Stability numbers in K-contact manifolds

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Abstract

The aim of this paper is to study the stability of the characteristic vector field of a compact K-contact manifold with respect to the energy and volume functionals when we consider on the manifold a two-parameter variation of the metric. First of all, we multiply the metric in the direction of the characteristic vector field by a constant and then we change the metric by homotheties. We will study to what extent the results obtained in [3] for Sasakian manifolds are valid for a general K-contact manifold. Finally, as an example, we will study the stability of Hopf vector fields on Berger spheres when we consider homotheties of Berger metrics.

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1 Introduction

A smooth vector field V on a Riemannian manifold (M, g) can be seen as a map into its tangent bundle endowed with the Sasaki metric, g^S, defined by g. The volume of V is the volume of $V(M)$ considered as a submanifold of (TM, g^S) . Analogously, we can define the energy of V as the energy of the map $V : (M, g) \longrightarrow (TM, g^S)$.

On a compact manifold M, the critical points of both functionals should be parallel with respect to the Levi-Civita connection defined by g, so it is usual to restrict these functionals to the submanifold of unit vector fields. Obviously, if M admits unit parallel vector fields, they are the absolute minimizers of both functionals, but for many natural manifolds admitting smooth unit vector fields but not parallel ones, the value of the infimum and the regularity of minimizers is now an open problem.

The geometrically simplest manifolds admitting unit vector fields but not parallel ones are odd-dimensional spheres. Hopf vector fields defined as those tangent to the fibres of the Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbb{C}P^m$ are very special unit vector fields.

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In [9], Gluck and Ziller showed that Hopf vector fields on the 3-dimensional round spheres are the absolute minimizers of the volume and the analogous result for the energy was shown by Brito in [5]. For higher dimension, they are unstable critical points of the energy ([7], [14] and [15]).

All these results are independent of the radius of the sphere, but as concerns the stability as critical points of the volume, Borrelli and Gil-Medrano have shown in [4] that for $m > 1$ there exists a critical value of the radius, such that, Hopf vector fields are stable critical points of the volume if and only if the radius is lower than or equal to this critical radius. By stable we mean that the Hessian of the functional is positive semi-definite.

Moreover, Borrelli has proved in [3] that this phenomenon is not particular of the sphere and it is occurs for the characteristic vector field of any Sasakian manifold when we change the metric by homotheties.

Besides, in [6] and [11], Gil-Medrano and the author have studied the behaviour of Hopf vector fields when we consider on the sphere another variation of the standard metric: the canonical variation of the Riemannian submersion given by the Hopf fibration. These metrics on the sphere are known as Berger metrics. So, a natural question arises: to study the stability of Hopf vector fields when we consider on the sphere the double variation, that is to say, homotheties of Berger metrics.

Motivated by this question and knowing the result of Borelli concerning Sasakian manifolds, in this paper we study the stability with respect to the energy and the volume of the characteristic vector field of a compact K-contact manifold (M, ξ, g), when we consider on M the following two-parameter variation of the metric: we re-scale the metric g in the direction of the characteristic vector field by a constant factor $\mu \neq 0$, and then we change these metrics g^μ by homotheties, obtaining the metrics $g^\mu_\lambda = \lambda\, g^\mu$, with $\lambda > 0$. If we consider these metrics on the manifold, we obtain a two-parameter family $(M, \xi^\mu_\lambda, g^\mu_\lambda)$ of almost contact metric structures with Killing characteristic vector field $\xi^\mu_\lambda = 1/\sqrt{|\mu|\lambda}\,\xi$.

The paper is organized as follows. In section 2 we recall the definitions and results that we will need in the sequel and we show that *the unit vector fields ξ^μ_λ are critical points of the energy and volume for all $\mu \neq 0$ and $\lambda > 0$.*

In section 3 we study the stability with respect to the energy of the characteristic vector field of a compact K-contact manifold when we consider on the manifold the metrics g^μ_λ. Since the energy is homogeneous in λ, the stability of the vector fields ξ^μ_λ with respect to the energy is independent of λ, but it could depend on μ. In fact, we show that *there exist $\mu^+_s \in [0, +\infty]$ and $\mu^-_s \in [-\infty, 0[$ such that ξ^μ is a stable critical point of the energy if and only if $0 < \mu \leq \mu^+_s$ or $\mu \leq \mu^-_s$.* We call μ^+_s and μ^-_s, the E-stability numbers of the K-contact manifold.

As concerns the volume, it is not homogeneous in λ. So, for each $\mu \neq 0$, in section 4 we study how the stability of ξ^μ_λ depends on λ. We show that *there exist $\lambda^\mu_{s_1} \leq \lambda^\mu_{s_2} \in [0, +\infty]$ such that ξ^μ_λ is stable minimal if and only if $\lambda \in [\lambda^\mu_{s_1}, \lambda^\mu_{s_2}]$*, and therefore the stability region in λ, when it is not empty, is an interval. We call $\lambda_{s_1}, \lambda_{s_2} : \mathbb{R} \setminus \{0\} \to [0, +\infty]$, where $\lambda_{s_i}(\mu) = \lambda^\mu_{s_i}$, the stability functions of the K-contact manifold. This result generalizes the corresponding one given in [3] for Sasakian manifolds.

In section 5, as an example, we compute the E-stability numbers and the stability functions of the odd-dimensional spheres equipped with its canonical K-contact structure

(S^{2m+1}, ξ, g). Here ξ is the Hopf vector field and g the usual metric. On the sphere, the metrics g^μ defined before are precisely the Berger metrics, so the E-stability numbers are computed in [6] and [11].

As concerns the stability functions, for (S^3, g^μ), we show that *if* $\mu \leq 1$ *then* $\lambda^\mu_{s_1} = 0$ *and* $\lambda^\mu_{s_2} = +\infty$ *and that if* $\mu > 1$, *then* $\lambda^\mu_{s_1} = \lambda^\mu_{s_2} = 0$. *That is to say, the stability of* ξ^μ_λ *as a minimal vector field of* (S^3, g^μ_λ) *is independent of* λ *and it is achieved if and only if* $\mu \leq 1$. For higher dimensional spheres, we are able to compute the values of $\lambda^\mu_{s_1}$ and $\lambda^\mu_{s_2}$ for $\mu \leq 1$ and $\mu \geq \mu_c = 1/2(1 + \sqrt{(m+1)/(m-1)})$. For example, we can show that *if* $m > 1$ *and* $1/\sqrt{2m-2} < \mu \leq 1$, $\lambda^\mu_{s_1} = 0$ *and* $\lambda^\mu_{s_2} = (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$, *that is to say,* ξ^μ_λ *is a stable minimal vector field if and only if* $\lambda \leq (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$. As a particular case, if $\mu = 1$, we obtain the result shown in [4] on the existance of a critical radius on round spheres. So the existence of this critical value of λ is not a property characteristic of round spheres but, surprisingly, as we will see, it is neither a property of all Berger spheres.

When $1 < \mu < \mu_c$, we only have a partial answer and there exist values of λ for which the behaviour of Hopf vector fields ξ^μ_λ is unknown.

We finish the paper showing that for Lorentzian Berger spheres, $\mu < 0$, $\lambda^\mu_{s_1} = \lambda^\mu_{s_2} = 0$, *that is to say, Hopf vector fields* ξ^μ_λ *are unstable for all* $\lambda > 0$.

2 Preliminaries

2.1 Energy and volume

Given a n-dimensional Riemannian manifold (M, g), the Sasaki metric g^S on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows:

$$g^S(\zeta_1, \zeta_2) = g(\pi_* \circ \zeta_1, \pi_* \circ \zeta_2) + g(\kappa \circ \zeta_1, \kappa \circ \zeta_2),$$

where $\pi : TM \to M$ is the projection and κ is the connection map of ∇. We will consider also its restriction to the tangent sphere bundle, obtaining the Riemannian manifold (T^1M, g^S).

Definition 2.1. *The energy of a vector field* V *is the energy of the map* $V : (M, g) \longrightarrow (TM, g^S)$ *and it is given by*

$$E(V) = \frac{n}{2} vol(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 \,\mathrm{dv}.$$

The relevant part of the energy, $B(V) = \frac{1}{2} \int_M \|\nabla V\|^2 \mathrm{dv}$, is known as the total bending of V and its restriction to unit vector fields has been widely studied by Wiegmink in [13], (see also [14]).

Definition 2.2. *The volume of a vector field* V *is the* n*-dimensional volume of the submanifold* $V(M)$ *of* (TM, g^S). *Since* $V^* g^S(X, Y) = g(X, Y) + g(\nabla_X V, \nabla_Y V)$

$$F(V) = \int_M f(V) \,\mathrm{dv} = \int_M \sqrt{\det L_V} \,\mathrm{dv},$$

where $L_V = \mathrm{Id} + (\nabla V)^t \circ \nabla V$.

If the manifold is compact, the critical points of both functionals should be parallel, so it is usual to restrict these functionals to the submanifold of unit vector fields.

The condition for a unit vector field to be a critical point of the energy for variations among unit vector fields and the second variation at a critical point have been computed in [13] and [7].

Proposition 2.3. *Given a unit vector field* V *on a Riemannian manifold* (M,g) *then*

1. V *is a critical point of the energy if and only if* $\omega_{(V,g)}(V^\perp) = \{0\}$, *where* $\omega_{(V,g)} = C_1^1 \nabla (\nabla V)^t$.
2. *If* V *is a critical point and* A *is orthogonal to* V *then*

$$(HessE)_V(A) = \int_M (\|\nabla A\|^2 + \|A\|^2 \omega_{(V,g)}(V))\,\mathrm{dv}.$$

Remark 2.4. For a $(1,1)$-tensor field K, if $\{E_i\}$ is a g-orthonormal local frame, we have

$$C_1^1 \nabla K(X) = \sum_i g((\nabla_{E_i} K)X, E_i).$$

For the volume, we have the following result

Proposition 2.5 ([7],[8])**.** *Let* V *be a unit vector field on a Riemannian manifold* (M,g). *Then*

1. V *is a critical point of the volume if and only if* $\omega_V(V^\perp) = \{0\}$, *where* $\omega_V = C_1^1 \nabla K_V$ *and* $K_V = \sqrt{\det L_V}\, L_V^{-1} \circ (\nabla V)^t$.
2. *If* V *is a critical point of the volume and* A *is orthogonal to* V

$$\begin{aligned}(HessF)_V(A) \;=\; & \int_M \|A\|^2 \omega_V(V)\,\mathrm{dv} + \int_M \frac{2}{\sqrt{\det L_V}}\, \sigma_2(K_V \circ \nabla A)\,\mathrm{dv} \\ - & \int_M \mathrm{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla V \circ K_V \circ \nabla A\right)\mathrm{dv} \\ + & \int_M \sqrt{\det L_V}\;\; \mathrm{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla A\right)\mathrm{dv},\end{aligned}$$

where $\sigma_2 = 1/2((\mathrm{tr}(K_V \circ \nabla A))^2 - \mathrm{tr}(K_V \circ \nabla A)^2)$.

Moreover, in [8] it was proved that a unit vector field is a critical point of F if and only if it defines a minimal immersion in (T^1M, g^S).

Remark 2.6. The Hessian of the volume at a vector field V defining a minimal immersion can be simplified if V is assumed to be a Killing vector field. Using Lemma 9 of [8] we obtain

$$\begin{aligned}(HessF)_V(A) \;=\; & \int_M \|A\|^2 \omega_V(V)\,\mathrm{dv} + \int_M \frac{2}{\sqrt{\det L_V}}\, \sigma_2(K_V \circ \nabla A)\,\mathrm{dv} \\ + & \int_M \sqrt{\det L_V}\;\; \mathrm{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ L_V^{-1} \circ \nabla A\right)\mathrm{dv}. \qquad (1)\end{aligned}$$

In a Lorentzian manifold, the energy is defined for all vector fields. Nevertheless, the volume of a reference frame (unit timelike vector field) V is not always defined, since the 2-covariant field V^*g^S can be degenerated. Due to this, we study the volume restricted to unit timelike vector fields for which V^*g^S is a Lorentzian metric on M. We will denote this set of vector fields by $\Gamma^-(T^{-1}M)$ and it is an open subset of the set of smooth reference frames. If V belongs to $\Gamma^-(T^{-1}M)$, then $\det L_V > 0$ and the volume is well defined.

The conditions for a reference frame to be a critical point of the energy and the volume on a Lorentzian manifold are the same conditions that those given in Propositions 2.3 and 2.5 for Riemannian metrics. If we compute the second variation, we obtain the following

Proposition 2.7 ([10])**.** *Let V be a unit timelike vector field on a compact Lorentzian manifold (M, g).*

1. *If V is a critical point of the energy, the Hessian of E at V acting on $A \in V^\perp$ is given by*
$$(HessE)_V(A) = -\int_M \|A\|^2 \omega_{(V,g)}(V)\,\mathrm{dv} + \int_M \|\nabla A\|^2\,\mathrm{dv}.$$

2. *For a unit timelike vector field $V \in \Gamma^-(T^{-1}M)$ defining a minimal immersion, the Hessian of F at V acting on $A \in V^\perp$ is given by*

$$\begin{aligned}(HessF)_V(A) \;=\; & -\int_M \|A\|^2 \omega_V(V)\,\mathrm{dv} + \int_M \frac{2}{\sqrt{\det L_V}}\,\sigma_2(K_V \circ \nabla A)\,\mathrm{dv}\\ & -\int_M \mathrm{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla V \circ K_V \circ \nabla A\right)\mathrm{dv}\\ & +\int_M \sqrt{\det L_V}\;\mathrm{tr}\left(L_V^{-1} \circ (\nabla A)^t \circ \nabla A\right)\mathrm{dv}.\end{aligned}$$

Remark 2.8. Let us point ut that if we compare the above expressions of the Hessian with those obtained for Riemannian metrics, the only difference is the minus sign of the first term of the expression of the Hessian.

2.2 K-contact manifolds

Let M be a manifold of dimension $2m+1$ and φ, ξ and η tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively.

Definition 2.9. *(φ, ξ, η) is called an almost contact structure on M if the followings are satisfied :*

1. $\eta(\xi) = 1$.
2. $\eta(\varphi(X)) = 0, \qquad X \in \Gamma(TM)$.
3. $\varphi^2(X) = -X + \eta(X)\xi, \qquad X \in \Gamma(TM)$.

We call ξ the characteristic vector field of the almost contact metric structure.

Definition 2.10. *$(\varphi, \xi, \eta, g, \varepsilon)$ is called an almost contact metric structure on M, if (φ, ξ, η) is an almost contact structure on M and g is a semi-Riemannian metric on M such that*

1. $g(\xi, \xi) = \varepsilon \qquad \varepsilon = 1$ *or* -1.
2. $\eta(X) = \varepsilon\, g(\xi, X), \qquad X \in \Gamma(TM)$.
3. $g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\, \eta(X)\eta(Y) \qquad X,\ Y \in \Gamma(TM)$.

Moreover, if $d\eta(X, Y) = g(\varphi X, Y)$ for all $X,\ Y \in \Gamma(TM)$ then $(\varphi, \xi, \eta, g, \varepsilon)$ is called a contact metric structure.

Definition 2.11. *A contact metric structure on M is said to be a K-contact structure if the characteristic vector field is Killing.*

Definition 2.12. *A contact metric structure on M is said to be normal if*

$$(\nabla_X \varphi)Y = \varepsilon\, \eta(Y)X - g(X, Y)\xi, \quad X, Y \in \Gamma(TM).$$

In this case we call M a Sasakian manifold.

It is easy to see that the characteristic field of a Sasakian manifold is a Killing vector field. So a Sasakian manifold can be seen as a particular case of a K-contact manifold.

Let $(M, \varphi, \xi, \eta, g)$ be a Riemannian K-contact manifold. In M we can consider the following 1-parameter family of metrics g^μ, with $\mu \neq 0$

$$g^\mu_{|\xi} = \mu\, g_{|\xi}, \qquad g^\mu_{|\xi^\perp} = g_{|\xi^\perp}, \qquad g^\mu(\xi, \xi^\perp) = 0.$$

When $\mu > 0$ the metric g^μ is Riemannian and if $\mu < 0$ the metric is Lorentzian and $\xi^\mu = 1/\sqrt{|\mu|}\,\xi$ is timelike. For regular K-contact manifolds the metrics so defined are the metrics obtained by performing the canonical variation of the Riemannian submersion given by the Boothby-Wang fibration (see [2]).

The Levi-Civita connection ∇^μ of g^μ is related to ∇ by

$$\nabla^\mu_X Y = \nabla_X Y, \quad \nabla^\mu_X \xi^\mu = \mu\, \nabla_X \xi^\mu, \quad \nabla^\mu_{\xi^\mu} X = \nabla_{\xi^\mu} X + (\mu - 1)\nabla_X \xi^\mu \quad \text{for } X, Y \in \xi^\perp.$$

In general if X, Y are vector fields on M,

$$\nabla^\mu_X Y = \nabla_X Y + (\mu - 1)g(X, \xi)\nabla_Y \xi + (\mu - 1)g(Y, \xi)\nabla_X \xi.$$

Moreover, for each $\mu \neq 0$, we can consider the metrics $g^\mu_\lambda = \lambda\, g^\mu$, with $\lambda > 0$. Then, we obtain a two-parameter family of semi-Riemannian metrics g^μ_λ in M.

If $\xi^\mu_\lambda = 1/\sqrt{\lambda\,|\mu|}\,\xi$ and $\eta^\mu_\lambda = \varepsilon_\mu\, g^\mu_\lambda(\xi^\mu_\lambda)$, it is easy to see that $(M, \varphi, \xi^\mu_\lambda, \eta^\mu_\lambda, g^\mu_\lambda, \varepsilon_\mu)$ is an almost contact metric structure on M with Killing characteristic vector field. In addition,

Proposition 2.13. *Let $(M, \varphi, \xi, \eta, g)$ be a Riemannian K-contact manifold. Then $(M, \xi^\mu_\lambda, g^\mu_\lambda)$ is a K-contact manifold if and only if $|\mu| = \lambda$.*

Proof. As can be seen in [2] and [12], if $(M, g^\mu_\lambda, \xi^\mu_\lambda)$ is a K-contact manifold with characteristic vector field ξ^μ_λ then $\varphi^\mu_\lambda = \varepsilon_\mu \nabla^\mu \xi^\mu_\lambda$. But,

$$(\nabla^\mu \xi^\mu_\lambda)^2(X) = (\nabla^\mu \xi^\mu_\lambda)(\varepsilon_\mu \sqrt{\frac{|\mu|}{\lambda}} \nabla_X \xi) = \frac{|\mu|}{\lambda} \varphi^2(X) = -\frac{|\mu|}{\lambda} X,$$

for all $X \in \xi^\perp$. Then, since $(\varphi^\mu_\lambda)^2 = -Id$ on $\xi^\perp$, we have that $|\mu| = \lambda$.

Conversely, if $|\mu| = \lambda$, since ξ^μ_λ is Killing

$$R^\mu_\lambda(\xi^\mu_\lambda, X, \xi^\mu_\lambda) = -\nabla^\mu_{\nabla^\mu_X \xi^\mu_\lambda} \xi^\mu_\lambda = \frac{|\mu|}{\lambda} X = X.$$

So, by the characterization of the characteristic vector field of a K-contact manifold in terms of the curvature shown in [2], $(M, g^\mu_\lambda, \xi^\mu_\lambda)$ is a K-contact manifold. □

Remark 2.14. As a particular case, (M, ξ^μ, g^μ) is a K-contact manifold if and only if $|\mu| = 1$.

Remark 2.15. If M^{2m+1} is a Riemannian manifold admitting a unit Killing vector field such that the sectional curvatures of the planes containing it are constant and equal each other, then in the family of semi-Riemannian manifolds obtained by deforming the metric in the direction of the Killing vector field as above, we only have a K-contact manifold.

It is known (see [13]) that if W is a unit Killing vector field, then it is a critical point of the energy if and only if the Ricci tensor verifies that $\mathrm{Ric}(W, X) = 0$, for all X orthogonal to W. In a K-contact manifold the characteristic vector field satisfies this condition, so it is a critical point of the energy.

In $(M, \xi^\mu_\lambda, g^\mu_\lambda)$, a straightforward computation shows that

$$\mathrm{Ric}^\mu_\lambda(\xi^\mu_\lambda, X) = \frac{1}{\sqrt{\lambda}} \mathrm{Ric}^\mu(\xi^\mu, X) = \varepsilon_\mu \sqrt{\frac{|\mu|}{\lambda}} \mathrm{Ric}(\xi, X) = 0,$$

so ξ^μ_λ is critical for the energy for all λ and μ.

For the volume, $L_{\xi^\mu_\lambda} = (1 + |\mu|/\lambda)\mathrm{Id}$ on $\xi^\perp$, $f(\xi^\mu_\lambda) = (1 + |\mu|/\lambda)^m$ and $K_{\xi^\mu_\lambda} = (1 + |\mu|/\lambda)^{m-1}(\nabla^\mu \xi^\mu_\lambda)^t$. Then, ξ^μ_λ is also a critical point of the volume.

Summarizing,

Proposition 2.16. *Let $(M, \varphi, \xi, \eta, g)$ be a Riemannian K-contact manifold. The unit Killing vector fields ξ^μ_λ are critical points of the energy and of the volume for all λ and μ.*

3 E-stability numbers

Let (M^n, g) be a semi-Riemannian manifold and V a unit vector field on (M^n, g). If we consider on M^n the metric λg with $\lambda > 0$ and the unit vector field $V_\lambda = 1/\sqrt{\lambda}\, V$, then the total bending verifies that $B_\lambda(V_\lambda) = \lambda^{\frac{n}{2}-1} B(V)$.

Therefore, the energy is homogeneous in λ and the harmonicity and stability of the vector fields V_λ are independent on λ.

Due to this, the stability of ξ^μ_λ as a critical point of the energy does not depend on λ, but it could depend on μ. In fact, we have

Proposition 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian K-contact manifold of dimension $2m+1$ and $\{(M, \xi^\mu, g^\mu)\}$ the family of manifolds obtained by multiplying the metric g by $\mu \neq 0$ in the direction of the characteristic vector field ξ. There exist $\mu_s^+ \in [0, +\infty]$ and $\mu_s^- \in [-\infty, 0[$ such that ξ^μ is a stable critical point of the energy if and only if $0 < \mu \leq \mu_s^+$ or $\mu \leq \mu_s^-$. Moreover, if $m > 1$ then $\mu_s^+ < +\infty$.*

Proof. By Propositions 2.3 and 2.7 , if $A \in \xi^\perp$

$$(HessE)_{\xi^\mu}(A) = \int_M (\|\nabla^\mu A\|^2 + \varepsilon_\mu \|A\|^2 \omega_{(\xi^\mu, g^\mu)}(\xi^\mu))\, \mathrm{dv}^\mu. \tag{2}$$

But

$$\omega_{(\xi^\mu, g^\mu)}(\xi^\mu) = \sum_{i=1}^{2m} g^\mu((\nabla^\mu_{E_i}(\nabla^\mu \xi^\mu)^t)\xi^\mu, E_i) = -\sum_{i=1}^{2m} g^\mu(\nabla^\mu_{E_i}\xi^\mu, \nabla^\mu_{E_i}\xi^\mu) = -2m|\mu|, \tag{3}$$

and

$$\begin{aligned} \|\nabla^\mu A\|^2 &= \|\nabla A\|^2 - \|\nabla_\xi A\|^2 + (\mu - 1)\|A\|^2 + \frac{1}{\mu}\|\nabla_\xi A + (\mu - 1)\nabla_A \xi\|^2 \\ &= \|\nabla A\|^2 + (\frac{1}{\mu} - 1)\|\nabla_\xi A\|^2 + (2\mu - 3 + \frac{1}{\mu})\|A\|^2 + 2\,\frac{\mu - 1}{\mu}\, g(\nabla_A \xi, \nabla_\xi A), \end{aligned} \tag{4}$$

from where

$$(HessE)_{\xi^\mu}(A) = \sqrt{|\mu|}\, H_A(\mu),$$

with

$$H_A(\mu) = \int_M \Big(\|\nabla A\|^2 + (\mu(2 - 2m) - 2)\|A\|^2 + \Big(\frac{1}{\mu} - 1\Big)\|\nabla_\xi A - \nabla_A \xi\|^2\Big)\, \mathrm{dv}.$$

The norms involved in the righthand side of equation (4) and in the definition of H_A are computed with the metric g.

To study the stability of ξ^μ we have to control the sign of $H_A(\mu)$. If we derive respect μ,

$$\frac{dH_A}{d\mu} = \int_M \Big((2 - 2m)\|A\|^2 - \frac{1}{\mu^2}\|\nabla_\xi A - \nabla_A \xi\|^2\Big)\, \mathrm{dv},$$

and, for each A, $H_A(\mu)$ is strictly decreasing.

If $\{\mu > 0;\ \xi^\mu$ stable for the energy $\} = \emptyset$, we set $\mu_s^+ := 0$, otherwise we put $\mu_s^+ := max\{\mu > 0 : \xi^\mu$ stable for the energy $\}$ (If ξ^μ is always stable, $\mu_s^+ = +\infty$ and this maximum is a supremum). By definition, if $\mu > \mu_s^+$, ξ^μ is unstable.

If there exists $\mu_1 < \mu_s^+$ such that ξ^{μ_1} is unstable, then there exists $A \in \xi^\perp$ such that $H_A(\mu_1) < 0$ which leads to a contradiction because $H_A(\mu_1) < H_A(\mu_s^+)$ and H_A is decreasing.

Moreover, if $m > 1$, $\lim_{\mu\to+\infty} H_A = -\infty$, for all $A \in \xi^\perp$ and $\mu_s^+ < +\infty$.

For the negative values of μ, if $\{\mu < 0;\ \xi^\mu$ stable for the energy $\} = \emptyset$, we set $\mu_s^- := -\infty$, otherwise we define $\mu_s^- := max\{\mu < 0 : \xi^\mu$ stable for the energy $\}$.

Analogously, we can prove that ξ^μ is stable for the energy if and only if $\mu \leq \mu_s^-$. Moreover,

$$\lim_{\mu\to 0^-} H_A = \lim_{\mu\to 0^-} \int_M (\|\nabla A\|^2 - 2\|A\|^2 + (\frac{1}{\mu} - 1)\|\nabla_A \xi - \nabla_\xi A\|^2)\, \mathrm{dv} = -\infty$$

and $\mu_s^- < 0$. □

Definition 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian K-contact manifold, we call μ_s^+ the E^+-stability number and we call μ_s^- the E^--stability number of the K-contact manifold.*

4 Stability functions

For each $\mu \neq 0$, we are going to study how the stability of ξ_λ^μ as a minimal unit vector field of (M, g_λ^μ) depends on λ. As in [3], it will be useful the following Lemma.

Lemma 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian K-contact manifold of dimension $2m+1$ and let $\{(M, \xi_\lambda^\mu, g_\lambda^\mu)\}$ be the family of almost contact metric manifolds obtained by multiplying the metric g^μ by $\lambda > 0$. Then*

$$(HessF)_{\xi_\lambda^\mu}(A) = \lambda^{m+\frac{1}{2}}\Big(1 + \frac{|\mu|}{\lambda}\Big)^{m-2}\Big((HessE)_{\xi^\mu}(A) + \frac{1}{\lambda}\mathcal{C}^\mu(A)\Big), \tag{5}$$

where

$$\mathcal{C}^\mu(A) = \int_M ((1-2m)\mu|\mu|\|A\|^2 + \mu\|\nabla_{\xi^\mu}^\mu A\|^2 + |\mu|\beta(A))\, \mathrm{dv}^\mu,$$

and $\beta(A) = |\mu|^{-1}(1+|\mu|)^{2-2m}\sigma_2(K_{\xi^\mu} \circ \nabla^\mu A)$.

Proof. Using (1) and Remark 2.8,

$$\begin{aligned}(HessF)_{\xi_\lambda^\mu}(A) = \quad & \varepsilon_\mu \int_M \|A\|_\lambda^2 \omega_{\xi_\lambda^\mu}(\xi_\lambda^\mu)\, \mathrm{dv}_\lambda^\mu \\ & + \int_M f(\xi_\lambda^\mu)\mathrm{tr}(L_{\xi_\lambda^\mu}^{-1} \circ (\nabla^\mu A)^t \circ L_{\xi_\lambda^\mu}^{-1} \circ (\nabla^\mu A))\, \mathrm{dv}_\lambda^\mu \\ & + \int_{S^{2m+1}} \frac{1}{f(\xi_\lambda^\mu)}\sigma_2(K_{\xi_\lambda^\mu} \circ \nabla^\mu A)\, \mathrm{dv}_\lambda^\mu.\end{aligned}$$

If $\{\xi, E_i, \varphi E_i\}$ is a (φ, g)-basis, then $\{\xi^\mu_\lambda, E_{i_\lambda}, E_{i*_\lambda}\}$ with $E_{i_\lambda} = 1/\sqrt{\lambda}\ E_i$ and $E_{i*_\lambda} = 1/\sqrt{\lambda}\ \varphi E_i$, is an adapted g^μ_λ-orthonormal local frame. Therefore,

$$L_{\xi^\mu_\lambda}(E_{i_\lambda}) = (1 + |\mu|/\lambda)E_{i_\lambda}, \quad L_{\xi^\mu_\lambda}(E_{i*_\lambda}) = (1 + |\mu|/\lambda)E_{i*_\lambda} \quad \text{and} \quad L_{\xi^\mu_\lambda}(\xi^\mu_\lambda) = \xi^\mu_\lambda,$$

so

$$f(\xi^\mu_\lambda) = (1 + |\mu|/\lambda)^m \quad \text{and} \quad K_{\xi^\mu_\lambda} = -(1 + |\mu|/\lambda)^{m-1}\varepsilon_\mu\sqrt{|\mu|/\lambda}\,\varphi.$$

By direct computation we obtain that,

$$\omega_{\xi^\mu_\lambda}(\xi^\mu_\lambda) = -2m\frac{|\mu|}{\lambda}\Big(1 + \frac{|\mu|}{\lambda}\Big)^{m-1}.$$

Moreover,

$$\mathrm{tr}(L^{-1}_{\xi^\mu_\lambda} \circ (\nabla^\mu A)^t \circ L^{-1}_{\xi^\mu_\lambda} \circ (\nabla^\mu A)) = \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-2}(\|\nabla^\mu A\|^2 + \frac{\mu}{\lambda}\|\nabla^\mu_{\xi^\mu_\lambda} A\|^2 + \frac{\mu|\mu|}{\lambda}\|A\|^2),$$

since

$$\begin{aligned}
g^\mu(L^{-1}_{\xi^\mu_\lambda} \circ (\nabla^\mu A)^t \circ L^{-1}_{\xi^\mu_\lambda} \circ \nabla^\mu_{E_i} A, E_i) &= \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-1} g^\mu(L^{-1}_{\xi^\mu_\lambda} \circ \nabla^\mu_{E_i} A, \nabla^\mu_{E_i} A) \\
&= \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-2}(g^\mu(\nabla^\mu_{E_i} A\, \nabla^\mu_{E_i} A) + \frac{\mu|\mu|}{\lambda} g(A, E_{i*})^2), \\
g^\mu(L^{-1}_{\xi^\mu_\lambda} \circ (\nabla^\mu A)^t \circ L^{-1}_{\xi^\mu_\lambda} \circ \nabla^\mu_{E_{i*}} A, E_{i*}) &= \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-1} g^\mu(L^{-1}_{\xi^\mu_\lambda} \circ \nabla^\mu_{E_{i*}} A, \nabla^\mu_{E_{i*}} A) \\
&= \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-2}(g^\mu(\nabla^\mu_{E_{i*}} A, \nabla^\mu_{E_{i*}} A) + \frac{\mu|\mu|}{\lambda} g(A, E_i)^2), \\
g^\mu(L^{-1}_{\xi^\mu_\lambda} \circ (\nabla^\mu A)^t \circ L^{-1}_{\xi^\mu_\lambda} \circ \nabla^\mu_{\xi^\mu} A, \xi^\mu) &= \Big(1 + \frac{|\mu|}{\lambda}\Big)^{-1} g^\mu(\nabla^\mu_{\xi^\mu} A, \nabla^\mu_{\xi^\mu} A).
\end{aligned}$$

Now, we only need to compute $\sigma_2(K_{\xi^\mu_\lambda} \circ \nabla^\mu A)$, but

$$K_{\xi^\mu_\lambda} \circ \nabla^\mu A = \frac{1}{\sqrt{\lambda}} \frac{(1 + \frac{|\mu|}{\lambda})^{m-1}}{(1 + |\mu|)^{m-1}}\, K_{\xi^\mu} \circ \nabla^\mu A,$$

and then,

$$\sigma_2(K_{\xi^\mu_\lambda} \circ \nabla^\mu A) = \frac{1}{\lambda} \frac{(1 + \frac{|\mu|}{\lambda})^{2m-2}}{(1 + |\mu|)^{2m-2}}\, \sigma_2(K_{\xi^\mu} \circ \nabla^\mu A).$$

Since $\mathrm{dv}^\mu_\lambda = \lambda^{m+\frac{1}{2}}\, \mathrm{dv}^\mu$, we have

$$\begin{aligned}
(HessF)_{\xi^\mu_\lambda}(A) &= \lambda^{m+\frac{1}{2}}\Big(1 + \frac{|\mu|}{\lambda}\Big)^{m-2} \int_M \Big(- 2m\mu\Big(1 + \frac{|\mu|}{\lambda}\Big)\|A\|^2 + \|\nabla^\mu A\|^2 \\
&\qquad + \frac{\mu}{\lambda}\|\nabla^\mu_{\xi^\mu} A\|^2 + \frac{\mu|\mu|}{\lambda}\|A\|^2 + \frac{|\mu|}{\lambda}\beta(A)\Big)\, \mathrm{dv}^\mu,
\end{aligned}$$

where, if $B_i^j = g(\nabla_{E_i} A, E_j)$

$$\begin{aligned} |\mu|\beta(A) &= (1+|\mu|)^{2-2m}\sigma_2(K_{\xi^\mu}\circ\nabla^\mu A) \\ &= |\mu|\Big(\Big(\sum_{i=1}^m (B_i^{i*} - B_{i*}^i)\Big)^2 - \sum_{i,j=1}^m (B_{i*}^j B_{j*}^i - B_{i*}^{j*} B_j^i - B_i^j B_{j*}^{i*} + B_i^{j*} B_j^{i*})\Big). \end{aligned}$$

Therefore by equations (2) and (3),

$$(HessF)_{\xi^\mu_\lambda}(A) = \lambda^{m+\frac{1}{2}}\Big(1+\frac{|\mu|}{\lambda}\Big)^{m-2}\Big((HessE)_{\xi^\mu}(A) + \frac{1}{\lambda}\mathcal{C}^\mu(A)\Big),$$

with

$$\mathcal{C}^\mu(A) = \int_M ((1-2m)\mu|\mu|\|A\|^2 + \mu\|\nabla^\mu_{\xi^\mu} A\|^2 + |\mu|\beta(A))\,\mathrm{dv}^\mu.$$

□

Theorem 4.2. *Let (M,φ,ξ,η,g) be a compact Riemannian K-contact manifold of dimension $2m+1$ and let $\{(M,\xi^\mu_\lambda,g^\mu_\lambda)\}$ be the family of almost contact metric manifolds obtained by multiplying the metric g^μ by $\lambda>0$. For all $\mu\neq 0$, there exist $\lambda^\mu_{s_1}\le\lambda^\mu_{s_2}\in[0,+\infty]$ such that ξ^μ_λ is stable minimal if and only if $\lambda\in[\lambda^\mu_{s_1},\lambda^\mu_{s_2}]$. Moreover, if ξ^μ is stable for the energy then $\lambda^\mu_{s_2}=+\infty$, that is to say, ξ^μ_λ is stable minimal if and only if $\lambda\ge\lambda^\mu_{s_1}$.*

Proof. We are going to prove that if there exist $\lambda_1<\lambda_2\in]0,+\infty[$ such that $\xi^\mu_{\lambda_1}$ and $\xi^\mu_{\lambda_2}$ are stable minimal then for all $\lambda\in[\lambda_1,\lambda_2]$, ξ^μ_λ is also stable minimal.

If we assume that there exist $\lambda_1<\lambda<\lambda_2$ such that ξ^μ_λ is unstable, then there exists $A\in\xi^\perp$ such that, $(HessF)_{\xi^\mu_\lambda}(A)<0$. For this A,

$$(HessE)_{\xi^\mu}(A) + \frac{1}{\lambda}\mathcal{C}^\mu(A) < 0, \tag{6}$$

$$(HessE)_{\xi^\mu}(A) + \frac{1}{\lambda_2}\mathcal{C}^\mu(A) \ge 0, \tag{7}$$

$$(HessE)_{\xi^\mu}(A) + \frac{1}{\lambda_1}\mathcal{C}^\mu(A) \ge 0. \tag{8}$$

Equations (6) and (7), imply that $\mathcal{C}^\mu(A)<0$ which leads to a contradiction if we compare (6) and (8).

In conclusion, the stability region in λ is an interval. If $\{\lambda:\ \xi^\mu_\lambda \text{ stable minimal }\}=\emptyset$, we set $\lambda^\mu_{s_1}=\lambda^\mu_{s_2}=0$, otherwise we define

$$\lambda^\mu_{s_1} := \inf\{\lambda:\ \xi^\mu_\lambda \text{ stable minimal }\},$$

$$\lambda^\mu_{s_2} := \sup\{\lambda:\ \xi^\mu_\lambda \text{ stable minimal }\}.$$

Accordingly with the definition $\lambda^{\mu}_{s_1}$ and $\lambda^{\mu}_{s_2}$ can take the values 0 or $+\infty$.

Moreover, if ξ^{μ} is stable for the energy and we suppose that $\lambda^{\mu}_{s_2} < +\infty$, there exist $\lambda > \lambda^{\mu}_{s_2}$ and $A \in \xi^{\perp}$ such that

$$(HessE)_{\xi^{\mu}}(A) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A) < 0.$$

Since ξ^{μ} is stable for the energy, $(HessE)_{\xi^{\mu}}(A) \geq 0$, so $\mathcal{C}^{\mu}(A) < 0$. But,

$$(HessE)_{\xi^{\mu}}(A) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A) > (HessE)_{\xi^{\mu}}(A) + \frac{1}{\lambda^{\mu}_{s_2}}\mathcal{C}^{\mu}(A) \geq 0,$$

which leads to a contradiction. Then, $\lambda^{\mu}_{s_2} = +\infty$. □

Therefore, when we study the stability of the vector fields ξ^{μ}_{λ} with respect to the volume, for each $\mu \neq 0$, the stability region in λ, when it is not empty, is a connect subset.

Remark 4.3. In the sequel, if ξ^{μ}_{λ} is unstable for all λ, we will set $\lambda^{\mu}_{s_1} = \lambda^{\mu}_{s_2} = 0$. In other words, we take $\lambda^{\mu}_{s_1} \in [0, +\infty[$.

Definition 4.4. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian manifold, we call $\lambda_{s_1}, \lambda_{s_2} : \mathbb{R} \setminus \{0\} \to [0, +\infty]$, where $\lambda_{s_i}(\mu) = \lambda^{\mu}_{s_i}$, the stability functions of the K-contact manifold.*

Remark 4.5. The Theorem 4.2 is more general that the result obtained in [3] since we only assume that (M, ξ, g) is a K-contact manifold, not necessarily Sasakian. Moreover, for $\mu \neq 1$, (M, ξ^{μ}, g^{μ}) is only an almost contact metric structure with Killing characteristic vector field, and this is the weakest hypothesis needed to conclude the result.

To obtain the above Theorem is essential the expression of the Hessian given by (5). If we have information about the sign of $\mathcal{C}^{\mu}(A)$, we can show

Corollary 4.6. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian K-contact manifold and $(M, \xi^{\mu}_{\lambda}, g^{\mu}_{\lambda})$ the family of almost contact metric manifolds obtained by multiplying the metric g^{μ} by $\lambda > 0$. If $\mathcal{C}^{\mu}(A) \geq 0$ for all $A \in \xi^{\perp}$ then we have the following*

- *a) If ξ^{μ} is stable for the energy then $\lambda^{\mu}_{s_1} = 0$ and $\lambda^{\mu}_{s_2} = +\infty$, in other words, ξ^{μ}_{λ} is stable minimal for all $\lambda > 0$.*
- *b) If ξ^{μ} is unstable for the energy then $\lambda^{\mu}_{s_1} = 0$ and $\lambda^{\mu}_{s_2} < +\infty$, that is to say, ξ^{μ}_{λ} is stable minimal if and only if $\lambda \leq \lambda^{\mu}_{s_2}$.*

Proof. Part a) is a direct consequence of (5) and the fact that, under the assumption, $(HessE)_{\xi^{\mu}}(A) \geq 0$ for all $A \in \xi^{\perp}$.

To show b), if ξ^{μ} is unstable for the energy, there exists $A \in \xi^{\perp}$ such that $(HessE)_{\xi^{\mu}}(A) < 0$ and

$$\lim_{\lambda \to +\infty} \left((HessE)_{\xi^{\mu}}(A) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A)\right) < 0.$$

So, $\lambda^{\mu}_{s_2} < +\infty$. If we suppose that $\lambda^{\mu}_{s_1} > 0$, then there exists $\lambda < \lambda^{\mu}_{s_1}$ and $A_\lambda \in \xi^{\perp}$ such that

$$(HessE)_{\xi^{\mu}}(A_\lambda) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A_\lambda) < 0. \tag{9}$$

Then,

$$(HessE)_{\xi^{\mu}}(A_\lambda) + \frac{1}{\lambda^{\mu}_{s_1}}\mathcal{C}^{\mu}(A_\lambda) < 0, \tag{10}$$

which is in contradiction with the definition of $\lambda^{\mu}_{s_1}$. □

Remark 4.7. We will see in the last section, that for odd-dimensional spheres with $0 < \mu \leq 1$, the above hypothesis on $\mathcal{C}^{\mu}$ is fulfilled.

Corollary 4.8. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Riemannian K-contact manifold and $(M, \xi^{\mu}_{\lambda}, g^{\mu}_{\lambda})$ the almost contact metric manifolds obtained by multiplying the metric g^{μ} by $\lambda > 0$. If ξ^{μ} is unstable for the energy and there exists $A_\mu \in \xi^{\perp}$ such that $\mathcal{C}^{\mu}(A_\mu) < 0$ then, either ξ^{μ}_{λ} is unstable minimal for all $\lambda > 0$, or $\lambda^{\mu}_{s_1} > 0$ and $\lambda^{\mu}_{s_2} < +\infty$.*

Proof. Since ξ^{μ} is unstable for the energy, there exists $A \in \xi^{\perp}$ such that $(HessE)_{\xi^{\mu}}(A) < 0$ and

$$\lim_{\lambda \to +\infty} \big((HessE)_{\xi^{\mu}}(A) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A)\big) < 0.$$

Moreover, by hypothesis there exists $A_\mu \in \xi^{\perp}$ such that $\mathcal{C}^{\mu}(A_\mu) < 0$ and consequently

$$\lim_{\lambda \to 0^+} \big((HessE)_{\xi^{\mu}}(A_\mu) + \frac{1}{\lambda}\mathcal{C}^{\mu}(A_\mu)\big) = -\infty,$$

Therefore, either ξ^{μ}_{λ} is unstable for all $\lambda > 0$ ($\lambda^{\mu}_{s_1} = \lambda^{\mu}_{s_2} = 0$) or $\lambda^{\mu}_{s_1} > 0$ and $\lambda^{\mu}_{s_2} < +\infty$. □

5 Odd-dimensional spheres

The Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbb{C}P^m$ determines a foliation of S^{2m+1} by great circles and a unit vector field can be chosen as a generator of this distribution. It is given by $V = JN$ where N represents the unit normal to the sphere and J the usual complex structure on $\mathbb{R}^{2m+2}$. V is the standard Hopf vector field.

It is well known that the sphere equipped with the usual metric carries on a contact structure such that (S^{2m+1}, V, g) is a Sasakian manifold, and then K-contact.

In S^{2m+1} we can consider the canonical variation g^{μ}, with $\mu \neq 0$, of the usual metric g

$$g^{\mu}|_{V^{\perp}} = g|_{V^{\perp}}, \quad g^{\mu}|_{V} = \mu\, g|_{V}, \quad g^{\mu}(V, V^{\perp}) = 0.$$

For $m = 1$ and $\mu > 0$ these metrics on the sphere are known as Berger metrics (see [1] pg. 252). For all $\mu \neq 0$ the map $\pi : (S^{2m+1}, g^{\mu}) \longrightarrow \mathbb{C}P^m$ is a semi-Riemannian submersion with totally geodesic fibers. The distribution determined by the fibers admits as a unit generator $V^\mu = \frac{1}{\sqrt{|\mu|}} JN$ that we will call also Hopf vector field.

In [6] and [10] the authors have studied the stability of the Hopf vector fields V^μ with respect to the energy and the volume on Berger spheres. The results related with the energy can be reformulated in terms of the E-stability numbers as follows

Proposition 5.1. *The E-stability numbers of (S^3, V, g) are $\mu_s^+ = 1$ and $\mu_s^- = -\infty$. On (S^{2m+1}, V, g) with $m > 1$, $\mu_s^+ = 1/\sqrt{2m-2}$ and $\mu_s^- = -\infty$.*

Now, we are going to study the stability of Hopf vector fields with respect to the volume when we consider on the sphere the metrics $g_\lambda^\mu = \lambda\, g^\mu$ with $\lambda > 0$. This problem is equivalent to that of studying the behavior of Hopf vector fields on Berger spheres of any radius.

The second variation of the energy and the volume at Hopf vector fields on Berger spheres has been computed in [6] obtaining the following

Proposition 5.2. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g^μ). For each vector field A orthogonal to V^μ we have*

$$a)\ (HessE)_{V^\mu}(A) = \int_{S^{2m+1}} \Big(-2m\mu\|A\|^2 + \|\nabla^\mu A\|^2 \Big)\,\mathrm{dv}^\mu.$$

$$b)\ (HessF)_{V^\mu}(A) = (1+|\mu|)^{m-2} \int_{S^{2m+1}} \Big(\|\nabla^\mu A\|^2 + \ \mu\|\nabla^\mu_{V^\mu} A + \varepsilon_\mu \sqrt{|\mu|} JA\|^2 + \mu(-2m - 2m|\mu| + 2\varepsilon_\mu + 2\varepsilon_\mu(m-\mu))\|A\|^2 \Big)\,\mathrm{dv}^\mu.$$

Using these expressions and equation (5), we have that on the sphere

$$\mathcal{C}^\mu(A) = \int_{S^{2m+1}} (|\mu|(2m+2)(1-\mu)\|A\|^2 + \mu\|\nabla^\mu_{V^\mu} A + \varepsilon_\mu \sqrt{|\mu|} JA\|^2)\,\mathrm{dv}^\mu. \qquad (11)$$

5.1 Riemannian Berger spheres

In the sequel, we will assume that the parameter μ is positive. The study of Lorentzian Berger metrics will be performed in last subsection.

It has been shown in [6] that on (S^3, g^μ), the Hopf vector fields are the only absolute minimizers of the volume when $\mu \leq 1$, otherwise they are unstable. For the metrics g_λ^μ the situation is the following

Proposition 5.3. *On (S^3, V, g) with $\mu > 0$, we have*

1. *If $\mu \leq 1$, $\lambda_{s_1}^\mu = 0$ and $\lambda_{s_2}^\mu = +\infty$, that is to say, V_λ^μ is a stable minimal vector field for all λ.*

2. *If $\mu > 1$, $\lambda_{s_1}^\mu = \lambda_{s_2}^\mu = 0$, that is to say, V_λ^μ is unstable minimal for all λ.*

Proof. By equations (5) and (11),

$$(HessF)_{V^\mu_\lambda}(A) = \lambda^{\frac{3}{2}}(1+\mu/\lambda)^{-1}\Big((HessE)_{V^\mu}(A) + \frac{1}{\lambda}\mathcal{C}^\mu(A)\Big),$$

with

$$\mathcal{C}^\mu(A) = \int_{S^3}(4\mu(1-\mu)\|A\|^2 + \mu\|\nabla^\mu_{V^\mu}A + \sqrt{\mu}JA\|^2)\,\mathrm{dv}^\mu.$$

When $\mu \leq 1$, V^μ is a stable critical point of the energy and $\mathcal{C}^\mu(A) \geq 0$. Then, V^μ_λ is minimal stable for all $\lambda > 0$.
To show 2), if i, j, k represent the imaginary unit quaternions and we take $V = iN$, $E_1 = jN$ and $E_2 = kN$, then $\{V^\mu, E_1, E_2\}$ is a g^μ-orthonormal frame. If we compute the Hessian on the direction E_1 we have,

$$\begin{aligned}(HessF)_{V^\mu_\lambda}(E_1) &= \lambda^{\frac{3}{2}}(1+\mu/\lambda)^{-1}\int_{S^3}(2(-\mu - 2\frac{\mu^2}{\lambda} + 2\frac{\mu}{\lambda}) + \|\nabla^\mu E_1\|^2 \\ &\qquad + \frac{\mu}{\lambda}\|\nabla^\mu_{V^\mu}E_1 + \sqrt{\mu}E_2\|^2)\,\mathrm{dv}^\mu,\end{aligned}$$

where

$$\begin{aligned}\|\nabla^\mu E_1\|^2 &= g^\mu(\nabla^\mu_{V^\mu}E_1, \nabla^\mu_{V^\mu}E_1) + g^\mu(\nabla^\mu_{E_2}E_1, \nabla^\mu_{E_2}E_1) \\ &= \frac{1}{\mu}\|\nabla_V E_1 + (\mu-1)E_2\|^2 + g^\mu(V,V) \\ &= \frac{(\mu-2)^2}{\mu} + \mu, \\ \|\nabla^\mu_{V^\mu}E_1 + \sqrt{\mu}E_2\|^2 &= \frac{1}{\mu}\|\nabla_V E_1 + (2\mu-1)E_2\|^2 = \frac{4}{\mu}(\mu-1)^2.\end{aligned}$$

Therefore,

$$(HessF)_{V^\mu_\lambda}(A) = 4\,\sqrt{\mu}\;\lambda^{3/2}(1+\frac{\mu}{\lambda})^{-1}\mathrm{vol}(S^3)(-\frac{\mu}{\lambda} + \frac{1}{\lambda} + \frac{1}{\mu} - 1) < 0,$$

for all $\lambda > 0$ and V^μ_λ is unstable. □

By the above Proposition the stability of the Hopf vector fields on (S^3, g^μ) is invariant by homotheties. For spheres of upper dimension, it has been shown in [4] that for $\mu = 1$, Hopf vector fields are stable minimal vector fields if and only if $\lambda < 1/(2m-3)$, so the situation for $m > 1$ will be quite different, except for some values of μ.

Proposition 5.4. *On (S^{2m+1}, g^μ) with $m > 1$ and $0 < \mu \leq 1/\sqrt{2m-2}$, $\lambda^\mu_{s_1} = 0$ and $\lambda^\mu_{s_2} = +\infty$. That is to say, the Hopf vector field V^μ_λ is stable as a critical point of the volume for all $\lambda > 0$.*

Proof. By Proposition 5.1, V^μ is stable for the energy. Since $\mathcal{C}^\mu(A) \geq 0$ for all $A \in V^\perp$, Corollary 4.6 give us the result. □

The instability results for the round spheres have been obtained by showing that the Hessian is negative when acting on the vector fields $A_a = a - f_a N - \bar{f}_a V$ with $a \in \mathbb{R}^{2m+2}$, $f_a = \langle a, N\rangle$ and $\bar{f}_a = \langle a, V\rangle$.

If we compute the value of the Hessian acting on these particular vector fields we obtain

Lemma 5.5. *Let (S^{2m+1}, g^μ) be a Berger sphere with $\mu > 0$ and $m > 1$. If $A_a = a - f_a N - \bar{f}_a V$ with $a \in \mathbb{R}^{2m+2}$, then*

$$(HessF)_{V^\mu_\lambda}(A_a) \quad = \quad \lambda^{m+\frac{1}{2}}(1+\frac{\mu}{\lambda})^{m-2} f(m,\mu,\lambda)\frac{\sqrt{\mu}m}{m+1}|a|^2 \mathrm{vol}(S^{2m+1}),$$

where

$$f(m,\mu,\lambda) = (1-2m)\mu + 2 + \frac{(\mu-1)^2}{\mu} + \frac{1}{\lambda}((2m-2)\mu(1-\mu)+1).$$

Proof. It is an immediate consequence of Lemma 5.2 of [6] and equation (5). □

Now, as a consequence, we have the following

Proposition 5.6. *On (S^{2m+1}, g^μ) with $m > 1$, we have*

a) If $1/\sqrt{2m-2} < \mu \leq 1$ then $\lambda^\mu_{s_1} = 0$ and $\lambda^\mu_{s_2} \leq (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$.

b) If $1 < \mu < \mu_c = 1/2(1 + \sqrt{(m+1)/(m-1)})$ then $\lambda^\mu_{s_2} \leq (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$.

c) If $\mu \geq \mu_c$, then V^μ_λ is unstable for the volume for all $\lambda > 0$, that is to say, $\lambda^\mu_{s_1} = \lambda^\mu_{s_2} = 0$.

Proof. If $1/\sqrt{2m-2} < \mu \leq 1$, then $(HessE)_{V^\mu}(A_a) < 0$ and $\mathcal{C}^\mu(A_a) > 0$. Therefore, if

$$\lambda > \frac{\mathcal{C}^\mu(A_a)}{-(HessE)_{V^\mu}(A_a)} = 1 + \frac{(2-2m)\mu^3 + \mu + 1}{(2m-2)\mu^2 - 1},$$

then $(HessE)_{V^\mu}(A_a) + 1/\lambda\ \mathcal{C}^\mu(A_a) < 0$ and V^μ_λ is unstable. Moreover, it is easy to see that $\mathcal{C}^\mu(A) \geq 0$ for all $A \in V^\perp$ and by Corollary 4.6 we get a).

To show b) and c) it is enough to write the condition

$$(HessF)_{V^\mu_\lambda}(A_a) < 0.$$

□

Now, we are going to show that, in some cases, the bound of $\lambda^\mu_{s_2}$ in a), b) above is attained. In order to do so, it is useful to consider the following expressions of the Hessian of the energy.

Proposition 5.7 ([6]). *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g^μ), for each vector field A orthogonal to V^μ we have:*

$$
\begin{aligned}
a)\ (HessE)_{V^\mu}(A) = &\int_{S^{2m+1}} \Big((2m+2-\mu(m^2+4m-1))\ \|A\|^2 \\
&+ \|\nabla^\mu_{V^\mu} A + m\sqrt{\mu} JA\|^2 + \frac{1}{2}\ \|\pi \circ D^C A\|^2_{V^\perp} \Big)\, \mathrm{dv}^\mu. \\
b)\ (HessE)_{V^\mu}(A) = &\int_{S^{2m+1}} \Big((-2m-2-\mu(m^2-1))\ \|A\|^2 \\
&+\ \|\nabla^\mu_{V^\mu} A - m\sqrt{\mu} JA\|^2 + \frac{1}{2}\ \|\bar{D}^C A\|^2_{V^\perp} \Big)\, \mathrm{dv}^\mu.
\end{aligned}
$$

Where, if $\bar{\nabla}$ denotes the Levi-Civita connection on $\mathbb{R}^{2m+2}$, D^C and $\bar{D}^C$ are the differential operators $D^C_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W$ and $\bar{D}^C_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W$, and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to V^\perp$ is the natural projections $\{x\} \times \mathbb{C}^{m+1} \to V^\perp_x$.

Proposition 5.8. *On (S^{2m+1}, g^μ) with $m > 1$ and $\mu^+_s < \mu \leq 1$, $\lambda^\mu_{s_2} = (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$, that is to say, V^μ_λ is a stable minimal vector field if and only if $\lambda \leq (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$.*

Proof. By Proposition 5.6 part a), we only need to show that under the hypothesis on μ, if $\lambda \leq (1 + ((2-2m)\mu^3 + \mu + 1)/((2m-2)\mu^2 - 1))$ the Hessian is non negative, when acting on any vector field A orthogonal to V^μ.

Let $A : S^{2n+1} \to (JN)^\perp \subset \mathbb{C}^{n+1}$, we set

$$A_l(p) = \frac{1}{2\pi} \int_0^{2\pi} A(e^{i\theta} p) e^{-il\theta} d\theta \in (JN)^\perp_p$$

so that the Fourier series of A is

$$A(p) = \sum_{l \in \mathbb{Z}} A_l(p).$$

Since $A_l(e^{i\theta} p) = e^{il\theta} A_l(p)$, we have

$$\nabla_{JN} A = \bar{\nabla}_{JN} A = \sum_{l \in \mathbb{Z}} il A_l$$

and, if $\mathcal{C}(p)$ denotes the fiber of the Hopf fibration passing through p,

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0,$$

if $l \neq q$. As in [4] we can show that

$$(HessF)_{V^\mu_\lambda}(A) = \sum_{l \in \mathbb{Z}} (HessF)_{V^\mu_\lambda}(A_l). \tag{12}$$

By Proposition 5.7 part a),

$$(HessE)_{V^\mu}(A_l) \geq e_1(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{dv}^\mu,$$

with

$$\begin{aligned} e_1(m,\mu,l) &= \mu(1-m^2-4m)+2m+2+\frac{1}{\mu}(l-1+\mu(m+1))^2 \\ &= \mu(2-2m)+2l(m+1)+\frac{1}{\mu}(l-1)^2. \end{aligned}$$

Then, if $\mu \leq 1$, $e_1(m,\mu,l) \geq 0$ for all $l \geq 1$. Moreover, $\mathcal{C}^\mu(A) \geq 0$ when $\mu \leq 1$. Under the hypothesis on μ and λ for $l=0$, we have

$$(HessF)_{V^\mu_\lambda}(A_0) \;\; \geq \;\; \lambda^{m+\frac{1}{2}}(1+\frac{\mu}{\lambda})^{m-2} f(m,\mu,\lambda) \int_{S^{2m+1}} \|A_0\|^2 \,\mathrm{dv}^\mu,$$

where

$$f(m,\mu,\lambda) = \mu(2-2m)+\frac{1}{\mu}+\frac{1}{\lambda}(\mu^2(2-2m)+(2m-2)\mu+1) \geq 0.$$

Therefore, $(HessF)_{V^\mu_\lambda}(A_l) \geq 0$ for all $l \geq 0$. Now, if we use Proposition 5.7 part b),

$$(HessE)_{V^\mu}(A_l) \geq e_2(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{dv}^\mu,$$

with

$$\begin{aligned} e_2(m,\mu,l) &= \mu(1-m^2)-2m-2+\frac{1}{\mu}(l-1+\mu(1-m))^2 \\ &= \mu(2-2m)+2l(1-m)-4+\frac{1}{\mu}(l-1)^2. \end{aligned}$$

For $\mu \leq 1$,

$$e_2(m,\mu,l) \geq 2-2m+2l(1-m)-4+(l-1)^2 = 2m(-l-1)-2-2l+l^2+1 \geq 0,$$

for all $l < 0$ and $\mathcal{C}^\mu(A) \geq 0$, so $(HessF)_{V^\mu_\lambda}(A_l) \geq 0$.

Equation (12) give us that V^μ_λ is volume stable. □

Remark 5.9. For $\mu = 1$, the above Proposition is the result shown in [4] on the existence of a critical radius of round spheres. So, Proposition 5.8 shows that the existence of a critical value of λ is not a property characteristic of round spheres but, surprisingly, it is neither a property of all Berger spheres.

From Propositions 5.4, 5.6 and 5.8 we know the values of $\lambda^\mu_{s_1}$ and $\lambda^\mu_{s_2}$ except for $1 < \mu < \mu_c$. In this case, we can construct a vector field C_2 (see description in next subsection) such that $\nabla^\mu_{V^\mu} C_2 = (\mu-2)/\sqrt{\mu} J C_2$ and consequently,

$$\begin{aligned} \mathcal{C}^\mu(C_2) &= \int_{S^{2m+1}} ((2m+2)\mu(1-\mu) + 4(\mu-1)^2)\|C_2\|^2 \, \mathrm{dv}^\mu \\ &= (2(1-\mu)((m-1)\mu+2)) \int_{S^{2m+1}} \|C_2\|^2 \, \mathrm{dv}^\mu < 0. \end{aligned}$$

By Corollary 4.8, we know then that for $1 < \mu < \mu_c$, $\lambda^\mu_{s_1} = \lambda^\mu_{s_2} = 0$ or $\lambda^\mu_{s_1} > 0$ and $\lambda^\mu_{s_2} < +\infty$.

Since

$$\lim_{\lambda \to 0^+} \Big((HessE)_{V^\mu}(C_2) + \frac{1}{\lambda} \mathcal{C}^\mu(C_2) \Big) = -\infty,$$

we can obtain a lower bound of $\lambda^\mu_{s_1}$ (it can also be zero if V^μ_λ is always unstable) that, even jointly with the upper bound for $\lambda^\mu_{s_2}$ showed in Proposition 5.6, give us only a partial information. Therefore, for some values of $1 < \mu < \mu_c$, the stability of V^μ_λ is still an open problem.

5.2 Lorentzian Berger spheres

In [11] it has been shown that Hopf vector fields V^μ on Lorentzian Berger spheres are unstable critical points of the energy and of the volume for all $\mu < 0$. So,

Proposition 5.10. *On (S^{2m+1}, g), $\mu_s^- = -\infty$.*

The key to prove the above result is to consider vector fields $C_{2s} = \mathrm{grad}^\mu f_{2s} + V^\mu(f_{2s}) f_{2s}$, where f_{2s} is a polynomial of degree $2s$ in $\mathbb{R}^{2m+2}$ such that its restriction to the sphere is a simultaneous eigenfunction of the Laplacian and of the vertical Laplacian of the sphere. These vector fields verify that $\nabla^\mu_{V^\mu} C_{2s} = (\mu - 2s)/\sqrt{-\mu}\; JC_{2s}$ and we have shown

Lemma 5.11 ([11])**.** *Let V^μ be the Hopf vector field on (S^{2m+1}, g^μ) with $\mu < 0$. Then for each $s > 0$ there exists a vector field $C_{2s} = \mathrm{grad}^\mu f_{2s} + V^\mu(f_{2s}) V^\mu$ such that*

a) $(HessE)_{V^\mu}(C_{2s}) = \dfrac{2}{\mu}(\mu^2(1-m) + \mu(2s-1)(m+1) + 2s^2) \displaystyle\int_{S^{2m+1}} \|C_{2s}\|^2 \, \mathrm{dv}^\mu.$

b) $(HessF)_{V^\mu}(C_{2s}) = \dfrac{2}{\mu}(1-\mu)^{m-2} f(s,m,\mu) \displaystyle\int_{S^{2m+1}} \|C_{2s}\|^2 \, \mathrm{dv}^\mu,$

where

$$f(s,m,\mu) = \mu^3(m-1) + \mu^2(4s-2m) + \mu((2s-1)(m+1) - 2s^2) + 2s^2.$$

Using the same arguments as in [11], we can show

Proposition 5.12. *On (S^{2m+1}, g^μ) with $\mu < 0$, $\lambda^\mu_{s_1} = \lambda^\mu_{s_2} = 0$. That is to say, the Hopf vector fields V^μ_λ are unstable for all $\lambda > 0$.*

Proof. From equation (5),

$$(HessF)_{V^\mu_\lambda}(A) = \lambda^{m+\frac{1}{2}}(1-\mu/\lambda)^{m-2}\Big((HessE)_{V^\mu}(A) + \frac{1}{\lambda}\mathcal{C}^\mu(A)\Big),$$

with

$$\mathcal{C}^\mu(A) = \int_{S^{2m+1}} \mu\big((\mu(2m+2)-(2m+2))\|A\|^2 + \|\nabla^\mu_{V^\mu}A - \sqrt{-\mu}JA\|^2\big)\,\mathrm{dv}^\mu.$$

Using Lemma 5.11, for each $s \in \mathbb{N}$, there exists a vector field $C_{2s} = \mathrm{grad}^\mu f_{2s} + V^\mu(f_{2s})V^\mu$, such that

$$(HessE)_{V^\mu}(C_{2s}) = \frac{2}{\mu}(\mu^2(1-m)+\mu(4s-1)(m+1)+8s^2)\int_{S^{2m+1}}\|C_{2s}\|^2\,\mathrm{dv}^\mu.$$

Therefore,

$$(HessF)_{V^\mu_\lambda}(C_{2s}) = \lambda^{m+\frac{1}{2}}(1-\frac{\mu}{\lambda})^{m+2}f(m,\mu,\lambda,s)\int_{S^{2m+1}}\|C_{2s}\|^2\,\mathrm{dv}^\mu,$$

where

$$f(m,\mu,\lambda,s) = \frac{2}{\mu}\Big(\mu^2(1-m)+\mu(4s-1)(m+1)+8s^2+\frac{1}{\lambda}\big(\mu^3(m-1)+\mu^2(8s-1-m)-8s^2\mu\big)\Big).$$

Since $\lim_{s\to+\infty} f(m,\mu,\lambda,s) = -\infty$, for each $\mu < 0$ and $\lambda > 0$, we only have to choose s big enough to obtain that $(HessF)_{V^\mu_\lambda}(C_{2s}) < 0$ and then V^μ_λ unstable. □

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