* The proof is obtained using part a1) of Proposition 6.5 and the fact that

$$(Hess\tilde{B})_{V^\mu}(A) = \int_{S^{2m+1}} \|\nabla^\mu_A V^\mu + \nabla^\mu_{V^\mu} A\|^2 dv_\mu + (HessE)_{V^\mu}(A).$$

For the 3-dimensional sphere we can do better because, although the inequality (3.2) fails on a Lorentzian manifold, we have shown in [7] that

$$\tilde{B}(X) \geq \frac{1}{2}\int_{S^3} Ric_\mu(X,X) dv_\mu,$$

for all timelike unit vector fields, with equality if and only if $h_{11} = h_{22}$ and $h_{12} = -h_{21}$. The Ricci tensor verifies $Ric_\mu(X,X) \geq -2\mu = Ric_\mu(V^\mu, V^\mu)$ for all unit timelike vector fields X, with equality if and only if $X = V^\mu$. Therefore, we have shown the following

Proposition 6.6. *On any Lorentzian Berger* 3*-sphere, the Hopf vector field is, up to sign, the only minimizer of the spacelike energy.*

References

[1] V. Borrelli, F. Brito and O. Gil-Medrano, *The infimum of the energy of unit vector fields on odd-dimensional spheres*, Ann. Global Anal. Geom. 23 (2003) 129-140.

[2] V. Borrelli and O. Gil-Medrano, *A critical radius for unit Hopf vector fields on spheres*, Preprint.

[3] F. Brito, *Total bending of flows with mean curvature correction*, Diff. Geom. and its Appl. 12 (2000), 157-163.

[4] F. Brito, P. Chacón and A. M. Naveira, *On the volume of unit vector fields on spaces of constant sectional curvature*, Commentarii Math. Helv. 79 (2004), 300-316.

[5] F. Brito and P. Walczak, *On the Energy of Unit vector Fields with Isolated Singularities*, Ann. Math. Polon. 73 (2000), 269-274.

[6] O. Gil-Medrano, *Relationship between volume and energy of vector fields*, Differential Geom. Appl. 15 (2001), 137-152.

[7] O. Gil-Medrano and A. Hurtado, *Spacelike energy of timelike unit vector fields on a Lorentzian manifold*, J. Geom. and Phys. 51 (2004), 82-100.

[8] O. Gil-Medrano and E. Llinares-Fuster, *Second variation of volume and energy of vector fields. Stability of Hopf vector fields*, Math. Ann. 320 (2001), 531-545.

[9] O. Gil-Medrano and E. Llinares-Fuster, *Minimal unit vector fields*, Tohôku Math. J. 54 (2002), 71-84.

[10] H. Gluck and W. Ziller, *On the volume of a unit vector field on the three-sphere*, Comment. Math. Helv. 61 (1986), 177-192.

[11] J. C. González-Dávila and L. Vanhecke, *Energy and volume of unit vector fields on three-dimensional Riemannian manifolds*, Differential Geom. Appl. 16 (2002), 225-244.

[12] A. Hurtado, *Volume, energy and spacelike energy of vector fields on Lorentzian manifolds*, Proceedings of the II International Meeting on Lorentzian Geometry Murcia 2003, Publ. de la RSME, to appear.

[13] W. A. Poor, *Differential geometric structures*, Mc Graw-Hill, New-York, 1981.

[14] G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. 303 (1995), 325-344.

[15] C.M.Wood, *On the energy of a unit vector field*, Geometriae Dedicata 64 (1997), 319-330.

Olga Gil-Medrano and Ana Hurtado
Departamento de Geometría y Topología
Universitat de València
46100 Burjassot, Valencia, España
email : Olga.Gil@uv.es, Ana.Maria.Hurtado@uv.es